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Inequivalent representations of matroids having no $U_{3,6}$-minor[☆]

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Abstract

It is proved that, for any prime power q, a 3-connected matroid with no $U_{3,6}$-minor has at most $(q-2)!$ inequivalent representations over GF(q).

1. Introduction

Recall that a *representation* of a matroid M over a field $\mathbb{F}$ is a matrix A whose columns are labelled by the elements of M with the property that a set of columns of A is linearly independent if and only if the labels of that set of columns are independent in M. Recall also that two representations of a matroid are *equivalent* if one can be obtained from the other via a sequence of the following operations: elementary row operations; multiplying a column by a nonzero scalar; interchanging two columns along with their labels; deleting zero rows; and applying a field automorphism to all elements of the matrix. Finally, recall that M is *uniquely representable* over $\mathbb{F}$ if all $\mathbb{F}$-representations of M are equivalent.

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In his seminal paper [3], where he proves that 3-connected matroids are uniquely representable over GF(4); Kahn conjectured that for every prime power q, there is an integer μ_q such that no 3-connected GF(q)-representable matroid has more than μ_q inequivalent representations over GF(q). At the time, the conjecture was known to be true for GF(2), GF(3) and GF(4). Moreover, Oxley, Vertigan and Whittle [6] verified Kahn's Conjecture for GF(5). However, counterexamples are given in [6] that show that Kahn's Conjecture fails for all fields with at least seven elements.

In this paper, we prove that Kahn's Conjecture holds so long as we exclude the uniform matroid $U_{3,6}$ as a minor. Let $n_q(M)$ denote the number of inequivalent representations of a matroid over GF(q). Specifically we prove

Theorem 1.1. *Let M be a 3-connected matroid. If M has no $U_{3,6}$-minor then* $n_q(M) \leqslant (q-2)!$

The counterexamples given in [6] contain no uniform minor that is larger than $U_{3,6}$ so that Theorem 1.1 is best possible in that no analogue of it holds for excluding a larger uniform matroid. Nonetheless, we do believe that Kahn's Conjecture can be recovered in full generality for matroids whose 3-separations are controlled in an appropriate way, in particular for 4-connected matroids. Moreover, we also believe that the results of this paper will be of value in establishing Kahn's Conjecture for 4-connected matroids.

It is assumed that the reader is familiar with the theory of matroids as set forth in Oxley [4]. In particular, it is assumed that the reader is familiar with the theory of matroid connectivity and matroid representation. Terminology and notation follow [4] with the exception that we denote the simplification and cosimplification of a matroid M by si(M) and co(M), respectively. A *line* of M is a rank-2 flat of M and a *coline* of M is a rank-2 flat of M^*.

To prove Theorem 1.1, we use the theory of totally free matroids developed in [2] and the theory of segment–cosegment exchanges introduced by Oxley, Semple and Vertigan [5]. While we restate enough material from [2,5] to make this paper essentially self-contained, familiarity with these papers would be an advantage.

2. Totally free matroids

In this section, we review material from [2] that will be needed for the results of this paper. Two elements of a matroid are clones if they cannot be distinguished by matroidal properties. More precisely, the elements e and e' are *clones* in the matroid M if the function that exchanges e and e' and acts as the identity on $E(M) - \{e, e'\}$ is an automorphism of M. Recall that a *cyclic flat* of the matroid M is a flat that is also a union of circuits.

Proposition 2.1. [2, Proposition 4.9] *Two elements e and e' are clones in M if and only if the set of cyclic flats containing e is equal to the set of cyclic flats containing e'.*

Evidently two elements are clones in M if and only if they are clones in M^*. It is also easily seen that if e and e' are clones in M, then they are clones in any minor of M that contains both e and e'. The relation of being clones is clearly an equivalence relation on the elements of M. The equivalence classes of this relation are known as *clonal classes*. A *clonal pair* (respectively, *clonal triple*) is a set of two (respectively, three) elements that is contained in a clonal class.

Let e be an element of a matroid M. If the matroid M' is a single-element extension of M on the ground set $E(M) \cup \{e'\}$ such that e and e' are clones in M', then we say that M' is obtained from M by *cloning* e *with* e'. As noted in [2], it is always possible to clone e with e' by adding e' in parallel to e, or, if e is a loop of M, adding e' as a loop. However, it is not always possible to clone e with e' in such a way that $\{e, e'\}$ is independent in M'. If $\{e, e'\}$ is independent in M', then we say that e has been *independently cloned with* e'. If e is an element of a matroid M such that e cannot be independently cloned, then e is *fixed* in M.

For example, note that any loop of M, or any element of M that is contained in a parallel pair is fixed in M. Also, any element of M that lies on the intersection of two non-trivial lines is fixed. The next two lemmas provides useful ways to verify that elements are fixed.

