Lecture 1s

Finding the Line of Intersection of Two Planes

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Now suppose we were looking at two planes P_1 and P_2, with normal vectors $\vec{n}_1$ and $\vec{n}_2$. We saw earlier that two planes were parallel (or the same) if and only if their normal vectors were scalar multiples of each other. But what if two planes are not parallel? Then they intersect, but instead of intersecting at a single point, the set of points where they intersect form a line. Let's call the line L, and let's say that L has direction vector $\vec{d}$. Then since L is contained in P_1, we know that $\vec{n}_1$ must be orthogonal to $\vec{d}$. And, similarly, L is contained in P_2, so $\vec{n}_2$ must be orthogonal to $\vec{d}$ as well. That means that to find $\vec{d}$, we need to find a vector that is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$. Luckily we know how to do that now...

Example: Find a vector equation of the line of intersections of the two planes $x_1 - 5x_2 + 3x_3 = 11$ and $-3x_1 + 2x_2 - 2x_3 = -7$.

First we read off the normal vectors of the planes: the normal vector $\vec{n}_1$ of $x_1 - 5x_2 + 3x_3 = 11$ is $\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}$, and the normal vector $\vec{n}_2$ of $-3x_1 + 2x_2 - 2x_3 = -7$ is $\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$.

Next, we find the direction vector $\vec{d}$ for the line of intersection, by computing $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \times \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} (-5)(-2) - (3)(2) \\ (3)(-3) - (1)(-2) \\ (1)(2) - (-5)(-3) \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -13 \end{bmatrix}$.

Now, those steps were all straightforward. But there is still something very important missing–a point on the line! That is, we need to find a point P that is on both planes. A quick trick to narrow down the possibilities is to set the value of one of the variables. So I'll choose to fix $x_3 = 0$, and then I am looking for a point satisfying both $x_1 - 5x_2 + 0 = 11$ and $-3x_1 + 2x_2 + 0 = -7$. But $x_1 - 5x_2 + 0 = 11$ means that $x_1 = 11 + 5x_2$. Plugging this fact into the second equation gives us $-3(11 + 5x_2) + 2x_2 = -7 \Rightarrow x_2 = -2$. And so $x_1 = 1$, and we see that $P(1, -2, 0)$ is a point on both planes. (We can plug P in to the given equations of the plane to double check: $(1) - 5(-2) + 3(0) = 11$ and $-3(1) + 2(-2) - 2(0) = -7$.)

So, we have a point $(1, -2, 0)$ on the line, and a direction vector $\begin{bmatrix} 4 \\ -7 \\ -13 \end{bmatrix}$ for the line, so a vector equation of the line is

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -7 \\ -13 \end{bmatrix}, \; t \in \mathbb{R}$$

Want to double check this? Setting $t = 1$, we get the point $Q(5, -9, 13)$ on the line. We can plug Q into the given equations of the planes to verify that Q is also on both planes.