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Matchings, Matroids and Unimodular Matrices

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Abstract

We focus on combinatorial problems arising from symmetric and skew–symmetric matrices. For much of the thesis we consider properties concerning the principal submatrices. In particular, we are interested in the property that every nonsingular principal submatrix is unimodular; matrices having this property are called *principally unimodular*. Principal unimodularity is a generalization of total unimodularity, and we generalize key polyhedral and matroidal results on total unimodularity. Highlights include a generalization of Hoffman and Kruskal's result on integral polyhedra, a generalization of Tutte's results on regular matroids, and partial results toward a decomposition theorem.

Quite separate from the study of principal unimodularity we consider a particular skew–symmetric matrix of indeterminates associated with a graph. This matrix, called the Tutte matrix, was introduced by Tutte to study matchings. By considering the rank of an arbitrary submatrix of the Tutte matrix we discover a natural generalization of the maximum matching problem. We generalize Edmonds' description of the matching polyhedra, Cunningham and Marsh's theorem on total dual integrality, and the Tutte–Berge min–max formula. Interestingly, our proofs do not require the use of augmenting paths.

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Contents

Chapter 1

Introduction

Total unimodularity, matching and matroids are cornerstones of combinatorial optimization. We consider generalizations of these subjects, that arise through the study of symmetric and skew–symmetric matrices. Our focus is mainly on "principal unimodularity" (a generalization of total unimodularity). Our treatment of principal unimodularity occupies, in some capacity, Chapters 2 through 7, though this may not be apparent amidst Chapters 3, 4 and 5 which introduce "delta–matroids". Chapter 8, explores a generalization of matching that has little relation to either principal unimodularity or delta–matroids, so we postpone its introduction until the end of this chapter. We begin by reviewing key results concerning total unimodularity and regular matroids.

A matrix is *totally unimodular* if every square nonsingular submatrix is unimodular (that is, has determinant ± 1). Hoffman and Kruskal [43] noticed the following connection between totally unimodular matrices and integral polyhedra.

(1) *An m by n integral matrix A is totally unimodular if and only if, for every $b \in \mathbf{Z}^m$, each vertex of the polyhedron $\{x \in \mathbf{R}^n : Ax \leq b, x \geq 0\}$ is integral.*

Motivated by this fundamental result of Hoffman and Kruskal in integer programming, researchers obtained a number of results giving conditions for total unimodularity; see Padberg [57] for a survey. We focus on the matroidal study of total unimodularity, which culminates in Tutte's excluded minor characterization [68] and Seymour's decomposition theorem [61].

Regular matroids

By *matroid*, we mean a pair $(V, \mathcal{B})$ where V is a finite set and $\mathcal{B}$ is a collection of subsets, called *bases*, of V satisfying:

Basis exchange axiom. For $B, B' \in \mathcal{B}$ and $x \in B' \setminus B$, there exists $y \in B \setminus B'$ such that $B \Delta \{x, y\} \in \mathcal{B}$.

(Here $A \Delta B$ denotes the symmetric difference of A and B, that is, $(B \setminus A) \cup (A \setminus B)$.) In particular, if V is the set of columns of a matrix A over a field $\mathbf{F}$, and $\mathcal{B}(A)$ is the collection of maximal linearly independent subsets of V, then $(V, \mathcal{B}(A))$ is a matroid; such matroids are called *representable*. A matroid is called *regular* if it can be represented over the reals by a totally unimodular matrix.

The following result of Camion [16] shows that the correspondence between regular matroids and totally unimodular matrices is essentially one–to–one.

(2) *Let A, A' be totally unimodular matrices, such that $A \equiv A'$ modulo 2. Then A can be obtained from A' by multiplying certain rows and columns by -1.*

Given a $(0, \pm 1)$–matrix A we construct a matrix $A_1 = (I, A)$, where I is the identity matrix. A_1 is totally unimodular if and only if A is. Thus if $\mathcal{B}(A_1)$ is not regular, then A is not totally unimodular. So we assume that there exists a totally unimodular matrix A'_1 such that $\mathcal{B}(A_1) = \mathcal{B}(A'_1)$. By row operations, we may assume that $A'_1 = (I, A')$ for some matrix A'. It is easy to prove that $A \equiv A'$ modulo 2. Then, by (2), A is totally unimodular if and only if A can be got from A' by multiplying certain rows and columns by -1.

The following theorem of Tutte [71] is more interesting in its own right than it is as a characterization of total unimodularity.

(3) *For a matroid M, the following are equivalent:*

(i) M is regular,

(ii) M is representable over every field, and

(iii) M is representable over $GF(2)$ and $GF(3)$.

