446/646 Assignment 1:

Due January 27, in class. (All rings are assumed to be commutative)

1) Let $f : M \to N$ be a bijective R-module homomorphism. Show that f^{-1} is an R-module homomorphism.
2) Show that $\oplus_{i=1}^{\infty}\mathbb{Z} \not\cong \prod_{i=1}^{\infty}\mathbb{Z}$ as $\mathbb{Z}$-modules.
3*) Show in fact that $\prod_{i=1}^{\infty}\mathbb{Z}$ is not free as a $\mathbb{Z}$-module.
4) Show that every vector space has a basis and show that any two bases for a vector space have the same cardinality.
5) Show that $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ for R-modules M, N, P, where all tensor products are taken over R.
6) Show that tensoring is right-exact: i.e., if $M \to N \to P \to 0$ is exact then so is $M \otimes C \to N \otimes C \to P \otimes C \to 0$.
7) Let F be a field. Show that $F^n \otimes_F F^m \cong F^{mn}$.
8) Let R be a ring and let I and J be proper ideals. Show that $R/I \otimes_R R/J \cong R/(I + J)$.
9) Let $N \subseteq M$ be R-modules. Prove that there is a bijective correspondence between R-submodules of M/N and R-submodules of M that contain N.
10) An R-module M is said to have a basis if there is a subset $S \subseteq M$ such that every element of M can be expressed uniquely as a finite R-linear combination of elements from S. That is, if $m \in M$ then there exist $s_1, \ldots, s_d \in S$ and $r_1, \ldots, r_d \in R$ such that $m = r_1s_1 + \cdots + r_ds_d$ and if $u_1, \ldots, u_m \in S$ are distinct and $r_1u_1 + \cdots + r_mu_m = 0$ then we have $r_1 = \cdots = r_m = 0$. Show that an R-module M has a basis if and only if $M \cong R^{\oplus J}$ for some index set J.
11) Is $\mathbb{R}$ a free $\mathbb{Z}$-module?
12) View $\mathbb{C}$ be an $\mathbb{R}$-module via the rule $\alpha \cdot \beta = \alpha\beta$ for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{R}^4$
13) Now view $\mathbb{C}$ as an $\mathbb{R}[x]$-module by viewing $\mathbb{C}$ as $\mathbb{R}[x]/(x^2 + 1)$. What is $\mathbb{C} \otimes_{\mathbb{R}[x]} \mathbb{C}$ isomorphic to? Express your answer as a direct sum of modules of the form $\mathbb{R}[x]/(p(x)^d)$ with $p(x)$ irreducible and $d \geq 1$.
14) Let $R = \mathbb{C}[x, y]$ and let $M = \mathbb{C}$ with the R-module structure given by $p(x, y)\alpha = p(0, 0)\alpha$ for all $p(x, y) \in \mathbb{C}[x, y]$ and let $N = \mathbb{C}$ with the R-module structure given by $p(x, y) \cdot \alpha = p(1, 1)\alpha$ for all $\alpha \in \mathbb{C}$. Show that M and N are not isomorphic as R-modules.
15) Give an example of a ring R and a finitely generated R-module M such that M is not a free R-module but there is some finitely generated R-module N such that $M \oplus N$ is free.
16**) (No hints will be given for this question) Let R denote the ring of real-valued convergent sequences and let S denote the ring of real-valued sequences. Prove or disprove: $R \cong S$.