ON ROTA'S BASIS CONJECTURE

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ABSTRACT. Rota conjectured that, if $(B_1, \ldots, B_n)$ are disjoint bases in a rank-n matroid M, then there are n disjoint transversals of $(B_1, \ldots, B_n)$ that are bases of M. We prove the weaker result that there are $O(\sqrt{n})$ disjoint transversals of $(B_1, \ldots, B_n)$ that are bases. We also prove that, if $(B_1, \ldots, B_k)$ are disjoint bases of a rank-n matroid with $n > \binom{k+1}{2}$, then there are n disjoint independent transversals of $(B_1, \ldots, B_k)$.

1. Introduction

In 1989, Rota conjectured that, given n bases of a rank-n matroid, there is an n by n grid such that the rows contain the given bases and each column also contains a basis; see Huang and Rota [1]. By possibly adding parallel elements to the matroid, we can assume that the original n bases are disjoint, and thus we have the following equivalent conjecture.

Conjecture 1.1 (Rota's Basis Conjecture). *If $(B_1, \ldots, B_n)$ are disjoint bases in a rank-n matroid M, then there are n disjoint transversals of $(B_1, \ldots, B_n)$ that are bases of M.*

We prove the following related results.

Theorem 1.2. *For $n \geq k^2 - k + 1$, if $(B_1, \ldots, B_n)$ are disjoint bases in a rank-n matroid, then there are k disjoint transversals of $(B_1, \ldots, B_n)$ that are bases.*

Theorem 1.3. *If $(B_1, \ldots, B_k)$ are disjoint bases in a rank-n matroid where $n \geq \binom{k+1}{2} + 1$, then there are n disjoint independent transversals of $(B_1, \ldots, B_k)$.*

We hope that the quadratic bounds in Theorems 1.2 and 1.3 will be improved to linear functions.

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2. Disjoint independent transversals

In this section we prove Theorem 1.2. For sets $(S_1, \dots, S_n)$ and $X \subseteq \{1, \dots, n\}$, we let $S(X)$ denote $\bigcup(S_i : i \in X)$. We use the following result; see Rado [3] or Oxley [2, pp. 388].

Theorem 2.1 (Rado's Theorem)**.** *Let $(S_1, \dots, S_n)$ be sets in a matroid. Then there is an independent transversal of $(S_1, \dots, S_n)$ if and only if $r(S(X)) \geq |X|$ for each $X \subseteq \{1, \dots, n\}$.*

As a corollary we have the following lemma.

Lemma 2.2. *Let $t \leq n$ and let $(S_1, \dots, S_n)$ be independent sets of a matroid. If $|S_i| \geq \min(i, n-t)$ for each $i \in \{1, \dots, n\}$ and there are disjoint subsets $Y_1, \dots, Y_t$ of $\{1, \dots, n\}$ such that $S(Y_1), \dots, S(Y_t)$ each have rank at least n, then there is an independent transversal of $(S_1, \dots, S_n)$.*

Proof. Let $X \subseteq \{1, \dots, n\}$. If $Y_i \subseteq X$ for some $i \in \{1, \dots, t\}$, then $r(S(X)) \geq r(S(Y_i)) \geq n \geq |X|$. If Y_i is not contained in X for each i, then $|X| \leq n - t$. Let k be the maximum index in X in this case. Then $|X| \leq k$, and since $r(S(X)) \geq r(S_k) = |S_k| \geq \min(k, n-t)$, it again follows that $r(S(X)) \geq |X|$. Hence, by Rado's Theorem, there is an independent transversal of $(S_1, \dots, S_n)$. □

Proof of Theorem 1.2. Let $l = \binom{k}{2}$; note that $n \geq 2l+1$. We can choose bases $X_1, \dots, X_l$ such that $X_i \subseteq B_i \cup B_{n-i}$ and $|X_i \cap B_i| = n - i$. Let $S = (B_1 \cup \cdots \cup B_n) - (X_1 \cup \cdots \cup X_l)$. Now let $A_1 = \emptyset$ and, for each $i \in \{2, \dots, k\}$, let $A_i = \left\{1, \dots, \binom{i}{2}\right\}$. We claim that there are disjoint independent transversals $T_1, \dots, T_k$ of $(B_1, \dots, B_n)$ with $T_i \subseteq S \cup X(A_i)$ for each $i \in \{1, \dots, k\}$. Certainly there exists an independent transversal $T_1 \subseteq S$ of $(B_1, \dots, B_n)$. Assume that, for some $t \in \{1, \dots, k-1\}$, we have found disjoint independent transversals $T_1, \dots, T_t$ of $(B_1, \dots, B_n)$ with $T_i \subseteq S \cup X(A_i)$ for each $i \in \{1, \dots, t\}$. Let $T' = T_1 \cup \cdots \cup T_t$, let $S' = S \cup X(A_{t+1}) - T'$, and let $r = \binom{t+1}{2}$. Consider the independent sets $(S_1, \dots, S_n)$, where

$$S_i = \begin{cases} B_{r+i} \cap S', & \text{if } 1 \leq i < n - 2r; \\ B_{2r-n+1+i} \cap S' = B_{2r-n+1+i} - T', & \text{if } n - 2r \leq i < n - r; \\ B_i \cap S' = B_i - T', & \text{if } n - r \leq i \leq n. \end{cases}$$

