Lecture 4f

Calculating the Determinant Using Row Operations

(pages 268-9)

So far, we've seen that determinant calculations get easier when a matrix has zero entries. And it is particularly easy to calculate the determinant of triangular matrices (either upper- or lower-). Most recently we've seen that it is easy to keep track of the changes that occur to a determinant when we apply elementary row operations to a matrix. So, what we'll do now is use elementary row operations to find a row equivalent matrix whose determinant is easy to calculate, and then compensate for the changes to the determinant that took place.

Summarizing the results of the previous lecture, we have the following:

Summary: If A is an $n \times n$ matrix, then

(1) if B is obtained from A by multiplying one row of A by the non-zero scalar r, $\det B = r\det A$

(2) if B is obtained from A by interchanging two rows, then $\det B = -\det A$

(3) if B is obtained from A by adding a scalar multiple of one row to another, then $\det B = \det A$.

Notice that with the third operation, there is no change to the determinant:

Example: Let $A = \begin{bmatrix} 3 & 7 & -9 \\ 6 & 1 & 4 \\ -9 & 5 & 2 \end{bmatrix}$. Then we can get a row echelon form matrix B that is row equivalent to A as follows:

$$A = \begin{bmatrix} 3 & 7 & -9 \\ 6 & 1 & 4 \\ -9 & 5 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 3 & 7 & -9 \\ 0 & -13 & 22 \\ 0 & 26 & -25 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_2 \end{matrix}$$
$$\sim \begin{bmatrix} 3 & 7 & -9 \\ 0 & -13 & 22 \\ 0 & 0 & 19 \end{bmatrix} = B.$$

B is an upper-triangular matrix, so $\det B$ is the product of its diagonal entries: $(3)(-13)(19) = -741$. And since the only row operations used to get from A to B where operations of type (3), we have that $\det A = \det B$, and so $\det A = -741$.

In theory, you can get a matrix into row echelon form only using operations of type (3), but the reality of such a process can get quite complicated. So let's look at an example using all the types of operations.

Example: Let $A = \begin{bmatrix} 0 & 5 & -2 & -4 \\ 2 & 4 & -2 & 8 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix}$. Then we can get a row echelon form matrix B that is row equivalent to A as follows:

$$A = \begin{bmatrix} 0 & 5 & -2 & -4 \\ 2 & 4 & -2 & 8 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \\ \end{matrix} \sim B_1 = \begin{bmatrix} 2 & 4 & -2 & 8 \\ 0 & 5 & -2 & -4 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \\ \\ \\ \end{matrix}$$

$$\sim B_2 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ -3 & 4 & -1 & 1 \\ 5 & 5 & -8 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_1 \\ R_4 - 5R_1 \end{matrix} \sim B_3 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 10 & -4 & 13 \\ 0 & -5 & -3 & -11 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \end{matrix}$$

$$\sim B_4 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & 0 & 21 \\ 0 & 0 & -5 & -15 \end{bmatrix} \begin{matrix} \\ \\ \\ R_3 \updownarrow R_4 \end{matrix} \sim B = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & -2 & -4 \\ 0 & 0 & -5 & -15 \\ 0 & 0 & 0 & 21 \end{bmatrix}.$$

Since B is an upper-triangular matrix, we know that $\det B = (1)(5)(-5)(21) = -525$. Now we need to get $\det A$ from $\det B$. To do this, we will look over the row operations we used. The first row operation we used was a row swap, which means we need to multiply the determinant by (-1), giving us $\det B_1 = -\det A$. The next row operation was to multiply row 1 by $1/2$, so we have that $\det B_2 = (1/2)\det B_1 = (1/2)(-1)\det A$. The next matrix was obtained from B_2 by adding multiples of row 1 to rows 3 and 4. Since these row operations do not change the determinant, we have $\det B_3 = \det B_2 = (1/2)(-1)\det A$. B_4 was obtained from B_3 by adding a multiples of row 2 to rows 3 and 4. Again, these row operations do not change the determinant, so we have $\det B_4 = \det B_3 = (1/2)(-1)\det A$. Our final matrix, B, was obtained from B_4 by swapping two rows. This means that $\det B = -\det B_4 = (-1)(1/2)(-1)\det A$.

