446 Assignment 2: Due February 27, in class.

1. Let R be a ring and let X and Y be index sets. Use ideas from proof of the result for isomorphisms of direct sums of tensor products to show that if $R^{\oplus X}$ and $R^{\oplus Y}$ are free R-modules with bases $\{x_\alpha\}_{\alpha\in X}$ and$\{y_\beta\}_{\beta\in Y}$ respectively, then $R^{\oplus X}\otimes_R R^{\oplus Y}\cong R^{\oplus(X\times Y)}$ with basis $\{x_\alpha\otimes y_\beta\colon(\alpha,\beta)\in X\times Y\}$.

2. Let R be a ring and let S and T be R-algebras with homomorphisms $\alpha: R\to S$ and $\beta: R\to T$. Show that $S\otimes_R T$ is an R-algebra with multiplication given by $(s_1\otimes t_1)\cdot(s_2\otimes t_2)=s_1s_2\otimes t_1t_2$ (and extending via linearity) and has identity $1_S\otimes 1_T$ and the homomorphism from $R\to S\otimes_R T$ is given by $r\mapsto\alpha(r)\otimes 1_T=1_S\otimes\beta(r)$. (Hint: the only thing that is nontrivial is to show that multiplication is well-defined; once one has this, it is straightforward to show that $S\otimes_R T$ is an R-algebra. A nice way to do this is to define a multilinear map from $S\times T\times S\times T\to S\otimes_R T$ via $(s_1,t_1,s_2,t_2)\mapsto s_1s_2\otimes t_1t_2$ (check this is multilinear). Show using the universal property that this gives a bilinear map from $m:(S\otimes_R T)\times(S\otimes_R T)\to S\otimes_R T$. Then this m is exactly our multiplication.)

3. Show this important and useful fact: if R is a ring and S and T and C are R-algebras then in $f: S\otimes_R T\to C$ is an R-module homomorphism that sends $1_S\otimes 1_T$ to 1_C and $f(s_1s_2\otimes t_1t_2)=f(s_1\otimes t_1)f(s_2\otimes t_2)$ for all $s_1,s_2\in S$ and $t_1,t_2\in T$ then f is an R-algebra homomorphism. This is not a hard fact, but it is useful to know and is part of a general strategy to construct homomorphisms for tensor products of R-algebras: use the universal property to get an R-module homomorphism and then check that the product condition above holds.

4. Show that if $f: A\to B$ and $g: C\to D$ are R-algebra homomorphisms then $(f\otimes g):(A\otimes_R C)\to(B\otimes_R D)$ is an R-algebra homomorphism.

5. Show that if A and B are subalgebras of an R-algebra C then there is an R-algebra homomorphism $f: A\otimes_R B\to C$ satisfying $f(a\otimes b)=ab\in C$ for $a\in A$ and $b\in B$.

6. Use the ideas from the preceding exercise to show that if S is an R-algebra then $S\otimes_R R[t_1,\ldots,t_d]\cong S[t_1,\ldots,t_d]$.

7. Let R and S be a noetherian rings. Show that $R\times S$ is a noetherian ring (you don't need to show it is a ring). But show that an infinite direct product of non-trivial rings is never noetherian.

8. Show that if S and T are finitely generated R-algebras then $S\otimes_R T$ is also finitely generated as an R-algebra. In particular, if R is noetherian and S and T are finitely generated R-algebras then $S\otimes_R T$ is noetherian.

9. Let $F\subseteq K$ be fields and suppose that A is a K-algebra. Show that A is also an F-algebra and that both F and K embed in A as subrings.

10. Let $F\subseteq K$ be fields. Show that if A and B are K-algebras, then there is a surjective F-algebra homomorphism $f: A\otimes_F B\to A\otimes_K B$ satisfying $a\otimes_F b\to a\otimes_K b$. (Hint: show that if a map from $A\times B$ to a K-module P is K-bilinear then it is necessarily F-bilinear.)

11. Let $F\subseteq E\subseteq K$ be fields and let $f: K\otimes_F K\to K\otimes_E K$ be as in question 9. Show that if $E\neq F$ then f is not injective. (Hint: pick $\lambda\in E\setminus F$. Show that $\lambda\otimes_F 1-1\otimes_F\lambda$ is nonzero in $K\otimes_F K$ using exercise 1 and show that it is in the kernel of f.)

12. We recall that a set S is a *partially ordered set* (or poset) if it has an order $\leq$ with the properties that $a\leq a$ for all $a\in S$; if $a\leq b$ and $b\leq a$ then $a=b$; and if $a\leq b$ and $b\leq c$ then $a\leq c$. But we do not assume that all elements can be compared. For example, subsets of $\{1,\ldots,n\}$ form a partially ordered set under inclusion. Show that if $F\subseteq K$ are fields then the set $\mathcal{F}(K/F)$ of fields L with $F\subseteq L\subseteq K$ is a poset under inclusion. Show that if R is a ring then the set $\mathcal{I}(R)$ of ideals in R is a poset under inclusion.

13. Show that if $F \subseteq K$ are fields then there is a map Φ from the poset $\mathcal{F}(K/F)$ to the poset of ideals in $K \otimes_F K$ given by $L \in \mathcal{F}(K/F) \mapsto \Phi(L) := \ker(f)$, where $f : K \otimes_F K \to K \otimes_L K$ is as described in 10. Show that if $L_1 \subseteq L_2$ then $\Phi(L_1) \subseteq \Phi(L_2)$ (we call this property *inclusion-preserving*) and if $L_1 \subseteq L_2$ and $\Phi(L_1) = \Phi(L_2)$ then $L_1 = L_2$.

14. Show that if $F \subseteq L \subseteq K$ are fields and L is not finitely generated as a field extension of F then there is a non-terminating ascending chain of fields

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots,$$

with each L_i satisfying $F \subseteq L_i \subseteq K$. Use the preceding exercise to show that if $K \otimes_F K$ is noetherian then every subfield of K containing F is finitely generated as a field extension of F.

15. Let $R = \mathbb{C}[x, y]$. This is a noetherian ring, as we shall see. Let $J = (x(x-1), y^3)$. Find a finite (multi)-set of prime ideals containing J whose product is contained in J.