We now consider a deeper theorem, also due to Tutte [68]. First, we need some definitions. Let $M = (V, \mathcal{B})$ be a set–system. For $X \subseteq V$, we denote $M - X$ the set–system $(V \setminus X, \{B \subseteq V \setminus X : B \in \mathcal{B}\})$; we refer to this operation on M as the *deletion of X from M*. By *twisting by X* we mean the operation that converts M to $(V, \{B \Delta X : B \in \mathcal{B}\})$ which we denote by $M \Delta X$. The *dual of M* is the set–system $M \Delta V$, and the *contraction of X in M* is the set–system $(M \Delta X) - X$. A *minor* of M is a non–empty set–system obtained from M by deletions and contractions. Finally, the *Fano matroid* is the binary matroid represented by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(4) *Let M be a binary matroid. Then M is regular if and only if M does not contain a minor isomorphic to the Fano matroid or its dual.*

Tutte also found an excluded minor characterization for GF(2)–representability [68]; thus, with (4), we have a complete excluded minor characterization for regular matroids.

The results (2) and (4) both allow us to demonstrate that a given binary matroid is not regular. We now state a much deeper result that allows us to demonstrate that a binary matroid is regular; it also leads to an efficient recognition algorithm. Given a connected graph $G = (V, E)$, the edge sets of spanning trees of G define a matroid; such matroids are called *graphic matroids*. Graphic matroids are in fact regular. The following theorem of Seymour [61] shows that all regular matroids are essentially graphic.

(5) *Every regular matroid can be obtained by 1–, 2– and 3–sums of R_{10}, graphic matroids, and the duals of such matroids.*

Here R_{10} is a particular regular matroid, and 1–, 2– and 3–sums are operations for composing two matroids; for precise definitions see [56].

We generalize the results (1) through (4). While we do not find a generalization of Seymour's decomposition, we use it as motivation for other results. For instance, Theorem 5.17 generalizes the binary part of Seymour's "splitter theorem", which is an important step in the proof of the decomposition theorem.

Principal unimodularity

We call a square matrix *principally unimodular* if every nonsingular principal submatrix is unimodular. While principal unimodularity sounds weaker than total unimodularity, it is in fact a generalization. Indeed, a matrix A is totally unimodular if and only if $\begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$ is principally unimodular. In Chapter 2 we generalize Hoffman and Kruskal's polyhedral characterization of total unimodularity. Our characterization of principal unimodularity is in terms of integral "basic solutions" to the "linear complementarity problem". The *linear complementarity problem* is stated as follows: *Given an n by n matrix M and a vector $q \in \mathbf{R}^n$, find $z \in \mathbf{R}^n$ satisfying $z \geq 0$, $q + Mz \geq 0$ and $z^T(q + Mz) = 0$.* There is a large literature concerning the linear complementarity problem (see Cottle, Pang and Stone [19] for a survey); it is a generalization of linear programming that also contains bimatrix games and first–order optimality conditions for quadratic programming. As is the case with linear programming, the linear complementarity problem has a "basic" feasible solution whenever there exists a feasible solution, if M is symmetric or skew–symmetric. (This does not hold for arbitrary square matrices). Consequently, if M is a symmetric or skew–symmetric integral principally unimodular matrix, then, for each q for which the linear complementarity problem has a feasible solution, it has an integral feasible solution. This fundamental result, which appears to be new, motivates the further study of symmetric and skew–symmetric, integral, principally unimodular matrices.

Delta–matroids

Let V be a finite set, and A be a V by V symmetric or skew–symmetric matrix. We denote by $A[X]$ the principal submatrix of A indexed by $X \subseteq V$. Now define $\mathcal{F}(A) = \{X \subseteq V : A[X] \text{ is nonsingular}\}$. Bouchet [8], proved that $\mathcal{F}(A)$ satisfies the following:

Symmetric exchange axiom. For $F, F' \in \mathcal{F}$ and $x \in F \Delta F'$, there exists $y \in F \Delta F'$ such that $F \Delta \{x, y\} \in \mathcal{F}$.

A *delta–matroid* is a pair $(V, \mathcal{F})$ where V is a finite set and $\mathcal{F}$ is a collection of subsets of V, called *feasible sets*, satisfying the symmetric exchange axiom. The delta–matroids got from symmetric or skew–symmetric matrices are called *representable.* Delta–matroids were introduced by Bouchet [4] for the purpose of studying principal unimodularity (which he also introduced [7, 11]).

It is easily seen that a set–system obtained from a delta–matroid by the operations twisting and deletion (defined above) is also a delta–matroid. We redefine the term *minor*, for a set–system M, to be any set–system got from M by twisting and deleting. Thus usual "matroidal–minors" and duals are minors in this new sense. For matroidal properties that are closed under duality, like regularity and representability, our definition is quite convenient.