We claim that $|S_i| \geq \min(i, n-t)$ for each $i \in \{1, \dots, n\}$. First consider the case that $1 \leq i \leq l - r$. Then

$$S_i = B_{r+i} \cap S' = B_{r+i} - B_{r+i} \cap X_{r+i} - B_{r+i} \cap T'.$$

Since $r + i \notin A_t$, the sets X_{r+i} and T' are disjoint, and therefore $|S_i| = n-(n-(r+i))-t = i+r-t \geq i$. Similarly, if $n-l-r \leq i < n-2r$, then

$$S_i = B_{r+i} \cap S' = B_{r+i} - B_{r+i} \cap X_{n-(r+i)} - B_{r+i} \cap T',$$

and again $|S_i| = i + r - t \geq i$. It remains to consider the case that either $l - r < i < n - l - r$ or $n - 2r \leq i \leq n$. Here $S_i = B_j - T'$ for some $j \in \{1, \ldots, r\} \cup \{l + 1, \ldots, n - l - 1\} \cup \{n - r, \ldots, n\}$, and thus $|S_i| = n - t$. Hence $|S_i| \geq \min(i, n - t)$ for each $i \in \{1, \ldots, n\}$.

Let $y_i = \binom{t}{2} + i$ for each $i \in \{1, \ldots, t\}$. Then $y_i \in A_{t+1} - A_t$, and thus the basis X_{y_i} is contained in $(B_{y_i} \cap S') \cup (B_{n-y_i} \cap S') = S_{n-2r-1+y_i} \cup S_{n-y_i}$. Hence if $Y_i = \{n - 2r - 1 + y_i, n - y_i\}$, then $S(Y_i)$ has rank n for each $i \in \{1, \ldots, t\}$. By Lemma 2.2, there is an independent transversal $T_{t+1} \subseteq S'$ of $(B_1, \ldots, B_n)$, and we therefore inductively obtain the required transversals. □

3. Partitioning into independent transversals

In this section we prove Theorem 1.3 using the following lemma.

Lemma 3.1. *Let $(S_1, \ldots, S_k)$ be disjoint k-element sets in a matroid and let $(Y_1, Y_2, \ldots, Y_{k-1})$ be disjoint independent sets such that Y_i is an i-element transversal of $(S_1, \ldots, S_i)$ for each $i \in \{1, \ldots, k - 1\}$. If $(S_1 \cup \cdots \cup S_k) - (Y_1 \cup \cdots \cup Y_{k-1})$ is independent, then there are k disjoint independent transversals of $(S_1, \ldots, S_k)$.*

Proof. Let $Z = (S_1 \cup \cdots \cup S_k) - (Y_1 \cup \cdots \cup Y_{k-1})$ and let $Y_0 = \emptyset$. For each $i \in \{0, \ldots, k - 1\}$, there is a set $X_i \subseteq Z$ such that $|X_i| = |Y_i|$ and $(Z - X_i) \cup Y_i$ is independent. We can now find disjoint sets $(W_{k-1}, W_{k-2}, \ldots, W_0)$, in that order, such that W_i is a transversal of $(S_1, \ldots, S_k)$ with $Y_i \subseteq W_i \subseteq (Z - X_i) \cup Y_i$, for each $i \in \{0, \ldots, k - 1\}$. Note that each W_i is independent since it is contained in $(Z - X_i) \cup Y_i$. □

Proof of Theorem 1.3. Since $n \geq \binom{k+1}{2} + 1$, we can find disjoint independent sets Z and Z' such that $|Z \cap B_i| = k + 1 - i$ and $|Z' \cap B_i| = i$ for each $i \in \{1, \ldots, k\}$. Let $m = (n - (k + 1)) - (k - 1)$. By Rado's Theorem, we can find disjoint independent transversals $(T_1, \ldots, T_m)$ of $(B_1 - (Z \cup Z'), \ldots, B_k - (Z \cup Z'))$.

Let $S = (B_1 \cup \cdots \cup B_k) - (Z \cup Z' \cup T_1 \cup \cdots \cup T_m)$. We can find disjoint independent subsets $(Y'_1, \ldots, Y'_{k-1})$ of S such that Y'_i is a transversal of $(B_1, \ldots, B_{k-i})$. Let $S' = S - (Y'_1 \cup \cdots \cup Y'_{k-1})$. We can then find disjoint independent subsets $(Y_1, \ldots, Y_{k-1})$ of S' such that Y_i is a transversal of $(B_{i+1}, B_{i+2}, \ldots, B_k)$; see Figure 1. By Lemma 3.1, we can partition

$Z \cup Y_1 \cup \cdots \cup Y_{k-1}$ and $Z' \cup Y'_1 \cup \cdots \cup Y'_{k-1}$ into independent transversals of $(B_1, \ldots, B_k)$. □

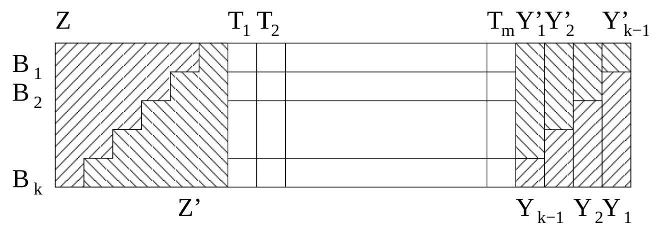

FIGURE 1. Proof of Theorem 1.3.

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