Lemma 2.2. *Let F_1 and F_2 be cyclic flats of the matroid M and x be in $F_1 \cap F_2$. If $r(F_1) + r(F_2) = r(F_1 \cup F_2) + 1$, then x is fixed in M.*

Proof. Let M' be a matroid obtained by cloning x with x'. Since F_1 is cyclic, $x \in \mathrm{cl}(F_1 - \{x\})$. Since x' is a clone of x, $x' \in \mathrm{cl}(F_1 - \{x\})$. This shows that $r(F_1 \cup \{x'\}) = r(F_1)$, and similarly $r(F_2 \cup \{x'\}) = r(F_2)$. Now, by submodularity

$$r(\{x, x'\}) \leqslant r(F_1 \cup \{x'\}) + r(F_2 \cup \{x'\}) - r(F_1 \cup F_2 \cup \{x'\}) = 1,$$

so that $\{x, x'\}$ is not independent in M' and hence x is fixed in M. $\square$

Lemma 2.3. [2, Lemma 6.4] *Suppose that e is in a triangle T of M. Then e is fixed in M if and only if there exists a circuit C of M such that $\mathrm{cl}(C) \cap T = \{e\}$.*

An element e is *cofixed* in M if e is fixed in M^*.

Proposition 2.4. [2, Proposition 4.8] *Let e and e' be clones in M. If $\{e, e'\}$ is independent, then e is fixed in neither M nor $M \backslash e'$. Dually, if $\{e, e'\}$ is coindependent, then e is cofixed in neither M nor M/e'.*

A matroid M is *totally free* if:

(i) M is 3-connected; and
(ii) if e is fixed in M, then $\mathrm{co}(M \backslash e)$ is not 3-connected, and if e is cofixed in M, then $\mathrm{si}(M/e)$ is not 3-connected.

Note that the definition of a totally free matroid is self-dual, so that M is totally free if and only if M^* is totally free.

It is easily seen that if e is fixed in M, then, $n_q(M) \leqslant n_q(M\backslash e)$, and if e is cofixed in M, then $n_q(M) \leqslant n_q(M/e)$. In other words, extending by fixed elements and coextending by cofixed elements cannot increase the number of inequivalent representations of a matroid. Using these facts it can be shown that if we wish to find a bound on the number of inequivalent representations a 3-connected matroid may have over a finite field, we need only consider the totally free minors of that matroid.

Lemma 2.5. [2, Theorem 2.4] *Let M be a 3-connected matroid that is representable over the finite field $\mathrm{GF}(q)$. Then $n_q(M)$ is bounded above by the maximum, taken over all totally free minors M' of M, of $n_q(M')$.*

The next result shows that totally free matroids cannot occur sporadically in a minor-closed class.

Lemma 2.6. [2, Theorem 8.12] *Let M be a totally free matroid with $|E(M)| \geqslant 5$. If e is an element of M such that either $M\backslash e$ is 3-connected but not totally free, or M/e is 3-connected but not totally free, then*
(i) *e has a unique clone e' in M;*
(ii) *$M\backslash e/e' = M/e\backslash e'$ is totally free; and*
(iii) *both $M\backslash e$ and M/e are 3-connected.*

Corollary 2.7. [2, Corollary 8.13] *Let M be a totally free matroid such that $|E(M)| \geqslant 5$ and, for all e in $E(M)$, neither $M\backslash e$ nor M/e is totally free. Then the ground set of M is the union of 2-element clonal classes. Moreover, if $e \in E(M)$ then both $M\backslash e$ and M/e are 3-connected, and if e' is the unique clone of e in M, then $M/e\backslash e'$ is totally free.*

We note some further properties of totally free matroids that will be needed for this paper.

Proposition 2.8. [2, Lemma 8.8] *If $\{e, f, g\}$ is a triangle or a triad of a totally free matroid M, then $\{e, f, g\}$ is a clonal triple.*

Proposition 2.9. [2, Proposition 8.9] *Let e be an element of the totally free matroid M. Then either $M\backslash e$ or M/e is 3-connected.*

Corollary 2.10. *Let M be a totally free matroid such that $|E(M)| \geqslant 5$, and let e be an element of $E(M)$. If e is an element of a triangle of M, then $M\backslash e$ is totally free, and if e is an element of a triad of M, then M/e is totally free.*

Proof. Say that e is an element of a triangle. By Proposition 2.9 either $M\backslash e$ or M/e is 3-connected. Since M/e contains a parallel pair, $M\backslash e$ must be 3-connected. If $M\backslash e$ is not totally free, then by Lemma 2.6 the element e has a unique clone in M. But this contradicts Proposition 2.8, which asserts that e is a member of a clonal triple in M. $\square$

