A Matter of Degree:
Improved Approximation Algorithms for Degree-Bounded Minimum Spanning Trees

J. Könemann

(joint work with R. Ravi)

Talk Outline

- The Problem
- Minimum-Degree MST's
- A Lagrangean Formulation
- Our Algorithm
- The Analysis

Degree-Bounded MST's

Given an undirected graph G=(V,E), a non-negative cost function

$$c : E \rightarrow \Re^+$$

and a parameter B.

Let $d_T(v)$ be the degree of node v and

$\Delta(T)$ be the maximum node degree in T

(BMST)

Find spanning tree T of G
with $\Delta(T) \leq B$
such that c(T) is minimized

Our Result

Theorem: Given G=(V,E) and positive parameter B, we compute T with

$$1.\qquad c(T) \le (1+\frac{1}{w}) \cdot \mathrm{opt}$$

$$2.\quad \Delta(T) \le r(1+w)B + \log_r n$$

where r>1, ω>0 and opt is the cost of the optimum degree-B-bounded MST.

e.g.: ω=1 and r=2 yields T with

$$c(T) \le 2 \cdot \mathrm{opt} \ \text{ and } \ \Delta(T) \le 4B + \log_2 n$$

Previous Work

[Ravi et. al., 93] show how to compute spanning tree T with

1. $c(T) \le O(\log n) \cdot opt$
2. $\Delta(T) \le O\left(\log n\right) \cdot B$

The authors extend their results to

- Steiner trees and generalized Steiner forests
- Node-weighted case

Talk Outline

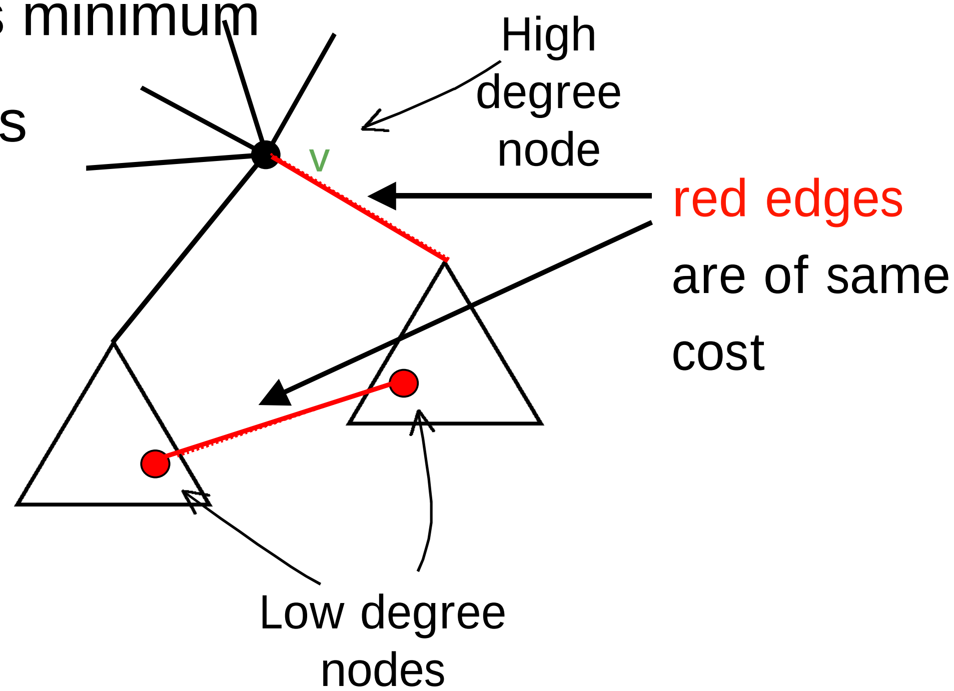
- The Problem
- Minimum-Degree MST’s
- A Lagrangean Formulation
- Our Algorithm
- The Analysis

Minimum-Degree MST's

Problem: Given an undirected graph G=(V,E) and a non-negative cost function c on the edges.

Find a min-cost spanning tree T such that its maximum degree Δ(T) is minimum

Idea: locally optimal trees



Call this an Improvement for v

Minimum-Degree MST's

Definition[Fürer, Raghavachari]

A tree T is called pseudo optimal if no improvement is applicable to any node node v with

$$d_T(v) \geq \Delta(T) - \log |V|$$

Denote application of this procedure to tree T by Plocal(T)

Fractional Trees

Let T^a be a convex combination of minimum-cost spanning trees for cost function c

$$T^a = \sum_{\substack{\text{min-cost} \\ \text{spanning tree } T}} a_T \cdot T$$

← incidence vector of tree T

Let the fractional degree of T^a at v be

$$d_c^a(v) = \sum_{\substack{\text{min-cost} \\ \text{spanning tree } T}} a_T d_T(v)$$

Define minimum maximum fractional degree as

$$\Delta_c^* = \min_{\text{convex comb. } a} \max_v d_c^a(v)$$

A Key Lemma

Modification of theorem by [Fischer, Fürer, Raghavachari] due to Éva Tardos

Key Degree Lemma A pseudo-optimal min-cost spanning tree T can be computed in poly-time and

$$\Delta(T) \leq r\Delta_c^* + \lceil \log_r n \rceil$$

where r>1.

