446 Assignment 3: Due March 15, in class.

1. Let k be a field. Show that if P is a nonzero prime ideal of $k[x]$ then P is maximal and $k[x]/P$ is a finite extension of k.

2. Let k be a finite field. Show that $k[x]$ has zero Jacobson radical and that it is a Jacobson ring (we did this for infinite k in class). (For this, you will need material from 348—namely, the following fact might be useful: for every prime power q and $n \geq 1$, there exists an irreducible polynomial in $\mathbb{F}_q[x]$ of degree n.)

3. Let $R \subseteq S$ be integral domains and suppose that S is a finitely generated R-module. Show that $1_R = 1_S$ and that if $b \in S$ show that there is some $n \geq 1$ and $r_0, \ldots, r_{n-1} \in R$, not all zero, such that $b^n + \cdots + r_0 = 0$.

4. Let $R \subseteq S$ be integral domains with $1_R = 1_S$ and suppose that S is finitely generated as an R-module (remember that this is much stronger than being finitely generated as an R-algebra). Show that S is a field if and only if R is a field. (Hint: the preceding question might be useful.)

5. Let $R \subseteq S$ be rings with $1_R = 1_S$ and suppose that is S a finite R-module. Show that if T is a multiplicatively closed set of R consisting of elements that are regular in S (and hence in R) then $T^{-1}S$ is a finite $T^{-1}R$-module.

6. Let F be an uncountable algebraically closed field. (If you prefer, just take $F = \mathbb{C}$.) Show that if K is a field extension of F and $[K : F] \leq \aleph_0$ then $K = F$. (Hint: first show that if $t \in K \setminus F$ is not algebraic over F then the subset $\{1/(t - \lambda) \colon \lambda \in F\} \subseteq K$ is linearly independent over F.)

7. Let F be an uncountable algebraically closed field and let A be a countably generated F-algebra (that is, A is an F-algebra that can be generated by a countable set of elements, so $A \cong F[x_1, x_2, \ldots]/I$). Show that the Nullstellensatz holds for A: that is, show that if M is a maximal ideal of A then $A/M \cong F$.

8. Let F be a field and let A be an F-algebra. We say that $x \in A$ is *algebraic* over F if there is some $n \geq 1$ and some $c_0, \ldots, c_n \in F$, not all zero, such that

$$c_0 + c_1 x + \cdots + c_n x^n = 0.$$

Show that if $x \in J(A)$ is algebraic over F then x is nilpotent.

9. Let F be an uncountable algebraically closed field and let A be a countably generated F-algebra. Show that $J(A)$, the Jacobson radical of A, is a nil ideal. Conclude that A is a Jacobson ring.

10. Let R be a ring. Show that the units group of $R[x]$ consists of all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ where a_0 is a unit of R and $a_1, \ldots, a_n$ are in the nil radical of R. (Hint: think of geometric series to show that these elements are units; to show that any unit is of this form, let P be a prime ideal of R. Show $PR[x]$ is a prime ideal of $R[x]$ and $p(x)$ mod $PR[x]$ is constant.)

11. Let R be a ring whose nil radical is (0) (the nil radical is just the radical ideal of (0)). Use the preceding exercise to show that $J(R[t])$ is zero.

    The next two problems are useful in algebraic number theory as they apply to what are called number rings.

12. Let R be an integral domain and suppose that the group $(R, +)$—R under addition–is a finitely generated abelian group. Show that if P is a maximal ideal of R then R/P is a finite field. (Hint: Nullstellensatz.)

13. Let R be an integral domain and suppose that the group $(R, +)$ is a finitely generated abelian group. Show that if P is a prime ideal and P is not maximal then $P = (0)$. (Hint: think of question 3.)

14. Let R be a local ring with unique maximal ideal P and let M and N be R-modules such that $M \oplus N \cong R^n$ for some n. Show that there is some i such that $M \cong R^i$ and $N \cong R^{n-i}$. (Hint: Let $F = R/P$ and show that $M/PM \oplus N/PN \cong F^n$ and conclude that $M/PM \cong F^i$ for some i. Use Nakayama's lemma to show that M can be generated by i elements and N can be generated by $n-i$ elements. Show that these elements are independent! Observe that one doesn't really need the free module to have finite rank to get this argument to work.)