3. The generalized $\Delta - Y$ exchange

The generalized $\Delta - Y$ exchange was introduced and studied by Oxley, Semple and Vertigan [5]. We begin by recalling how Brylawski [1] approached the usual $\Delta - Y$ exchange of matroids. Recall that a flat F of a matroid M is a *modular flat of* M if $r(F) + r(F') = r(F \cup F') + r(F \cap F')$ for every flat F' of M. Let M_1 and M_2 be matroids such that $E(M_1) \cap E(M_2) = T$ and $M_1|T = M_2|T$. Let $N = M_1|T$. If T is a modular flat of M_1 then the *generalized parallel connection*, denoted $P_N(M_1, M_2)$ is well defined. It is the matroid with ground set $E(M_1) \cup E(M_2)$, whose flats are all subsets $X \subseteq E(M_1) \cup E(M_2)$ such that $X \cap E(M_i)$ is a flat of M_i for $i \in \{1, 2\}$.

Let $M_1 \cong M(K_4)$ and let $T \subseteq E(M_1)$ be a triangle of M_1. Then T is a modular flat of M_1, so if M_2 is a matroid such that $E(M_1) \cap E(M_2) = T$ and $N = M_2|T \cong U_{2,3}$, then $P_N(M_1, M_2)$ is well defined. The matroid $P_N(M_1, M_2)\backslash T$ is said to be produced by performing a $\Delta - Y$ exchange on M_2. This is the operation that is generalized in [5] and we outline that generalization now.

Firstly, a matroid Θ_k is introduced which generalizes the role played by $M(K_4)$ in the $\Delta - Y$ exchange. The ground set of Θ_k consists of a k-element line and a k-element coline with the property that each $(k-1)$-element subset of the coline forms a circuit with an element of the coline. Gluing Θ_k onto a k-element line of another matroid M and then deleting the line has the effect of replacing the line of M by a coline. More precisely, for all $k \geqslant 2$, the rank-k matroid Θ_k is defined on the ground set $A \cup B$, where $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. In Θ_2 the sets A and B are independent and $\{a_1, b_2\}$ and $\{a_2, b_1\}$ are both parallel pairs. For $k > 2$ the non-spanning circuits of Θ_k are as follows:

(i) All 3-element subsets of A; and
(ii) All subsets $(B - \{b_i\}) \cup \{a_i\}$ for $i \in \{1, \ldots, k\}$.

If X is a set of the matroid M such that $|X| \geqslant 2$ and $M|A \cong U_{2,|A|}$ then X is a *segment of* M. A *cosegment of* M is a subset of $E(M)$ that is a segment in M^*. Now A is a modular flat of Θ_k [5, Lemma 2.4], so if M is a matroid such that $E(\Theta_k) \cap E(M) = A$ and A is a segment of M, then $P_A(\Theta_k, M)$ is well defined. In order to preserve the dual nature of the exchange we require A to be coindependent in M. In this case A is a *strict segment of* M. If A is a strict segment of M, then $P_A(\Theta_k, M)\backslash A$ is said to have been obtained from M by a *segment–cosegment exchange on* A, and is denoted by $\Delta_A(M)$. It will be convenient for M and $\Delta_A(M)$ to have the same ground set, so for $i \in \{1, \ldots, k\}$ the element $b_i \in E(\Delta_A(M))$ is relabelled a_i.

Dually, a *strict cosegment* of M is an independent cosegment of M. In this case a segment–cosegment exchange on A may be performed on M^*. Now $\nabla_A(M)$ is defined to be $(\Delta_A(M^*))^*$. The matroid $\nabla_A(M)$ is said to be obtained from M by a *cosegment–segment exchange on* A. In [5, Lemma 2.11] it is proved that these operations are inverse to each other, that is, $\nabla_A(\Delta_A(M)) = M$.

Note that if A is a strict segment containing exactly two elements then $\Delta_A(M) \cong M$, and dually, if A is a strict cosegment of size two then $\nabla_A(M) \cong M$. In both cases, the isomorphism is simply the function that exchanges the two members of A and fixes every other element.

It follows from [5, Lemma 3.5] that if M is representable over the field $\mathbb{F}$ and A is a strict segment of M then there is a canonically associated representation of $P_A(M, \Theta_k)$ over $\mathbb{F}$. Thus segment–cosegment and cosegment–segment exchange preserve representability. It is also noted in [5, Corollary 3.6] that segment–cosegment exchange preserves the number of inequivalent representations. Thus, we have

Lemma 3.1. *If M is $GF(q)$-representable and A is a strict segment of M, then $\Delta_A(M)$ is $GF(q)$-representable and furthermore, $n_q(\Delta_A(M)) = n_q(M)$.*

It is well known that $n_q(M^*) = n_q(M)$ for any matroid M and any finite field $\mathrm{GF}(q)$ so it follows that if M' is obtained from M by a sequence of segment–cosegment and cosegment–segment exchanges then $n_q(M') = n_q(M)$.

