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(joint work with R. Ravi)
Talk Outline

- The Problem
- Minimum-Degree MST’s
- A Lagrangean Formulation
- Our Algorithm
- The Analysis
Degree-Bounded MST’s

Given an undirected graph $G=(V,E)$, a non-negative cost function $c : E \rightarrow \mathbb{R}^+$
and a parameter $B$.

Let $\delta_T(v)$ be the degree of node $v$ and
$\Delta(T)$ be the maximum node degree in $T$.

**(BMST)**

Find spanning tree $T$ of $G$
with $\Delta(T) \leq B$
such that $c(T)$ is minimized.
Our Result

**Theorem:** Given $G=(V,E)$ and positive parameter $B$, we compute $T$ with

\[ nB^rT \leq \log(1+\omega) \cdot \text{opt} \]

where $r>1$, $\omega>0$ and $\text{opt}$ is the cost of the optimum degree-$B$-bounded MST.

**e.g.:** $\omega=1$ and $r=2$ yields $T$ with

\[ c(T) \leq 2 \cdot \text{opt} \quad \text{and} \quad \Delta(T) \leq 4B + \log_2 n \]
Previous Work

[Ravi et al., 93] show how to compute spanning tree $T$ with

1. $c(T) \leq O(\log n) \cdot opt$
2. $\Delta(T) \leq O(\log n) \cdot B$

The authors extend their results to

- Steiner trees and generalized Steiner forests
- Node-weighted case
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Minimum-Degree MST’s

**Problem:** Given an undirected graph $G=(V,E)$ and a non-negative cost function $c$ on the edges.

Find a min-cost spanning tree $T$ such that its maximum degree $\Delta(T)$ is minimum.

**Idea:** locally optimal trees

Call this an Improvement for $v$
Minimum-Degree MST’s

Definition [Fürer, Raghavachari]
A tree $T$ is called \textit{pseudo optimal} if no improvement is applicable to any node node $v$ with

$$\delta_T(v) \geq \Delta(T) - \log |V|$$

Denote \textit{application} of this procedure to tree $T$ by $P_{\text{local}}(T)$
Fractional Trees

Let $T^\alpha$ be a convex combination of minimum-cost spanning trees for cost function $c$

$$T^\alpha = \sum_{\text{min-cost spanning tree } T} \alpha_T \cdot T$$

Let the fractional degree of $T^\alpha$ at $v$ be

$$\delta_{c \alpha} (v) = \sum_{\text{min-cost spanning tree } T} \alpha_T \delta_T (v)$$

Define minimum maximum fractional degree as

$$\Delta^*_c = \min_{\text{convex comb. } \alpha} \max_v \delta_{c \alpha} (v)$$
A Key Lemma

Modification of theorem by [Fischer, Fürer, Raghavachari] due to Éva Tardos

**Key Degree Lemma** A pseudo-optimal min-cost spanning tree $T$ can be computed in poly-time and

$$\Delta(T) \leq r\Delta^* + \left\lceil \log_r n \right\rceil$$

where $r > 1$. 

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A Lagrangean Formulation

IP formulation of the our problem
\[
\text{opt} = \min \quad c(T) \\
\text{s.t.} \quad \text{spanning tree } T
\]

dualize  \quad \delta_T(v) \leq B \quad \forall v \in V

Lagrangean relaxation
\[
(LR (B)) \quad z_{LR} = \max_{\lambda \geq 0} \quad \min_{\text{spanning tree } T} \quad c(T) + \sum_{v \in V} \lambda_v (\delta_T(v) - B)
\]

Fact: (weak duality) \quad z_{LR} \leq \text{opt}
A few Observations

We can rewrite objective function of Lagrangean

$$
\sum_{uv \in T} c_{uv} + \sum_{v \in V} \lambda_v (\delta_T(v) - B)
$$

Think of $\lambda_v$ is being added to each edge incident to $v$

$$
\sum_{uv \in T} (c_{uv} + \lambda_u + \lambda_v) - B \sum_{v \in V} \lambda_v
$$
More Observations

\[
(LR(B)) \max_{\lambda \geq 0} \min_{\text{spanning tree } T} c^\lambda(T) - B \sum_v \lambda_v
\]

Inner minimum is just an MST computation! We denote this by \(\text{MST}(c^\lambda)\).

**Theorem:** Solution to \((LR(B))\) can be computed in poly-time.
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Our Algorithm

Given: graph $G=(V,E)$, $c : E \rightarrow \mathbb{R}^+$ and $B > 0$

1. $\lambda = \text{Solve}( LR((1+\omega)B) )$
2. $T^\lambda = \text{MST}( c^\lambda )$
3. $T = \text{Plocal}(T^\lambda )$
4. Output $T$

notice weakened degree constraints
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Lagrangean Properties

Recall

\[
\max_{\lambda \geq 0} \quad \min_{\text{spanning tree } T} \quad c^\lambda (T) - (1 + \omega) B \sum_{v} \lambda_v
\]

We can formulate this as an LP!

\[
\max \quad \eta \\
\text{s.t.} \quad \eta \leq c^\lambda (T) - (1 + \omega) B \sum_{v \in V} \lambda_v \quad \forall \text{sp - trees } T \\
\lambda \geq 0
\]
LP formulation

\[
\begin{align*}
(P) & \quad \text{max} & \quad \eta \\
\text{s.t.} & & \eta \leq c^\lambda(T) - (1 + \omega)B \sum_{v \in V} \lambda_v & \forall \text{sp-trees } T \\
& & \lambda \geq 0
\end{align*}
\]

\[
(D) & \quad \text{min} & \quad c(T^\alpha) \\
\text{s.t.} & & \delta^\alpha(v) \leq (1 + \omega)B & \forall v \in V \\
& & \alpha \quad \text{convex comb.}
\]
Lagrangean Properties

**Proposition:** Let $\lambda$ be an optimum solution to

$$ (LR \ ((1 + \omega)B)) $$

then there is a convex combination

$$ T^\alpha = \sum_{\text{min-cost spanning tree } T} \alpha_T T $$

such that

1. $\forall v \in V : \delta^\alpha_{c^\lambda}(v) \leq (1 + \omega)B$
2. $\lambda_v > 0 \implies \delta^\alpha_{c^\lambda}(v) = (1+\omega)B$

**Proof:** complementary slackness.

**Corollary [Tardos]**

$$ \Delta^*_{c^\lambda} \leq (1 + \omega)B $$
Step 3 of our algorithm applies \textsf{Plocal} to tree $T^\lambda = \text{MST}(c^\lambda)$.

Final tree $T$ has degree at most
\[ r\Delta^*_c + \left\lceil \log_r n \right\rceil \]
by key degree lemma.

...and
\[ \Delta^*_c \leq (1+\omega)B \]
by the last Corollary.
Theorem: \( c(T) \leq (1+1/\omega) \cdot \text{opt} \)

Proof-Sketch:

\[
c(T) = c(T) + \sum_{v} \lambda_{v} (\delta_{T}(v) - B) + \sum_{v} \lambda_{v} (B - \delta_{T}(v)) \leq \text{opt} + \sum_{v} \lambda_{v} (B - \delta_{T}(v))
\]

Now bound \( B \sum_{v} \lambda_{v} \) by \( \frac{\text{opt}}{\omega} \).

Uses the fact that \( \lambda \) is optimum for \((1+\omega)B\) instead of just \( B \) critically.
Open Questions and Conclusion

Can the presented framework be generalized?
Can the result be extended to Steiner networks?
What about individual node degrees?