446 Assignment 4: Due April 3, in class.

1. Let A and B be finitely generated k-algebras with $B \subseteq A$ and A finitely generated as a B-module. Show that $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$. (Hint: Write $A = Bb_1 + \cdots + Bb_d$. Notice that $b_i b_j = \sum c_{i,j,\ell} b_\ell$ for some $c_{i,j,\ell} \in A$.)

2. Let k be a field and let A be a finitely generated k-algebra with generators $a_1, \ldots, a_m$. Suppose that there is a nonzero polynomial $p(x_1, \ldots, x_m)$ such that $p(a_1, \ldots, a_m) = 0$. Show that there is a a substitution
$$b_1 = a_1 + a_m^{N_1}, \ldots, b_{m-1} = a_{m-1} + a_m^{N_{m-1}}, b_m = a_m$$
such that $p(a_1, \ldots, a_m) = p(b_1 - b_m^{N_1}, \ldots, b_{m-1} - b_m^{N_{m-1}}, b_m)$ is a polynomial of the form
$$cb_m^N + \sum_{j<N} q_j(b_1, \ldots, b_{m-1}) b_m^j$$
with c a nonzero constant in k. (Hint: Let D be the total degree of $p(x_1, \ldots, x_m)$ and let $N_i = D^i$; show that a monomial $a_1^{j_1} \cdots a_m^{j_m}$ under the substitution gets transformed to something of
$$b_m^{j_m + Dj_1 + \cdots + D^{m-1} j_{m-1}} + \text{lower degree terms in } b_m.$$
Think of base D-expansions now.)

3. Let R be a ring and let S be a multiplicatively closed set of nonzero divisors. Show that the Krull dimension of $S^{-1}R$ is at most the Krull dimension of R.

4. Let $R = k[x_1, \ldots, x_d]$. Show that for every $m \leq d$ with $m \geq 0$ there is a subset S such that the Krull dimension of $S^{-1}R$ is exactly m.

5. A ring of finite Krull dimension ($= d$) is said to be *catenary* if every maximal chain of prime ideals (that is, a chain to which one cannot add in any more prime ideals) has length d. Let $R = \mathbb{C}[x, y]$ and let $P_1 = (x)$ and $P_2 = (x - 1, y)$. Show that $S = R \setminus (P_1 \cup P_2)$ is multiplicatively closed and that $S^{-1}R$ has Krull dimension 2 but that the chain $(0) \subset S^{-1}P_1$ is a maximal chain of prime ideals. Thus $S^{-1}R$ is not catenary.

6. Let k be a field and let A and B be finitely generated k algebras. Show that the Krull dimension of $A \otimes_k B$ is the sum of the Krull dimensions of A and B.

7. Let $R = \mathbb{C}[t]$ and $S = \mathbb{C}[t^2, t^3]$. Show that R and S are not isomorphic. Show that the map from $\mathrm{Spec}(R)$ to $\mathrm{Spec}(S)$ given by $P \mapsto P \cap S$ is a homeomorphism, so $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ are homeomorphic but $R \not\cong S$.

8. Let R be a noetherian local ring with maximal ideal M and let I be a proper ideal. Show that
$$L := \bigcap_{n \geq 1} I^n = (0).$$

9. Let R be an integral domain and let P be a height one prime. Show that $\mathrm{Spec}(R_P)$ consists of two points, one of which is closed and the other of which is dense.

10. Let R and S be rings and let $f : R \to S$ be a ring homomorphism. Show that f induces a continuous map $\phi_f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ given by $\phi_f(P) = f^{-1}(P)$. Show that if $g : S \to T$ is another homomorphism then $\phi_{g \circ f} = \phi_f \circ \phi_g : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$. Show that if f is an isomorphism then ϕ_f is a homeomorphism.

11. Let R be a ring and let $S = R[x]$. Then we have maps $f : R \to S$ given by $f(r) = r$ and $g : S \to R$ given by $g(p(x)) = p(0)$. Then $g \circ f = \mathrm{id}_R$. Let ϕ_f and ϕ_g be as in the preceding exercise. Which of ϕ_f, ϕ_g are injective, surjective?

12. Suppose that R is an infinite noetherian ring. Show that $|\mathrm{Spec}(R)| \leq |R|$. If R is finite, show that $|\mathrm{Spec}(R)| \leq 2^{|R|}$.

13. Give an example of a countable (necessarily non-noetherian) ring R such that $\mathrm{Spec}(R)$ is an uncountable topological space.

14. Show that if R is noetherian then $\mathrm{Spec}(R)$ is noetherian as a topological space. Is the converse true?

15. Show that if A is a finitely generated k-algebra that is an integral domain and B is a finitely generated subalgebra with $\mathrm{GKdim}(B) = \mathrm{GKdim}(A)$ then the field of fractions of A is a finite extension of the field of fractions of B.