Talk Outline

- The Problem
- Minimum-Degree MST's
- A Lagrangean Formulation
- Our Algorithm
- The Analysis

A Lagrangean Formulation

IP formulation of the our problem

$$\text{opt} = \min \quad c(T)$$

$$s.t. \quad \text{spanning tree T}$$

dualize → $d_T(v) \le B \quad \forall v \in V$

Lagrangean relaxation

$$(LR(B)) \quad z_{LR} = \max_{l \ge 0} \min_{\text{spanning tree } T} c(T) + \sum_{v \in V} l_v (d_T(v) - B)$$

Fact: (weak duality) $z_{LR} \le \text{opt}$

A few Observations

We can rewrite objective function of Lagrangean

$$\sum_{uv \in T} c_{uv} + \sum_{v \in V} l_v (d_T(v) - B)$$

Think of l_v is being added to each edge incident to v

$$\sum_{uv \in T} (c_{uv} + l_u + l_v) - B \sum_{v \in V} l_v$$

c^l_{uv}

More Observations

$$(LR(B)) \quad \max_{l \geq 0} \quad \min_{\text{spanning tree } T} \quad c^l(T) - B\sum_v l_v$$

Constant

Inner minimum is just an MST computation! We denote this by MST(c^l).

Theorem: Solution to (LR(B)) can be computed in poly-time.

Talk Outline

- The Problem
- Minimum-Degree MST's
- A Lagrangean Formulation
- Our Algorithm
- The Analysis

Our Algorithm

Given: graph G=(V,E), $c: E \rightarrow \Re^{+}$
and B>0

1. λ=Solve(LR((1+ω)B))
2. T^{l} =MST(c^{l})
3. T =Plocal(T^{l})
4. Output T

notice weakened degree constraints

Talk Outline

- The Problem
- Minimum-Degree MST's
- A Lagrangean Formulation
- Our Algorithm
- The Analysis

Lagrangean Properties

Recall

$$(LR((1+w)B)) \quad \max_{l \geq 0} \min_{\text{spanning tree T}} c^l(T) - (1+w)B \sum_v l_v$$

We can formulate this as an LP !

h

$$\text{(P)} \quad \begin{aligned} \max \quad & h \\ \text{s.t.} \quad & h \leq c^l(\mathrm{T}) - (1+w)\mathrm{B} \sum_{v \in V} l_v \quad \forall \text{sp-trees T} \\ & l \geq 0 \end{aligned}$$

LP formulation

(P)

$$\begin{aligned} \max \quad & h \\ \text{s.t.} \quad & h \le c^{l}(\mathrm{T}) - (1+w)\mathrm{B}\sum_{v\in V} l_v \quad \forall \text{sp-trees T} \\ & l \ge 0 \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & c(T^{a}) \\ \text{s.t.} \quad & d^{a}(v) \le (1+w)\mathrm{B} \quad \forall v \in V \\ & a \quad \text{convex comb.} \end{aligned}$$

Lagrangean Properties

Proposition: Let λ be an optimum solution to

$(LR((1+w)B))$ then there is a convex combination

$$T^{a} = \sum_{\substack{\text{min-cost spanning} \\ \text{tree } T \text{ for } c^{l}}} a_T T$$

such that 1. $\forall v \in V : d^{a}_{c^{l}}(v) \leq (1+w)B$

2. $l_v > 0 \Rightarrow d^{a}_{c^{l}}(v) = (1+w)B$

Proof: complementary slackness.

Corollary [Tardos] $\Delta^{*}_{c^{l}} \leq (1+w)B$

Degree of output tree

Step 3 of our algorithm applies Plocal to tree T^l =MST(c^l)

Final tree T has degree at most

$$r\Delta^*_{c^l} + \lceil \log_r n \rceil$$

by key degree lemma

...and

$$\Delta^*_{c^l} \leq (1+w)B$$

by the last Corollary.

Proving low Cost

min. cost of any degree-B-bounded spanning tree

<u>Theorem:</u> $c(T) \leq (1+1/w)\cdot \text{opt}$

<u>Proof-Sketch:</u>

$$\begin{aligned} c(T) &= \text{c}(T) + \sum_v l_v(d_T(v) - B) + \sum_v l_v(B - d_T(v)) \\ &\leq \text{opt} + \sum_v l_v(B - d_T(v)) \end{aligned}$$

Now bound $B\sum_v l_v$ by $\dfrac{\text{opt}}{w}$.

Uses the fact that λ is optimum for $(1+\omega)$B instead of just B critically.

Open Questions and Conclusion

Can the presented framework be generalized?

Can the result be extended to Steiner networks?

What about individual node degrees?