4. Quasi-lines

A matroid M is a *quasi-line* if, for some $k \geqslant 4$, M can be obtained from $U_{2,k}$ by a sequence of segment–cosegment and cosegment–segment exchanges. A detailed study of quasi-lines is given in [5]. In particular, it is shown that quasi-lines can be associated with certain labelled trees. We now outline some material from [5].

It is easily seen that, for $k \geqslant 4$, the matroid $U_{k-2,k}$ is a quasi-line. A *del–con tree* is a tree T for which every vertex v of T is labelled either (E_v, del) or (E_v, con) such that the following conditions are satisfied:

(i) each E_v is a finite, possibly empty, set;
(ii) if u and v are distinct vertices, then E_u and E_v are disjoint;
(iii) if v is a degree-1 vertex of T, then $|E_v| \geqslant 2$; and
(iv) if two vertices of T are adjacent then the second coordinates of their labels are different.

We also make the assumption that $|\bigcup_{v \in V(T)} E_v| \geqslant 4$. For any vertex v of T we call E_v a *vertex class*. If v is labelled (E_v, del) then v is a *del vertex of T* and E_v is a *del class of T*. *Con vertices* and *con classes* are defined analogously.

Suppose that u is a degree-1 vertex of T, and that u is labelled (E_u, σ). Then the unique neighbour v of u in T is labelled (E_v, τ) where $\{\sigma, \tau\} = \{\mathrm{del}, \mathrm{con}\}$. We can obtain a new tree by deleting u from T and leaving the label of every other vertex unchanged except for v, which is relabelled $(E_u \cup E_v, \tau)$. This operation is called *shrinking u into v*.

With any del–con tree T we may canonically associate a quasi-line $M(T)$ as follows. We can find a sequence of del–con trees $T_1, T_2, \ldots, T_n$ such that $T_n = T$ and for $1 \leqslant i \leqslant n-1$, T_i has i vertices and is obtained from T_{i+1} by shrinking a degree-1 vertex into its unique neighbour. Let $E = \bigcup_{v \in V(T)} E_v$. The tree T_1 consists of a single vertex labelled either (E, del) or (E, con). If the single vertex of T_1 is a del vertex, let

M_1 be the matroid on the ground set E that is isomorphic to $U_{2,|E|}$. If T_1 consists of a single con vertex then let M_1 be the matroid on the set E that is isomorphic to $U_{|E|-2,|E|}$. For $1 \leqslant i \leqslant n-1$, if T_i is obtained by shrinking the degree-1 con vertex v of T_{i+1} into its unique neighbour, then let $M_{i+1} = \Delta_{E_v}(M_i)$. If T_i is obtained from T_{i+1} by shrinking the degree-1 del vertex v into its neighbour, let $M_{i+1} = \nabla_{E_v}(M_i)$. Define $M(T) = M_n$.

Let $d_T(v)$ denote the degree of a vertex v of T. A del–con tree T is *reduced* if it satisfies the following properties:

(i) If $d_T(v) = 1$ then $|E_v| \geqslant 3$; and
(ii) If $d_T(v) = 2$ then E_v is not empty.

If T is not reduced it can be made so by a sequence of the following operations:

(i) If v is a degree-1 vertex of T and $|E_v| = 2$ then shrink v into its neighbour.
(ii) If v is adjacent to only two vertices u and w, and v is labelled (E_v, σ) where E_v is empty, then contract the edges uv and wv. In the resulting graph the vertices are u, v and w are identified as a single vertex, which is labelled $(E_u \cup E_w, \tau)$, where $\{\sigma, \tau\} = \{\text{del}, \text{con}\}$.

It is proved in [5, Lemma 4.6] that if T' is obtained from T by the above two operations then $M(T) = M(T')$. Furthermore, if M is a quasi-line with at least five elements, then M can be represented by a reduced del–con tree that is unique up to isomorphism [5, Lemma 4.16]. It is easily seen that the dual of a quasi-line is a quasi-line. Indeed, if T' is obtained from T by interchanging the labels del and con at each vertex, then $M(T') = (M(T))^*$. We now note some further elementary properties of quasi-lines.

Proposition 4.1. *Let v be a degree-1 del vertex of the reduced del–con tree T. Then E_v is a line of $M(T)$.*

Proposition 4.2. [5, Lemma 4.11] *Suppose that T is a reduced del–con tree. Then e and e' are clones in $M(T)$ if and only if they are contained in the same vertex class of T.*

Proposition 4.3. [5, Corollary 4.12] *Let T be a reduced del–con tree. If Z is a triangle or triad of $M(T)$, then the elements of Z are contained in a single vertex class of T.*

It follows from [5, Lemma 4.3] that quasi-lines are 3-connected. The 3-separations of quasi-lines are also characterized, [5, Lemmas 4.13 and 4.14].