We call a set–system containing the empty set a *normal* set–system. Note that every representable delta–matroid is normal. Bouchet [8] showed that every normal minor of a representable delta–matroid is representable. We call a delta–matroid whose feasible sets all have the same parity (that is, cardinality modulo two) an *even delta–matroid*. We note that every skew–symmetric matrix of odd size is singular, so every delta–matroid that is representable by a skew–symmetric matrix is even.

It is not difficult to prove that matroids are delta–matroids. Except for trivial matroids, representable matroids are not normal, and hence not representable delta–matroids. However, for any base B of a representable matroid M, the set–system $M \Delta B$ is a representable delta–matroid. In Chapter 3 we shall see that a number of well–known matroidal results generalize to delta–matroids.

Regular delta–matroids

A *regular delta–matroid* is a delta–matroid that is representable by a skew–symmetric, principally unimodular matrix. An *equable delta–matroid* is a delta–matroid that is representable by an integral, symmetric, principally unimodular matrix. Interestingly, if M is a normal delta–matroid, then M is equivalent under twisting to a regular matroid if and only if M is both regular and equable. This dichotomy also extends to a near partitioning of the interesting properties of regular matroids.

For equable delta–matroids, results (2), (3) and (4) all generalize cleanly. Our generalization of Tutte's excluded minor characterization, was obtained by a generalization of Gerards' graphical proof of Tutte's theorem [38]. Our original proof was quite long. We present a shorter proof that we obtain by using a theorem of Truemper [65]. The situation is not so nice with regard to generalizing Seymour's decomposition. There appears to be no nontrivial way in which to decompose an equable delta–matroid; we do not even have an appropriate definition for a "2–sum".

For regular delta–matroids, only result (3) generalizes cleanly, see Theorem 4.13. However regular delta–matroids have a very rich matroidal structure and there is some hope that Seymour's decomposition theorem may generalize. To begin with, regular delta–matroids are even, and even delta–matroids have more structure than general delta–matroids. We will also see that the class of regular delta–matroids is preserved under a natural generalization of 1– and 2–sums. This 2–sum is the cause of much difficulty in generalizing (2); we are obliged to consider "3–connected" regular delta–matroids. In joint work with Bouchet and Cunningham we obtained the result that a 3–connected regular delta–matroid has a "unique" representation by a principally unimodular skew–symmetric matrix. The "uniqueness" factors out negating the matrix and the multiplication of a row and its corresponding column by -1. To prove the uniqueness theorem we introduce a tool, called a "blocking sequence", for studying 3–connected, binary, even delta–matroids. These blocking sequences also enable us to generalize Seymour's "splitter theorem" to binary, even delta–matroids.

A final interesting point concerning regular delta–matroids is that there exists a large natural class; namely "Eulerian delta–matroids". A *circle graph* is the intersection graph of chords of a circle, and an *Eulerian delta–matroid* is a binary delta–matroid representable by the adjacency matrix of a circle graph. Bouchet proved that Eulerian delta–matroids are regular [7, 11]. Bouchet also found a nice characterization of Eulerian delta–matroids that neatly distinguishes them from regular delta–matroids; however, for the purpose of a decomposition, it would be preferable to have an excluded minor characterization. De Fraysseix [26] proved that, if B is a base of a matroid M, then M is a planar matroid if and only if $M \Delta B$ is an Eulerian delta–matroid. (Here by *planar matroid* we mean the graphic matroid of a planar graph.) In Chapter 5 we use blocking sequences to prove interesting results about circle graphs.

Matching

In the final chapter we consider a generalization of matching. The problem arises most naturally by considering a certain skew–symmetric matrix of indeterminates; however we begin by stating the problem graphically: *Given a graph $G = (V, E)$ and equicardinal subsets S, T of V, find a set P of $|S|$ vertex disjoint (S, T)–paths and a perfect matching of the vertices that are not covered by any path in P.* When S and T are both empty, the problem is to find a perfect matching. The other extreme is also interesting; when S, T partition V, then the problem is to find a perfect matching in the bipartite graph induced by the edges in the cut (S, T). In the general case, the problem is an interesting blend of network flows and matchings.

The connection to skew–symmetric matrices is the following. Let $G = (V, E)$ be a graph, and let $\{x_{ij} : ij \in E\}$ be a set of algebraically independent indeterminates. We construct a skew–symmetric matrix $A = (a_{ij})$ such that $a_{ij} = \pm x_{ij}$ for $ij \in E$ and $a_{ij} = 0$ otherwise. Tutte [67] observed that A is nonsingular if and only if G has a perfect matching. In similar fashion we show that our generalized matching problem is equivalent to deciding whether $A[V \setminus S, V \setminus T]$ is nonsingular. (Here $A[X, Y]$ denotes the submatrix of A indexed by rows X and columns Y.)

