$d < N$, then by Theorem 0.1.27 any element of $V^N$ can be expressed as a linear combination of elements of the form $w_1 w_2^m w_3$ with length$(w_1 w_2^m w_3) \leq N$. By the Cayley-Hamilton theorem $w_2^m$ can be expressed as an $F$-linear combination of smaller powers of $w_2$ and hence $w_1 w_2^n w_3$ is in fact in $V^{N-1}$. It follows that $V^N = V^{N-1}$, and so $V^{n+1} = V^n$ for all $n \geq N$. ∎

## 0.2 Structure theory

### 0.2.1 Structure theory for Artinian rings

To understand affine algebras of GK dimension zero, it is necessary to introduce the concept of an Artinian ring.

**Definition 0.2.1** *A ring $R$ is said to be* left Artinian *(respectively* right Artinian*), if $R$ satisfies the descending chain condition on left (resp. right) ideals; that is, for any chain of left (resp. right) ideals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

*there is some $n$ such that*

$$I_n = I_{n+1} = I_{n+2} = \cdots .$$

*A ring that is both left and right Artinian is said to be* Artinian.

A related concept is that of a *Noetherian ring.*

**Definition 0.2.2** *A ring $R$ is said to be* left Noetherian *(respectively* right Noetherian*) if $R$ satisfies the ascending chain condition on left (resp. right) ideals. Just as in the Artinian case, we declare a ring to be* Noetherian *if it is both left and right Noetherian.*

An equivalent definition for a left Artinian ring is that every non-empty collection of left ideals has a minimal element (when ordered under inclusion). Similarly, a left Noetherian ring can be defined as a ring in which every non-empty collection of left ideals has a maximal element. A left Noetherian ring has the property that every left ideal is generated by a finite number of elements as a left $R$-module. In the commutative case the relation between the Artinian property and the Noetherian property is the following.

**Theorem 0.2.3** *A commutative ring $R$ is Artinian if and only if it is Noetherian and has Krull dimension zero.*

**Proof.** See [12] page 75 Theorem 2.14. ∎

In the noncommutative case, the relationship is not so simple. A theorem of Hopkins (see [16]) shows that Artinian rings are Noetherian just as in the commutative case; however, the relationship one might expect, namely that the GK dimension of an Artinian ring should be zero when considered an algebra over its center, does not hold. (See for example 6.6.18 on page 205 of [22].) It is an open problem whether an affine Artinian ring is necessarily finite dimensional and whether an Artinian ring can have GK dimension lying strictly between zero and infinity when considered as an algebra over its center.

It is clear that any finite dimensional $F$-algebra is Artinian, since a descending chain of left (or right) ideals is a descending chain of $F$-vector spaces. Thus an affine algebra of GK dimension zero is Artinian.

We shall now develop the Artin-Wedderburn theory of Artinian rings. To do this we must first introduce the concept of a *primitive ring*. Given a ring $R$, we say that a left $R$-module $M$ is *faithful* if

$$r \cdot M \;=\; 0 \qquad \text{implies} \qquad r \;=\; 0.$$

We say that $M$ is *simple* if it is nonzero and has no proper nonzero submodules. If $R$ has no nonzero proper ideals, then we say $R$ is a *simple ring*.

**Definition 0.2.4** *A ring $R$ is said to be (left)* primitive *if it has a faithful, simple left $R$-module. An ideal $P$ of $R$ is primitive, if $R/P$ is a primitive ring and a ring $R$ is said to be* semiprimitive *if* $(0)$ *is the intersection of primitive ideals of $R$.*

One can also define the idea of right primitivity. Bergman [8] has constructed examples of rings that are left primitive but not right primitive. We give some examples of primitive rings.

**Example 0.2.5** *A simple ring is primitive.*

**Proof.** Notice if $R$ is a simple ring and $\mathcal{M}$ is a maximal left ideal of $R$, then $M := R/\mathcal{M}$ is simple as a left $R$-module. If $r \cdot M = 0$, then $(RrR) \cdot M = 0$. Since $R$ has no nonzero proper ideals and $1 \cdot M \neq 0$, we conclude that $r = 0$. Thus $M$ is also faithful as a left $R$-module. Hence $R$ is primitive. ∎

**Example 0.2.6** *Let $D$ be a division ring. Then $M_n(D)$ is simple and hence primitive.*

Notice that $n \times 1$ vectors with entries in $D$ is a left $M_n(D)$ module that is faithful and simple.

