Lecture 4j

Matrix Inverse by Cofactors

(pages 274-6)

While I'm sure you all think that computing the cofactors of every entry of a matrix is fun, you must be wondering why we would need the cofactors of every entry. After all, we only need to compute the cofactors of one row or column in order to compute the determinant. But our goal now is not to find the determinant of the matrix, but instead to find the inverse of the matrix. To that end, we start by noticing the following:

Theorem 5.3.1 (False Expansion Theorem): If A is an $n \times n$ matrix and $i \neq k$, then

$$a_{i1}C_{k1} + \cdots + a_{in}C_{kn} = 0$$

Proof of Theorem 5.3.1: Let B be the matrix obtained from A by replacing the k-th row of A with the i-th row of A, so that B is identical to A except that the k-th row of B is the same as the i-th row of A instead of the k-th row of A. Of course, the i-th row of B is also the i-th row of A, and so the i-th and k-th rows of B are the same. So, since two rows of B are equal, we know that det $B = 0$. But what if we were to compute the determinant of B by expanding along its k-th row? Then we would have

$$b_{k1}C^*_{k1} + \cdots + b_{kn}C^*_{kn} = \det B = 0$$

where C^*_{ij} is the cofactor of b_{ij} in B. The first thing we notice is that row k of B is the same as row i of A, so $b_{kj} = a_{ij}$, giving us

$$a_{i1}C^*_{k1} + \cdots + a_{in}C^*_{kn} = 0$$

The next thing to remember is that the only place that A and B differ is in the k-th row. And since all the cofactors C^*_{kj} and C_{kj} omit the k-th row, they are in fact the same. And so we can replace C^*_{kj} with C_{kj} in our formula, getting us the desired result:

$$a_{i1}C_{k1} + \cdots + a_{in}C_{kn} = 0$$

Why is this called the "false expansion" theorem? Well, a "true" calculation of the determinant would involve combining the terms of row i with the cofactors of row i, as in

$$a_{i1}C_{i1} + \cdots a_{in}C_{in} = \det\ A$$

while our "false" expansion combines the terms of row i with the cofactors of row k ($k \neq i$), as in

$$a_{i1}C_{k1} + \cdots a_{in}C_{kn} = 0$$

Another way to look at these equations is as the dot product of two vectors. If we let $\vec{a}_i^T$ be the i-th row of A, and $\vec{C}_i^T$ be the i-th row of cof A, then we have

$$\det\ A = \vec{a}_i^T \cdot \vec{C}_i^T \quad \text{and} \quad 0 = \vec{a}_i^T \cdot \vec{C}_k^T \quad (i \neq k)$$

In this way, we know the value of the dot product of any row of A with any row of cof A. But what can we do with this information? Well, we can calculate the product of A and (cof $A)^T$, since the product of two matrices is comprised of the dot products of the rows of the first matrix with the columns of the second. (This is why we need to look at (cof $A)^T$–so that the rows of cof A become columns.) Our formulas are now summarized as follows:

$$(A(\text{cof}\ A)^T)_{ii} = \vec{a}_i^T \cdot \vec{C}_i^T = \det\ A \quad \text{and} \quad (A(\text{cof}\ A)^T)_{ik} = 0 \quad (i \neq k)$$

This means that the matrix $A(\text{cof}\ A)^T$ has det A on its diagonal entries, and all other entries are 0. That is, it looks like the identity matrix, except with det A instead of 1 on the diagonal. We can write this fact as:

$$A(\text{cof}\ A)^T = (\det\ A)I$$

Now, we've seen that A is invertible if and only if det $A \neq 0$. So, if A is invertible, we can divide both sides of this equation by det A, getting

$$A\left(\frac{1}{\det\ A}(\text{cof}\ A)^T\right) = I$$

and thus we have

$$A^{-1} = \frac{1}{\det\ A}(\text{cof}\ A)^T$$

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$. In the previous lecture, we found that cof $A = \begin{bmatrix} -7 & -4 \\ -3 & 2 \end{bmatrix}$. If we compute det $A = (2)(-7) - (4)(3) = -14 - 12 = -26$, then we know

$$A^{-1} = \frac{1}{\det\ A}(\text{cof } A)^T = \frac{1}{-26}\begin{bmatrix} -7 & -4 \\ -3 & 2 \end{bmatrix}^T = -\frac{1}{26}\begin{bmatrix} -7 & -3 \\ -4 & 2 \end{bmatrix}$$

Example: Let $B = \begin{bmatrix} 7 & 1 & 3 \\ 4 & -2 & -5 \\ 9 & 8 & -3 \end{bmatrix}$. In the previous lecture, we found that $\text{cof } B = \begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}$. And since we have already computed the cofactors of B, it is easy to calculate $\det\ B = b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13} = 7(46) + 1(-33) + 3(50) = 439$. And thus we have

$$\begin{aligned} B^{-1} &= \tfrac{1}{\det\ B}(\text{cof } B)^T \\ &= \tfrac{1}{439}\begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}^T \\ &= \tfrac{1}{439}\begin{bmatrix} 46 & 27 & 1 \\ -33 & -48 & 47 \\ 50 & -47 & -18 \end{bmatrix} \end{aligned}$$