We extend some fundamental results in matching theory to this generalized matching problem. We give a min–max formula for the rank of $A[V \setminus S, V \setminus T]$ that is essentially due to Lovász. This min–max theorem directly implies König's theorem and the Tutte–Berge formula (see [50]). Then we give a totally dual integral polyhedral description for the edge sets of these generalized matchings. As a consequence of the polyhedral description and the ellipsoid algorithm, we get an efficient algorithm for deciding whether such a generalized matching exists.

Ideally we would have liked to find an efficient combinatorial algorithm for solving the problem; despite promising partial results, this remains open. At first one may be tempted to try to calculate the determinant of $A[V \setminus S, V \setminus T]$; however the determinant may have an exponential number of terms. The next approach is to find an algorithm based on "alternating paths"; unfortunately we have been unable to find a satisfactory definition.

Conventions

This thesis is largely self–contained, since proofs rely on elementary linear algebra, and graph theory. However, while motivating certain results, we assume that the reader is familiar with Matroid Theory (see [56]) and Matching Theory (see [50]). With influences from so many areas in combinatorics, it has not always been possible to use standard notation; an index is provided to help minimize the confusion. All results are properly attributed, to the best knowledge of the author; appropriate reference can be found in the paragraph preceeding the statement of the result. Where attribution is missing, the result is claimed to be original research.

Chapter 2

The linear complementarity problem

The main goals of this short chapter are to introduce "principal unimodularity", and to to motivate the further study of principally unimodular symmetric and skew–symmetric matrices. We also introduce an important matrix operation called "pivoting". Let M be a V by V matrix. We call M *principally unimodular (PU)* if every principal submatrix of M is unimodular (that is, has determinant $0, \pm 1$). Principal unimodularity arises as a generalization of total unimodularity as follows: a matrix A is totally unimodular if and only if $\begin{pmatrix} 0 & A \\ \pm A^T & 0 \end{pmatrix}$ is PU.

Due to the connection with integrality in linear programming, totally unimodular matrices are of fundamental importance in combinatorial optimization. In this chapter, we will see that principal unimodularity plays an analogous role with respect to the linear complementarity problem. We give a terse treatment to the linear complementarity problem; for a detailed survey of the problem see Cottle, Pang and Stone [19], or Murty [52]. Applications of the linear complementarity problem include: linear programming, quadratic programming, and bimatrix games; the linear and quadratic programming applications involve skew–symmetric and symmetric matrices respectively.

Let V be a finite set, let M be a V by V matrix, and let q be a column vector indexed by V. The *linear complementarity problem*, with respect to q, M, is to find column vectors w, z indexed by V satisfying:

$$w = Mz + q, \tag{2.1}$$

$$w_v z_v = 0, \qquad (v \in V) \tag{2.2}$$

$$w, z \geq 0. \tag{2.3}$$

We denote the above problem by (q, M). Let w, z be column vectors indexed by V. We call w, z *complementary* if they satisfy (2.2), and w, z are *feasible* for (q, M) if they satisfy (2.1) and (2.3). Complementary feasible vectors for (q, M) are called *solutions* of (q, M). For a solution w, z of (q, M), w is uniquely determined by z, so we occasionally denote the pair z, w by the vector z.

Let X, Y be subsets of V. We denote by $M[X, Y]$ the X by Y submatrix of M, and we denote by $M[X]$ the principal submatrix $M[X, X]$. Suppose that $M[X]$ is nonsingular, for some subset X of V. There is a unique pair of vectors w', z' satisfying (2.1) such that $w'_X = 0$ and $z'_{\overline{X}} = 0$. Here v_X denotes the restriction of the vector v to the set X, and $\overline{X}$

denotes $V \setminus X$. The pair z', w' is defined as follows:

$$\begin{array}{rclrcl} z'_X & = & -(M[X])^{-1}q_X, & w'_X & = & 0, \\ z'_{\overline{X}} & = & 0, & w'_{\overline{X}} & = & q_{\overline{X}} - M[\overline{X}, X](M[X])^{-1}q_X. \end{array}$$

Note that w', z' are not necessarily nonnegative. Such complementary vectors are called *basic vectors* of (q, M) with respect to X. The main theorem of this chapter is the following.

Theorem 2.1 *Let M be a V by V integral matrix. Then the following are equivalent:*

1. *M is principally unimodular.*
2. *For every integral vector q, all basic solutions of (q, M) are integral.*

Unfortunately it is not the case, for an integral PU–matrix M, that (q, M) has an integral solution for every q for which (q, M) has a solution. Indeed, consider (q, M) where

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, q = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Note that M is PU. Let $z^* = (0, \frac{3}{2}, \frac{1}{2})^T$, and $w^* = (0, 0, 0)^T$; then z^*, w^* is a solution to (q, M). However, for any solution z, w to (q, M), we have $z_2 - z_3 - 1 \geq w_1$. Then, since $w, z \geq 0$, we must have $z_2 > 0$. So, by complementarity, $w_2 = 0$, and $2z_3 - 1 = 0$. Thus, z is not integral.