**Example 0.2.7** *A commutative ring $R$ is primitive if and only if it is a field.*

**Proof.** If $R$ is a field, then it is primitive by Example 0.2.5. If $R$ is primitive, then it has a faithful simple module $M$. Notice that since $M$ is simple,

$$M \;\cong\; R/\mathcal{M}$$

for some maximal ideal $\mathcal{M}$ of $R$. But $\mathcal{M} \cdot M = 0$ and since $M$ is faithful, we conclude that $\mathcal{M} = (0)$. Thus $R$ is a field. ∎

**Example 0.2.8** *Let $F$ be a field of characteristic zero. Then the Weyl algebra, $W(F)$, is simple and hence primitive.*

**Proof.** Let

$$T_x(r) \;=\; xr - rx \qquad \text{and} \qquad T_y(r) \;=\; yr - ry \qquad \text{for } r \in W(F). \tag{0.2.14}$$

By induction we have

$$y^j x \;=\; xy^j - jy^{j-1}$$

for all $j \geq 0$ and hence

$$T_x(x^i y^j) \;=\; x^i(jy^{j-1}). \tag{0.2.15}$$

Similarly

$$T_y(x^i y^j) \;=\; -(ix^{i-1})y^j. \tag{0.2.16}$$

Let $I$ be a nonzero ideal in $W(F)$. Let $a \in I$ be nonzero. We can write

$$a \;=\; \sum_{i,j} \beta_{i,j} x^i y^j$$

for some constants $\beta_{i,j} \in F$ with only finitely many of the $\beta_{i,j}$ nonzero. Let

$$N \;=\; \max\big\{i \mid \beta_{i,j} \neq 0 \text{ for some } j\big\}.$$

Notice that

$$T_y^N(a) \;=\; (-1)^N \sum_j \beta_{N,j} N! y^j \;\in\; I.$$

Let $\tilde{N}$ denote the largest value of $j$ such that $\beta_{N,j} \neq 0$. Then $T_x^{\tilde{N}}(T_y^N(a)) = (-1)^N \beta_{N,\tilde{N}} N!\tilde{N}! \neq 0$ is an element of $I$ and hence $I = W(F)$. ∎

Recall that $W(F)$ can be thought of as the ring of operators $F[t, d/dt]$. Let $V = F[t]$ and turn $V$ into an $F[t, d/dt]$-module by endowing it with the natural action; i.e.,

$$t \cdot p(t) \;=\; tp(t), \tag{0.2.17}$$

$$\big(d/dt\big) \cdot p(t) \;=\; p'(t). \tag{0.2.18}$$

We claim that $V$ is a faithful, simple module for $W(F)$ when $F$ has characteristic zero. To see that it is simple, notice that for any polynomial $p(t) = p_m t^m + \cdots p_0 \in V$ of degree $m$, we have

$$\frac{1}{m!p_m} t^j (d^m/dt^m) \cdot p(t) \;=\; t^j$$

and hence any element of $V$ generates $V$ as a $W(F)$-module. To see that $V$ is faithful, note that if a nonzero element $r \in W(F)$ annihilates $V$, then the two-sided ideal generated by $r$ must also annihilate $V$. But $W(F)$ is simple and so 1 must annihilate $V$ implying that $V$ is the zero module. Thus $V$ is indeed a faithful, simple $W(F)$-module.

The Jacobson density theorem is one of the most useful results for studying primitive rings. We give a proof of this result, but first we need a result due to Schur. Given a ring $R$ and a left $R$-module $M$, we denote by

$$\mathrm{End}_R(M) \tag{0.2.19}$$

the ring of all $R$-module homomorphisms from $M$ to $M$ with multiplication given by composition of maps. Notice that $M$ is a left $\mathrm{End}_R(M)$-module, with action given by $f \cdot x = f(x)$.

**Lemma 0.2.9** *(Schur's lemma) If $M$ is a simple $R$-module, then* $\mathrm{End}_R(M)$ *is a division ring.*

**Proof.** Let $f : M \to M$ be a nonzero homomorphism. Notice that the kernel of $f$ is a submodule of $M$ and is hence either $(0)$ or $M$ by the simplicity of $M$. Since $f$ is nonzero, we have that the kernel is trivial and so $f$ is injective. Notice that the image of $f$ is a nonzero submodule of $M$ and hence $f$ is surjective. Thus $f$ is a bijection. Take $g$ to be the inverse of $f$. Notice that if $x, y \in M$ and $r \in R$, then

$$f(g(x+y) - g(x) - g(y)) = f(g(x+y)) - f(g(x)) - f(g(y)) = x + y - x - y = 0.$$

Since $f$ is injective, we conclude that $g(x+y) = g(x)+g(y)$. Similarly, $g(rx) = rg(x)$, and so we see that $g$ is an $R$-module homomorphism. ∎