Lemma 4.4. *Let T be a reduced del–con tree with at least two vertices. If (X, Y) is a partition of $E(M(T))$ such that $|X|, |Y| \geqslant 3$, then (X, Y) is a 3-separation of $M(T)$ if* *and only if there is some vertex v of T such that if T' is any connected component of $T - v$, then the set $\bigcup_{w \in V(T')} E_w$ is contained in either X or Y.*

Proposition 4.5. [5, Lemma 4.8] *Suppose that $M(T)$ is a quasi-line, where T is a reduced del–con tree and where $|E(M(T))| \geqslant 5$. Let v be a vertex of T labelled (E_v, σ) where $\sigma \in \{del, con\}$. Suppose that $e \in E_v$. Let T' be the del–con tree obtained from T by relabelling v with $(E_v - e, \sigma)$.*

(i) *If v is a del vertex of T, then $M(T)\backslash e$ is a quasi-line and $M(T)\backslash e = M(T')$.*
(ii) *If v is a con vertex of T, then $M(T)/e$ is a quasi-line and $M(T)/e = M(T')$.*

Note that in the previous proposition T' may not be reduced. If u and v are degree-1 del vertices of the del–con tree T, then E_u and E_v are distinct lines of $M(T)$. Thus $r_{M(T)}(E_u \cup E_v) \in \{3, 4\}$. The case where the rank of this set is 3 is quite special.

Lemma 4.6. *Let P be a path of maximal length in the reduced del–con tree T such that the end vertices u and v of P are both del vertices. If $r(E_u \cup E_v) = 3$ then T is isomorphic to a path of length two. Moreover, if w is the internal vertex of this path, then $|E_w| = 1$.*

Proof. Assume that $|E(M(T)) - (E_u \cup E_v)| \geqslant 2$. We prove that $r(E_u \cup E_v) = 4$. Say $z \in E(M) - (E_u \cup E_v)$. By Proposition 4.5, either $M(T)\backslash z$ or $M(T)/z$ is a quasi-line. Let T' be a reduced del–con tree that represents this quasi-line. It is routinely checked that E_u and E_v are del classes of T' corresponding to end vertices of a maximal length path in T'. As $M(T')$ is a minor of $M(T)$, we have $r_{M(T')}(E_u \cup E_v) \leqslant r_{M(T)}(E_u \cup E_v)$. It follows that it suffices to prove that $r(E_u \cup E_v) = 4$ in the case that $|E(M(T)) - (E_u \cup E_v)| = 2$. In this case $E_u \cup E_v$ is spanning, as otherwise $M(T)$ is not 3-connected. Since E_u and E_v are lines of $M(T)$, $r(M(T)) > 2$. Assume that $r(M(T)) = 3$. Let $E(M(T)) - (E_u \cup E_v) = \{x, y\}$. Then, by Lemma 4.4, $(E_u \cup \{x\}, E_v \cup \{y\})$ is a 3-separation of $M(T)$. But $r(E_u \cup \{x\}) = r(E_v \cup \{y\}) = 3$, so that this partition is not a 3-separation. It follows from this contradiction that $r(E_u \cup E_v) = 4$.

The fact that T has the claimed structure if $r(E_u \cup E_v) = 3$ now follows easily. $\square$

We now work towards showing that quasi-lines are totally free. We first show that certain minors are not 3-connected.

Lemma 4.7. *Let v be an internal del vertex of the reduced del–con tree T and e be an element of E_v. Then $\mathrm{si}(M(T)/e)$ is not 3-connected.*

Proof. It is easily seen that there is a partition (X, Y) of $E(M) - \{e\}$ with the property that if T' is a component of $T - v$, then $\cup_{w \in V(T')} E_w$ is contained in either X or Y. By Lemma 4.4, $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both 3-separation of $M(T)$. By Proposition 4.5, $M(T)\backslash e$ is 3-connected. Hence $r(X) = r(X \cup \{e\})$ and

$r(Y) = r(Y \cup \{e\})$, and it follows that (X, Y) is a 2-separation of $M(T)/e$. Thus, $M(T)/e$ is not 3-connected. If $\mathrm{si}(M/e)$ is 3-connected, then either X or Y is a parallel class of $M(T)/e$. Assume that X is a parallel class. Then $r_{M(T)}(X) = 2$. For some degree-1 vertex w of T, X contains E_w. If w is a con class, then $r(E_w) > 2$, so w is a del class. Since E_w is a line of $M(T)$, we must have $E_w = X$. But, $e \in \mathrm{cl}(X)$ contradicting the fact that E_w is a flat of $M(T)$. Hence, $\mathrm{si}(M(T)/e)$ is not 3-connected as required. $\square$