For symmetric and skew–symmetric matrices the situation is nicer. For completeness, we will include a proof of the following result in a later section.

Theorem 2.2 (See Cottle, Pang, and Stone [19]) *Let M be a V by V symmetric or skew–symmetric matrix, and let q be a column vector indexed by V. If (q, M) has a solution, then there exists a basic solution to (q, M).*

As an immediate consequence of Theorems 2.1 and 2.2 we have the following result.

Corollary 2.3 *Let M be a symmetric or skew–symmetric V by V PU–matrix, and let q be an integral column vector indexed by V. If (q, M) has a solution, then there exists an integral solution to (q, M).* □

To prove Theorem 2.1, we need to introduce a matrix transformation, called "pivoting". However, one direction can be proved easily using the adjoint formula for the inverse of a matrix (see Horn and Johnson [44]). Let A be a nonsingular V by V matrix, where $V = \{1, \ldots, n\}$. We define a new V by V matrix (b_{ij}), denoted $adj(A)$, where

$$b_{ij} = (-1)^{i+j} \det(A[V - j, V - i]).$$

Then, the adjoint formula for the inverse of A is

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Proposition 2.4 *Let M be a V by V integral PU–matrix, let X be a subset of V such that $\det(M[X]) = \pm 1$, and let q be an integral column vector indexed by V. Then the basic vectors of (q, M) corresponding to X are integral.*

Proof It suffices to prove that $M[X]^{-1}$ is integral, which follows easily from the adjoint formula for the inverse. □

Linear programming

In this section we show how the linear complementarity problem arises as a generalization of linear programming.

Let X, Y be a partition of a finite set V. Let A be an X by Y matrix, c be a column vector indexed by Y, and b be a column vector indexed by X. We are interested in the following linear programming problem:

$$(P) - \begin{cases} \min & c^T z_1 \\ \text{s.t.} & \begin{aligned} Az_1 &\geq b \\ z_1 &\geq 0. \end{aligned} \end{cases}$$

The dual of (P) is

$$(D) - \begin{cases} \max & b^T z_2 \\ \text{s.t.} & \begin{aligned} A^T z_2 &\leq c \\ z_2 &\geq 0. \end{aligned} \end{cases}$$

A well–known result in linear programming is that, if z_1 is feasible to (P), and z_2 is feasible to (D), then z_1 is optimal to (P) and z_2 is optimal to (D), if and only if the following (complementary slackness) conditions are satisfied:

$$\begin{aligned} z_2^T(b - Az_1) &= 0 \\ z_1^T(c - A^T z_2) &= 0. \end{aligned}$$

Now, let

$$M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix}, \text{ and } q = \begin{pmatrix} c \\ -b \end{pmatrix}.$$

For a column vector z indexed by V, it is easy to verify that, z_Y is optimal to (P) and z_X is optimal to (D), if and only if z is a solution to the linear complementarity problem (q, M).

Theorem 2.1 generalizes the following well–known theorem in integer programming. A polyhedron $P \subseteq \mathbf{R}^V$ is *integral* if $\max(c^T x : x \in P)$ has an integral optimal solution whenever it has an optimal solution.

Theorem 2.5 (Hoffman and Kruskal [43]) *Let A be an integral X by Y matrix. Then the following are equivalent*

(1) A is totally unimodular.

(2) For every integral vector b indexed by X, the polyhedron $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ is integral. □

The assertion that (1) implies (2) is elementary and is easily implied by Theorem 2.1; however, it is not immediately obvious that the converse of Theorem 2.5 is a corollary of Theorem 2.1. If A is not totally unimodular, then Theorem 2.1 shows that there exists $b \in \mathbf{Z}^X$ and $c \in \mathbf{Z}^y$ such that either $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ or $\{y \in \mathbf{R}^X : A^T y \leq c, y \geq 0\}$ is not integral. If $\{x \in \mathbf{R}^Y : Ax \geq b, x \geq 0\}$ is not integral, then we are done. So we may assume that $\{y \in \mathbf{R}^X : A^T y \leq c, y \geq 0\}$ is not integral. Then there exists $b' \in \mathbf{Z}^X$ such that the value of the linear program (D) is fractional. Hence, by duality, $\{x \in \mathbf{R}^Y : Ax \leq b', x \geq 0\}$ is not integral. For a detailed discussion on total unimodularity and polyhedral theory, see Nemhauser and Wolsey [54].