**Theorem 0.2.10** *(Jacobson density theorem) Let $R$ be a primitive ring with a faithful simple module $M$. Let $D = \mathrm{End}_R(M)$. Then $R$ is dense in $\mathrm{End}_D(M)$; that is, given a $D$-linearly independent subset of $M$, $\{x_1, \ldots, x_n\}$, and another subset of $M$, $\{y_1, \ldots, y_n\}$, of the same size there exists an element $r \in R$ such that $rx_i = y_i$ for $1 \leq i \leq n$.*

**Proof.** We prove this theorem by induction on $n$. Notice that when $n = 1$, the result is true since $M$ is faithful. Suppose the claim is true for $n < m$ and consider the case $n = m$. Notice that $R(x_1, \ldots, x_{m-1}) \cong M \oplus M \oplus \cdots \oplus M$, where there are $m-1$ copies of $M$ appearing on the right hand side. If there exists an $r \in R$ such that $rx_i = 0$ for

$i < m$ and $rx_m \neq 0$, then we are done, since $rx_m$ generates $M$ as an $R$-module. Thus we may assume that $rx_m = 0$ whenever $rx_i = 0$ for $i = 1, \ldots, m-1$. It follows that we have a well-defined surjective map

$$\Phi : M^{m-1} \;\cong\; R(x_1, \ldots, x_{m-1}) \to M,$$

given by $(rx_1, \ldots, rx_{m-1}) \mapsto rx_m$. Let $f_j : M \to M^{(m-1)}$ be defined by

$$f_j(x) \;=\; (0, \ldots, 0, x, 0, \ldots, 0)$$

for $1 \leq j < m$. We have that $\Phi \circ f_j : M \to M$ is an element $\delta_j \in D$ for $1 \leq j < m$. Notice that for $j < m$, $\Phi \circ f_j(x_j) = r_j x_m$, where $r_j \in R$ satisfies

$$r_j x_i \;=\; \begin{cases} x_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus $(r_1 + \cdots + r_{m-1} - 1)x_i = 0$ for $1 \leq i \leq m$. It follows that $(r_1 + \cdots + r_{m-1} - 1)x_m = 0$ and so

$$(r_1 + \cdots + r_{m-1})x_m = x_m.$$

Thus

$$\begin{aligned} \delta_1 x_1 + \cdots + \delta_{m-1} x_{m-1} &= \Phi \circ f_1(x_1) + \cdots + \Phi \circ f_{m-1} x_{m-1} \\ &= (r_1 + \cdots + r_{m-1}) x_m \\ &= x_m. \end{aligned}$$

This contradicts the fact that $\{x_1, \ldots, x_m\}$ is a $D$-linearly independent subset of $M$. The result follows. ∎

An immediate corollary of this is the following.

**Theorem 0.2.11** *Let $R$ be a primitive, left Artinian ring. Then*

$$R \;\cong\; M_n(D)$$

*for some division ring $D$.*

**Proof.** Let $M$ be a faithful, simple $R$-module, and let $D = \text{End}_R(M)$. Notice that if $M$ is infinite dimensional over $D$, then we can find a countably infinite $D$-linearly independent subset $\{x_1, x_2, \ldots\} \subseteq M$. Let

$$I_n \;=\; \bigl\{r \in R \bigm| rx_i = 0 \text{ for } 1 \leq i \leq n\bigr\}.$$

Notice that

$$I_1 \;\supseteq\; I_2 \;\supseteq\; \cdots$$

is a descending chain of left ideals. By the density theorem there exists $r \in R$ such that $rx_1 = \cdots = rx_{n-1} = 0$ and $rx_n \neq 0$ and so $I_{n-1} \neq I_n$ for all $n \geq 2$. It follows that if $M$ is infinite dimensional over $D$, then $R$ is not Artinian. Hence we have that $M$ is finite dimensional over $D$, say this dimension is $n$. Then $M \cong D \oplus D \oplus \cdots \oplus D$, where there are $n$ copies of $D$. Hence $R$ is a dense subring of $\text{End}_D(M) \cong M_n(D)$. The only dense subring of $M_n(D)$ is the ring itself and so the result follows. ∎

In commutative algebra the concept of a prime ideal plays an important role. We now give the definition of a prime ideal in the noncommutative case. Given a ring $R$, we say that an ideal $P$ is a *prime* ideal if whenever $aRb \subseteq P$ we have that either $a$ or $b$ is an element of $P$. Equivalently, $P$ is prime if whenever $I$ and $J$ are ideals such that $IJ \subseteq P$, we necessarily have either $I$ or $J$ is contained in $P$. Notice this definition coincides with the definition of a prime ideal in a commutative ring. We say that a ring is *prime* if (0) is a prime ideal. Finally, we say that a ring is *semiprime* if (0) is the intersection of prime ideals in the ring. Equivalently, a semiprime ring is a ring with no nonzero nilpotent ideals.