Lemma 4.8. *Let T be a reduced del–con tree and e be an element of $M(T)$. If e belongs to a del class, then e is not fixed and if e belongs to a con class, then e is not cofixed.*

Proof. Say e belongs to the del class E_v. Consider the reduced del–con tree T' obtained from T by relabelling v with $(E_v \cup \{e'\}, \mathrm{del})$. By Proposition 4.2 $\{e, e'\}$ is an independent clonal pair in $M(T')$ and so by Proposition 2.4 we see that e is not fixed in $M(T')\backslash e' = M(T)$. The second statement follows by duality. $\square$

Lemma 4.9. *If M is a quasi-line then M is totally free.*

Proof. Let T be the reduced del–con tree such that $M = M(T)$. Now M is certainly 3-connected with at least four elements. Suppose that $e \in E_v$ is cofixed in $M(T)$. Then, as the members of E_v are clones, $|E_v| = 1$, so that v is not a degree-1 vertex. By Lemma 4.8, E_v is a del class. But then, by Lemma 4.7, $\mathrm{si}(M/e)$ is not 3-connected. Similarly, if e is fixed, then $\mathrm{co}(M\backslash e)$ is not 3-connected, and it follows that M is totally free. $\square$

To conclude the section we consider certain single-element extensions of quasi-lines.

Lemma 4.10. *Suppose that $\{e, f, g\}$ is a clonal triangle of M and that $M\backslash e = M(T)$ for some del–con tree T. Then M is a quasi-line and $M = M(T')$ where T' is obtained from T by relabelling v with $(E_v \cup e, \mathit{del})$, where $E_v \supseteq \{f, g\}$.*

Proof. By Proposition 4.2, f and g are in the same vertex class of $M\backslash e$, so that the matroid $M(T')$ is well defined and, again by Proposition 4.2, $\{e, f, g\}$ is a clonal triple of $M(T')$. It remains to prove that $M(T') = M$. Let $M_1 = M(T')$ and let $M_2 = M$.

For $\{i,j\} = \{1,2\}$ suppose that $C \subseteq E(M)$ is circuit of M_i. If $e \notin C$ then C is a circuit of $M_i\backslash e = M_j\backslash e$, and hence C is a circuit of M_j. Therefore assume $e \in C$. If $C = \{e, f, g\}$ then C is a circuit of both M_1 and M_2 so we may assume $C \neq \{e, f, g\}$. If $C \cap \{e, f, g\} = \{e\}$ then since e and f are clones in both M_i and M_j it follows that $(C - e) \cup f$ is a circuit of $M_i\backslash e = M_j\backslash e$, and therefore that C is a circuit of M_j. Similarly, if $C \cap \{e, f, g\} = \{e, x\}$ where $\{x, y\} = \{f, g\}$, then $(C - e) \cup y$ is a circuit of $M_i\backslash e = M_j\backslash e$ and hence C is a circuit of M_j. Therefore, we conclude that $M_1 = M_2$ and that the lemma holds. $\square$

5. Proof of the main theorem

We first prove

Theorem 5.1. *The set of totally free matroids that have no $U_{3,6}$-minor is exactly the set of quasi-lines.*

Proof. Assume that M is a quasi-line. By Lemma 4.9, M is totally free. The straightforward proof that M has no $U_{3,6}$-minor is given in [5, Lemma 6.1].

Now consider the converse. Assume that M is a totally free matroid with no $U_{3,6}$-minor and assume that the theorem fails. Among all counterexamples to the theorem, assume that M is chosen to have a minimum-sized ground set. First note that the only totally free matroids having fewer than six elements are $U_{2,4}$, $U_{2,5}$ and $U_{3,5}$, and that these are all quasi-lines. We next prove:

5.1.1. *M has no triangles or triads.*

Subproof. Suppose that $\{x,y,z\}$ is a triangle of M. By Proposition 2.8, $\{x,y,z\}$ is a clonal triple of M. By Corollary 2.10, $M\backslash x$ is totally free. Now by the minimality of M, $M\backslash x$ is a quasi-line. Thus, $M\backslash x = M(T)$ for some reduced del–con tree T. But now, by Lemma 4.10, M is a quasi-line. This contradiction shows that M has no triangles. It follows by duality that M has no triads. $\square$

An easy consequence of 5.1.1 is

5.1.2. *$r(M)>3$ and $r(M^*)>3$.*

5.1.3. *There is an element $e \in E(M)$ such that either $M\backslash e$ or M/e is totally free.*