Pivoting

Let M be a V by V matrix. For a subset X of V such that $M[X]$ is nonsingular, define matrices α, β, γ, δ, such that $\alpha = M[X]$ and $M = \left(\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}\right)$. Then define $M * X$ to be

$$\left(\begin{array}{c|c} \alpha^{-1} & -\alpha^{-1}\beta \\ \hline \gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{array}\right).$$

The operation that converts M to $M * X$ is called a *pivot.*

Let q be a column vector indexed by V. From the linear complementarity problem (q, M), we define a new problem $(q', M * X)$, which we denote $(q, M) * X$, where

$$\begin{aligned} q'_X &= -(M[X])^{-1} q_X, \\ q'_{\overline{X}} &= q_{\overline{X}} - M[\overline{X}, X](M[X])^{-1} q_X. \end{aligned}$$

The following lemma shows that the linear complementarity problems (q, M) and $(q, M)*X$ are essentially the same; the proof follows directly from the definitions.

Lemma 2.6 (See Cottle,Pang, and Stone [19]) *Let M be a V by V matrix, q be a column vector indexed by V, X be a subset of V such that $M[X]$ is nonsingular, and w, z be a solution of (q, M). Now, define w', z' such that $w'_X = z_X$, $z'_X = w_X$, $w'_{\overline{X}} = w_{\overline{X}}$, and $z'_{\overline{X}} = z_{\overline{X}}$. Then, w', z' is a solution of $(M, z) * X$.* □

Let w, z be a solution to (q, M), and let w', z' be the corresponding solution to $(q, M) * X$. It can be easily verified that w, z is a basic solution to (q, M) if and only if w', z' is a basic solution for $(q, M) * X$. Furthermore, properties like nonnegativity, complementarity and integrality are also preserved under such transformations.

Invariants

Pivoting in matrices preserves a number of interesting properties, like skew–symmetry, principal unimodularity, positive (semi–) definiteness, and positive principal minors, see Cottle, Pang, and Stone [19]. Note that, symmetry is *not* preserved under pivoting. However, if $M[X]$ is a nonsingular submatrix of a symmetric matrix M, then we can obtain a symmetric matrix from $M * X$ by multiplying the rows indexed by X by -1. We call a matrix that is obtained from a symmetric matrix by negating certain rows a *bisymmetric matrix.* Bisymmetry is preserved under pivoting.

The following theorem is fundamental to this dissertation, so we include a proof. For sets A, B, we define $A \Delta B$ to be the *symmetric difference* of A and B; that is, the union of $A \setminus B$ and $B \setminus A$.

Theorem 2.7 (Tucker [66]) *Let $M[X]$ be a nonsingular principal submatrix of a V by V matrix M. Then, for $S \subseteq V$,*

$$\det(M * X[S]) = \pm \det(M[X \Delta S]) / \det(M[X]).$$

Proof (Bouchet, personal communication) Let $Y = V \setminus X$, and let M be partitioned as follows:

$$M = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}.$$

Construct a copy $\tilde{V}$ of V, and for $Z \subseteq V$, denote by $\tilde{Z}$ the corresponding copy of Z. Now define M' to be

$$\begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & \tilde{X} & \tilde{Y} \end{array} \\ \begin{pmatrix} I & 0 & \alpha & \beta \\ 0 & I & \gamma & \delta \end{pmatrix} \end{array}.$$

For $R \subseteq V$, we have

$$\det M[R] = \pm \det M'[V, \tilde{R} \cup (V \setminus R)]. \tag{2.4}$$

Now define matrices

$$C = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha^{-1} & 0 \\ -\gamma\alpha^{-1} & I \end{pmatrix} \end{array},$$

and

$$B = CM' = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & \tilde{X} & \tilde{Y} \end{array} \\ \begin{pmatrix} \alpha^{-1} & 0 & I & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & I & 0 & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

Therefore

$$\begin{aligned} \det B[V, \tilde{R} \cup (V \setminus R)] &= \det M'[V, \tilde{R} \cup (V \setminus R)] \det C \\ &= \det M[V, \tilde{R} \cup (V \setminus R)] \det \alpha^{-1}. \end{aligned} \tag{2.5}$$

Now swapping the columns X and $\tilde{X}$ pairwise in B we get the matrix B',

$$B' = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & \tilde{X} & \tilde{Y} \end{array} \\ \begin{pmatrix} I & 0 & \alpha^{-1} & \alpha^{-1}\beta \\ 0 & I & -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

Then

$$\det B[V, \tilde{R} \cup (V \setminus R)] = \pm \det B'[V, (\tilde{R} \Delta \tilde{X}) \cup (V \setminus (R \Delta X)))]. \tag{2.6}$$

For $T \subseteq V$, we have

$$\det M * X[T] = \pm \det B'[V, \tilde{T} \cup (V \setminus T)]. \tag{2.7}$$

The result is obtained by combining equations 2.4 to 2.7. □

Elementary pivots

The following theorem about pivoting is implied by the quotient formula for the Schur complement (see Cottle *et al.* [19, pp. 76]).