**Proposition 0.2.12** *A primitive ring is prime.*

**Proof.** Let $R$ be a primitive ring and let $M$ be a faithful simple $R$-module. Suppose $aRb = 0$. Then $aRbM = 0$. Suppose $b \neq 0$. Then there exists $m \in M$ such that $bm \neq 0$ since $M$ is faithful. Since $M$ is simple, we have $RbM = M$. Thus $aM = 0$ and so $a = 0$. Hence either $a$ or $b$ is zero and so $R$ is prime. ∎

A prime ring need not be primitive. For example, take $R = F[t]$. Since $R$ is a domain, it is prime. By Example 0.2.7, $R$ is not primitive. Nevertheless, in the

Artinian case a prime ring is indeed primitive. We prove this result now.

**Proposition 0.2.13** *Let $R$ be a prime left Artinian ring. Then $R$ is primitive.*

**Proof.** Let $L$ be a minimal nonzero left ideal in $R$. Notice $L$ is a simple left $R$-module by the minimality of $L$. We claim also that $L$ is a faithful $R$-module. To see this, suppose that there exists some $a \in R$ such that $aL = 0$ and choose $b \neq 0 \in L$. Then $aRb = 0$ and since $R$ is prime we conclude that $a = 0$. Hence $L$ is faithful. Thus $R$ is primitive. ∎

Combining this result with Theorem 0.2.11 we see that a prime Artinian ring is isomorphic to matrices over a division ring. Hence every prime ideal in an Artinian ring is maximal, as the quotient is a simple ring. We now consider semiprime Artinian rings.

**Proposition 0.2.14** *A left Artinian ring has only finitely many prime ideals.*

**Proof.** Suppose that $\{P_i \mid i \geq 1\}$ is an infinite set of distinct prime ideals. Notice that the descending chain

$$P_1 \supseteq P_1 \cap P_2 \supseteq \cdots$$

must eventually terminate and hence there exists an $n$ such that

$$P_n \supseteq P_1 \cap P_2 \cap \cdots \cap P_{n-1}.$$

Letting $I_j = P_j + P_n$ for $1 \leq j \leq n$, we see that

$$P_n = I_1 \cap \cdots \cap I_{n-1}$$

and since $P_n$ is a prime ideal, we conclude that $I_j = P_n$ for some $j$. Hence $P_j \subseteq P_n$. Since $P_j$ is maximal, we have that $P_j = P_n$, which contradicts the assumption that $\{P_i \mid i \geq 1\}$ is a distinct set of primes. The result follows. ∎

We have just seen that an Artinian ring $R$ has only finitely many primes, say $P_1, \ldots, P_n$. If $R$ is also semiprime, then

$$(0) = \bigcap_{i=1}^{n} P_i.$$

Notice that by Theorem 0.2.11 and Proposition 0.2.13, a prime ideal in an Artinian ring is maximal. Thus $P_i$ and $P_j$ are comaximal for $i \neq j$; that is, $P_i + P_j = R$ for $i \neq j$. We can now employ the Chinese remainder theorem, which we quickly state.

**Theorem 0.2.15** *(Chinese remainder theorem) Let $I_1, \ldots, I_n$ be pairwise comaximal ideals in a ring $R$. Then*

$$R \Big/ \left( \bigcap_{i=1}^{n} I_i \right) \;\cong\; \prod_{i=1}^{n} R/I_i.$$

**Proof.** See Proposition 2.2.1 on page 162 of [28] ∎

By this theorem we have

$$R \;\cong\; R \Bigg/ \left( \bigcap_{i=1}^{n} P_i \right) \cong \prod_{i=1}^{n} R/P_i.$$

Thus we have the following theorem.

**Theorem 0.2.16** *(Artin-Wedderburn) A semiprime left Artinian ring is a finite product of matrix rings over division algebras.*

We have now completely determined the structure of semiprime Artinian rings. We continue our study of Artinian rings by introducing the concept of the Jacobson radical.

**Definition 0.2.17** *Given a ring $R$ we define the* Jacobson radical*, $J(R)$ to be*

$$J(R) \;=\; \bigcap \mathcal{M},$$

*where the intersection is taken over all maximal right ideals $\mathcal{M}$ of $R$.*

We now give some equivalent expressions for the Jacobson radical of a ring.