Subproof. Assume otherwise. Then, by Corollary 2.7 and 5.1.2, $E(M)$ is the union of 2-element clonal classes. Let $\{e,e'\}$ be a clonal class of M. Then, again by Corollary 2.7, $M/e\backslash e'$ is totally free. By the minimality of M, there is a reduced del–con tree T such that $M/e\backslash e' = M(T)$. By duality we may assume that T has a degree-1 del vertex v. Since E_v is the union of clonal classes of M, it follows that $|E_v| \geqslant 4$. Thus, $|E_v \cup \{e,e'\}| \geqslant 6$ and $M|(E_v \cup \{e,e'\})$ has rank 3 and no triangles, contradicting the fact that M has no $U_{3,6}$-minor. $\square$

By 5.1.3 and duality, we may assume that there is an element $e \in E(M)$ such that $M\backslash e$ is totally free. Thus $M\backslash e$ is a quasi-line. Let T_1 be the reduced del–con tree such that $M\backslash e = M(T_1)$. If T_1 has only one vertex, then $M\backslash e$ has no triangles and hence $r(M^*) = 3$, contradicting 5.1.2. Hence, T_1 has at least two vertices.

5.1.4. *M/e is totally free.*

Subproof. We first show that M/e is 3-connected. Assume otherwise and let (X, Y) be a 2-separation of M/e. As M has no triangles, $|X|, |Y| \geqslant 3$. An easy rank argument shows that (X, Y) is a 3-separation of $M\backslash e$ and $e \in \mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$. Since T_1 has more than one vertex, it follows from Lemma 4.4, that there is a vertex v of T_1 such that, if T' is any connected component of $T_1 - v$, then $\bigcup_{w \in V(T')} E_w$ is contained in either X or Y. Let u be a degree-1 vertex distinct from v. Without loss of generality, $E_u \subseteq X$. Let C be a 3-element subset of E_u. As $M\backslash e$ has no triangles, C is a triad of $M\backslash e$. But $e \in \mathrm{cl}(Y)$ and $Y \subseteq E(M) - C$, so C is a triad of M, contradicting the fact that M has no triads. Hence M/e is indeed 3-connected.

Assume that M/e is not totally free. Then, by Lemma 2.6, there is a unique clone e' of e in M. Let u and v be two degree-1 vertices of T_1. Without loss of generality $e' \notin E_u$. Let $\{x, y, z\}$ be a three elements subset of E_u. The set $\{x, y, z\}$ is a triad of $M\backslash e$, and therefore $\{x, y, z, e\}$ is a cocircuit of M. As $\{e, e'\}$ is a clonal pair, $\{x, y, z, e'\}$ is also a cocircuit. By cocircuit exchange $\{x, y, e, e'\}$ must be a cocircuit of M, and hence $\{x, y, e'\}$ is a triad of $M\backslash e$. Thus $e' \in E_u$ by Proposition 4.3, contradicting our hypothesis.

By 5.1.4 and the minimality of M, M/e is a quasi-line. Let T_2 be the reduced del–con tree such that $M/e = M(T_2)$. If T_2 has only one vertex then as M/e has no triads, $M/e \cong U_{2,k}$ and so $r(M) = 3$, contradicting 5.1.2. Thus T_2 has at least two vertices.

Let P be a maximal path in T_2 and let u and v be the end-vertices of P. Since M/e has no triads, u and v are del vertices of T_2. Let $F_u = E_u \cup e$ and $F_v = E_v \cup e$. F_u and F_v are rank-3 flats of M that meet in $\{e\}$. Thus $r_M(F_u \cup F_v) \leqslant 5$. Also, as E_u and E_v are distinct flats of M/e it follows that $r_M(F_u \cup F_v) \geqslant 4$. Suppose for a contradiction that $r_M(F_u \cup F_v) = 5$. Now F_u and F_v must both be cyclic flats, since $|F_u|, |F_v| \geqslant 4$ and M has no triangles. Also, $r(F_u) + r(F_v) = r(F_u \cup F_v) + 1$. So, by Lemma 2.2, e is fixed in M. This is a contradiction as $M\backslash e$ is 3-connected and M is totally free. Therefore $r_M(F_u \cup F_v) = 4$. Since $r_{M/e}(E_u \cup E_v) = 3$, we can apply Lemma 4.6 to deduce that T_2 is isomorphic to a path of length two, with end-vertices u and v and a single internal vertex w such that $|E_w| = 1$. Thus, M/e has the structure shown in Fig. 1.

We have shown that $r(M) = 4$. But M^* is also a minimal counterexample to the theorem, and $M^*\backslash e$ is also totally free. Therefore the same arguments show that $r(M^*) = 4$. It immediately follows that $|E(M)| = 8$ and that therefore M/e must be the matroid R_7 shown in Fig. 2.

Again, by applying the same arguments to M^*, we can show that M^*/e too is isomorphic to R_7. Thus $M\backslash e = (M^*/e)^*$ is isomorphic to R_7^*, shown in Fig. 3.