Theorem 2.8 *Let $M[X]$ be a nonsingular principal submatrix of a square matrix M, and let $M * X[Y]$ be a nonsingular principal submatrix of $M * X$. Then $(M * X) * Y = M * (X \Delta Y)$.* □

Let M be a V by V matrix. Suppose that $M[X]$ is a nonsingular principal submatrix of M, and there exists $X' \subseteq X$ such that $M[X']$ is nonsingular. Then, by Theorem 2.8, $M * X = M * X' * (X \setminus X')$. We call a nonempty set X an *elementary set of M* if $M[X]$ is nonsingular but there exists no proper subset X' of X such that $M[X']$ is nonsingular. We call $M * X$ an *elementary pivot* if X is an elementary set of M. Thus, any pivot is equivalent to a sequence of elementary pivots.

Proposition 2.9 *Let M be a V by V matrix, and let X be an elementary set of M. Then every row and column of $M[X]$ contains exactly one nonzero entry.*

Proof Easy. □

As an easy corollary of the previous proposition we have that, if M is symmetric or skew–symmetric, then all elementary sets have one or two elements.

Proofs

We now prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. If M is PU, then, by Proposition 2.4, for every integral q, all basic solutions of (q, M) are integral. We now prove the converse. We begin by proving the following claim.

Claim *Let X be a subset of V, such that $\det(M[X]) = \pm 1$. Then, $M * X$ is integral. Furthermore, if q, q' is a pair of vectors such that $(q', M * X) = (q, M) * X$, then q is integral if and only if q' is integral.*

Since $M[X]$ is unimodular and integral, $M[X]^{-1}$ is unimodular and integral. Therefore, $M * X$ is also integral. Thus, if q is integral, then q' is integral. The converse follows since pivoting is an involution. This proves the claim.

Suppose that M is not PU, and let Y be a minimum cardinality subset of V such that $M[Y]$ is not unimodular. Suppose that Y is not an elementary set of M. Since $M[Y]$ is nonsingular, there exists a subset Y' of Y, such that Y' is an elementary set of M. By our choice of Y, $M[Y']$ is unimodular. By the claim, it suffices to prove the theorem for $M * Y'$. Now $|Y' \Delta Y| < |Y|$, and by Theorem 2.7, $M * Y'[Y \Delta Y']$ is not unimodular. Thus, inductively, we may assume that Y is an elementary set.

We will create an integral vector q so that the basic solution w, z of (q, M), with respect to the set Y, is feasible but not integral. To be basic w, z, q must satisfy the following equations

$$M[Y]z_Y + q_Y = 0 \qquad (2.8)$$

$$M[\overline{Y}, Y]z_Y + q_{\overline{Y}} = w_{\overline{Y}}. \qquad (2.9)$$

By Proposition 2.9, every row and column of $M[Y]$ contains exactly one nonzero element. Therefore, every row and column of $M[Y]^{-1}$ contains exactly one nonzero element. Furthermore, since $M[Y]$ is integral but not unimodular, $M[X]^{-1}$ contains some non–integral entries. Thus, it is easy to find an integral q_Y such that the unique solution z_Y to (2.8) is both nonnegative and not integral. Given this z_Y, we can choose an integral $q_{\overline{Y}}$ sufficiently large so that the solution $w_{\overline{Y}}$ to (2.9) is nonnegative. Hence we have an integral q, and a nonintegral basic feasible solution w, z to (q, M), as required. □

Proof of Theorem 2.2. We assume that $M = (m_{ij})$ is skew–symmetric; the proof is essentially the same for bisymmetric matrices.

Let w, z be a solution to (q, M), and denote by X the support of z (that is, the set $\{v \in V : z_v \neq 0\}$). We prove the result by induction on $|X|$; if $|X| = 0$, then w, z is basic. Let $Y = \{v \in V : w_v = 0\}$. Note that, by complementarity, X is a subset of Y.

Suppose that $M[Y, X] = 0$. In particular, we have $M[X] = 0$. Choose some $x \in X$. Now define a new vector z' by fixing $z'_v = z_v$ for all $v \in V - x$, and decreasing z'_x as far as possible, while maintaining z' feasible to (q, M). Let $w' = Mz' + q$. Since $M[X] = 0$ and $z'_{\overline{X}} = 0$, we have

$$\begin{aligned} w'_X &= q_X, \\ w'_{\overline{X}} &= M[\overline{X}, X] z'_X + q_{\overline{X}}. \end{aligned}$$

However, since $w_X = 0$, we have $q_X = 0$. Therefore, w', z' are complementary, and hence w', z' is a solution to (q, M). If $z'_x = 0$, then z' has a smaller support than z, so the result follows inductively. Therefore, we may assume that $z'_x > 0$. Since we cannot reduce z'_x further while maintaining feasibility to (q, M), there exists $y \in V$ such that $w'_y = 0$, and $m_{xy} > 0$. Hence, by replacing w, z by w', z', and redefining Y accordingly, we get $M[X, Y] \neq 0$.