But we already know $\det B$; what we want to know is $\det A$. So, we solve for $\det A$, and get $\det A = (-1)(2)(-1)\det B = 2\det B = -1050$.

There are some important lessons to be learned from this example. The first is that we can ignore any row operations of type (3), since they do not affect the determinant. The second is that two row swaps cancel each other out, in terms of calculating the determinant, so we only care whether an odd or even number of row swaps take place. The third thing to notice is the most important, and it is the fact that if you multiply a row by r in your row reduction from A to B, then you will need to multiply $\det B$ by (1/r) to get to $\det A$. If the original theorem had been stated as $\det A = (1/r)\det B$ instead of $\det B = r\det A$, then this would have been more clear. And so, we have developed the following technique for calculating the determinant of A:

Algorithm for calculating a determinant: Let A be an $n \times n$ matrix. Use elementary row operations to obtain a matrix B from A whose determinant is easy to calculate. (B need not be in row echelon form. I'll do some examples of other types of B after this.) Calculate $\det B$, and then obtain $\det A$ from $\det B$ as follows:

(1) Add up the number of times you performed a row swap. If this number is even, then do nothing. If this number is odd, then multiply $\det B$ by -1.

(2) Let $r_1, \dots, r_m$ be a list of all the scalars used in an operation of the type "multiply a row by a non-zero scalar". Then take your result from step (1) and multiply it by $(1/r_1)(1/r_2)\cdots(1/r_m)$. You now have $\det A$. (Note: it does not matter what order the row multiplication took place, or what row was multiplied!)

Example: Let $A = \begin{bmatrix} 5 & -1 & -3 \\ -2 & 2 & 3 \\ 4 & 8 & 3 \end{bmatrix}$. At first glance, it is difficult to decide what our first row operation should be. There are no 1s in the first column, and none of the rows have all entries being a multiple of their first entry, so that we could easily divide to get a 1 in the first column. Even the first column entries do not have a common divisor. And so we could resign ourselves to using fractions (which, less face it, will sometimes happen). Or we can notice the following:

$$\begin{bmatrix} 5 & -1 & -3 \\ -2 & 2 & 3 \\ 4 & 8 & 3 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 5 & -1 & -3 \\ 3 & 1 & 0 \\ 9 & 7 & 0 \end{bmatrix} = B.$$

While B is not a triangular matrix, by expanding along the third column of B, we can still easily calculate its determinant:

$$\det B = (-3)(-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 9 & 7 \end{vmatrix} + 0 + 0 = -3(21-9) = -36.$$

And since the only row operations used to get from A to B were type (3), we have that $\det A = \det B = -36$.

Example: Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 5 & -7 & 4 \\ -3 & 5 & -9 & 2 \\ 4 & 4 & 6 & -1 \end{bmatrix}$. We can find B as follows:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 5 & -7 & 4 \\ -3 & 5 & -9 & 2 \\ 4 & 4 & 6 & -1 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 5 & -6 & 5 \\ 0 & 5 & -6 & 5 \\ 4 & 4 & 6 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 5 & -6 & 5 \\ 0 & 0 & 0 & 0 \\ 4 & 4 & 6 & -1 \end{bmatrix} = B.$$

Then $\det B = 0$, and since only row operations of type (3) were used to get from A to B, we have $\det A = \det B = 0$.

Worth noting, however, is that even if we had used row operations of type (1) or (2) to get B from A, we would still have $\det A = 0$, since $\det A$ would always be a scalar multiple of $\det B$, and anything times 0 would still be 0. We'll make use of this fact in the next lecture.