Since there are two disjoint triangles in M/e and none in M, it follows that e is in exactly two circuit-hyperplanes of M, and these flats meet exactly in $\{e\}$. Also, by examining $M\backslash e$ we see that there are exactly two circuit-hyperplanes of M that avoid e. Thus there are exactly four circuit-hyperplanes in M. Let these circuit-hyperplanes be $F_1, \ldots, F_4$. Clearly, every element is in at least one of these flats. Suppose that some element x is in exactly one. Then M/x has rank 3 and contains exactly one circuit-hyperplane. By deleting one element from this line we obtain a $U_{3,6}$-minor.

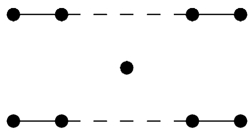

Fig. 1.

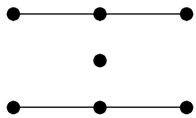

Fig. 2. $M/e \cong R_7$.

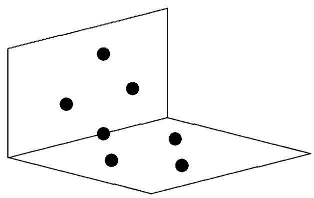

Fig. 3. $M \backslash e \cong R_7^*$.

Thus we must assume that every element is in at least two circuit-hyperplanes. There are four of these flats, so there are six possible intersections of circuit-hyperplanes. Since each of the eight elements of $E(M)$ is in at least one intersection, it follows by an application of the pigeonhole principle that there are two circuit-hyperplanes that meet in two elements x and y. Let these flats be F_1 and F_2. If $e \notin F_1 \cup F_2$ then there would exist two rank-3 flats of $M \backslash e$ that meet in two elements. As this is not the case, $e \in F_1 \cup F_2$. Since e is in exactly two circuit-hyperplanes that meet exactly in $\{e\}$, it follows that e is not in $F_1 \cap F_2$, and therefore we can say without loss of generality that $e \in F_1 - F_2$. Let F_3 be the circuit-hyperplane that meets F_1 exactly in $\{e\}$. Thus neither x nor y is in F_3. There is one remaining circuit-hyperplane F_4. If F_4 does not contain $\{x, y\}$ then without loss of generality say that $x \notin F_4$. Then, M/x contains exactly two non-trivial lines, and these lines meet in y. Thus, $M/x \backslash y \cong U_{3,6}$. Therefore, suppose that $\{x, y\} \subset F_4$. Now M/x has exactly three non-trivial lines and these lines meet in y. Again $M/x \backslash y \cong U_{3,6}$ contradicting the assumption that M does not have a $U_{3,6}$-minor. $\square$

Finally, we can prove Theorem 1.1 which for convenience we restate here.

Theorem 5.2. *Let M be a 3-connected $GF(q)$-representable matroid with no $U_{3,6}$-minor. Then M has at most $(q-2)!$ inequivalent $GF(q)$-representations.*

Proof. Let N be a totally free minor of M. By Theorem 5.1 N is a quasi-line. Say $|E(N)| = k$. By Lemma 3.1, N is $GF(q)$-representable if and only if $U_{2,k}$ is, that is, if and only if $k \leqslant q+1$. Furthermore, Lemma 3.1 also implies that $n_q(N) = n_q(U_{2,k})$. The number of inequivalent representations of $U_{2,k}$ over $\mathrm{GF}(q)$ is at most the number of ordered $(k-3)$-tuples of elements from $\mathrm{GF}(q) - \{0,1\}$ such that the elements of the $(k-3)$-tuple are pairwise distinct. Thus

$$n_q(U_{2,k}) \leqslant \frac{(q-2)!}{(q-k+1)!}$$

(In fact equality holds here for prime fields.) It follows that $n_q(N) \leqslant n_q(U_{2,q+1}) \leqslant (q-2)!$. Thus, all totally-free minors of M have at most $(q-2)!$ inequivalent representations and it now follows from Lemma 2.5 that M has at most $(q-2)!$ inequivalent representations. □

References

[1] T.H. Brylawski, Modular constructions for combinatorial geometries, Trans. Amer. Math. Soc. 203 (1975) 1–44.

[2] J. Geelen, J. Oxley, D. Vertigan, G. Whittle, Totally free expansions of matroids, J. Combin. Theory Ser. B 84 (2002) 130–179.

[3] J. Kahn, On the uniqueness of matroid representations over GF(4), Bull. London Math. Soc. 20 (1) (1988) 5–10.

[4] J. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

[5] J. Oxley, C. Semple, D. Vertigan, Generalized $\Delta - Y$ exchange and k-regular matroids, J. Combin. Theory Ser. B 79 (1) (2000) 1–65.

[6] J. Oxley, D. Vertigan, G. Whittle, On inequivalent representations of matroids over finite fields, J. Combin. Theory Ser. B 67 (2) (1996) 325–343.