Choose $x \in X$, and $y \in Y$ such that $m_{xy} \neq 0$. Now define S to be $\{x, y\}$. Since M is skew–symmetric, $M[S]$ is nonsingular. Recall that (q, M) has a basic solution if and only if $(q, M) * S$ has a basic solution. Now define z', w' such that

$$\begin{aligned} z'_S &= w_S, \quad w'_S = z_S, \\ z'_{\overline{S}} &= z_{\overline{S}}, \quad w'_{\overline{S}} = w_{\overline{S}}. \end{aligned}$$

Then, by Proposition 2.6, z', w' is a solution to $(q, M) * S$. However, since $S \subseteq Y$ and $S \cap X \neq \emptyset$, z' has a smaller support than z. Therefore, the result follows by induction. □

Chapter 3

Delta–matroids

This chapter is a general introduction to delta–matroids. Proofs of a number of known results are included for completeness, and to give the reader a feeling for the structure of delta–matroids.

A *set–system* is a pair $(V, \mathcal{F})$ where V is a finite set, and $\mathcal{F}$ is a set of subsets of V, called *feasible sets*. A *delta–matroid* is a set–system $(V, \mathcal{F})$ that satisfies the following axiom (see Bouchet [4] and Chandrasekaran and Kabadi [17]):

Symmetric exchange axiom For $X, Y \in \mathcal{F}$ and $x \in X \Delta Y$, there exists $y \in X \Delta Y$ such that $X \Delta \{x, y\} \in \mathcal{F}$.

Here $X \Delta Y$ denotes the *symmetric difference* of X and Y, that is, $(X \setminus Y) \cup (Y \setminus X)$. It is not difficult to prove that a nonempty set–system $(V, \mathcal{F})$ is a matroid (that is, $\mathcal{F}$ is the set of bases of a matroid) if and only if $(V, \mathcal{F})$ is a delta–matroid and all feasible sets are equicardinal. We recall that a set–system $(V, \mathcal{F})$ is a matroid if and only if $\mathcal{F}$ is nonempty and it satisfies the following axiom (see Oxley [56, pp. 17]):

Exchange axiom For $X, Y \in \mathcal{F}$ and $x \in Y \setminus X$, there exists $y \in X \setminus Y$ such that $X \Delta \{x, y\} \in \mathcal{F}$.

It is also the case that the independent sets of a matroid define a delta–matroid; however most important properties of delta–matroids generalize properties concerning the bases of matroids. For instance we will see that the most interesting delta–matroids are *even*, that is, all feasible sets have the same cardinality modulo 2.

Remark: An empty set–system (that is, a set–system with no feasible sets) is not a matroid, whereas it is a delta–matroid. This difference in convention is well–founded. For reasons of representability it is natural to require that the empty set is independent in a matroid. Similarly, it might be natural to require that the empty set is feasible in a delta–matroid. This condition would exclude matroids from being delta–matroids; instead we call a delta–matroid in which the empty set is feasible a *normal delta–matroid.*

Another interesting class of delta–matroids are those arising from the matchable sets of a graph. Let $G = (V, E)$ be a graph, and let $\mathcal{M}$ be the set of subsets X of V such that $G[X]$ (the subgraph of G induced by the vertex set X) has a perfect matching. Bouchet [9] proved that $(V, \mathcal{M})$ is a delta–matroid; we call this the *matching delta–matroid* of G.

Minors

Let $M = (V, \mathcal{F})$ be a set–system, and let $X \subseteq V$. We define $\mathcal{F}\Delta X = \{F\Delta X : F \in \mathcal{F}\}$, and refer to this operation as *twisting* on X. We refer to set–systems equivalent under twisting as *equivalent* set–systems. It is easy to see that $(V, \mathcal{F})$ is a delta–matroid if and only if $M\Delta X = (V, \mathcal{F}\Delta X)$ is a delta–matroid. Now define $\mathcal{F} - X$ to be $\{F \subseteq V \setminus X : F \in \mathcal{F}\}$; we refer to this as *deleting* X. If $(V, \mathcal{F})$ is a delta–matroid then $M - X = (V \setminus X, \mathcal{F} - X)$ is also a delta–matroid. Given $X, Y \subseteq V$, we call the set–system $(M\Delta X) - Y$ a *minor* of M; if $X \subseteq Y$ then we call the minor *rigid*. Note that any minor of M is equivalent to a rigid minor of M.

Let $M = (V, \mathcal{F})$ and $M' = (V', \mathcal{F}')$ be set–systems such that V and V' are disjoint. The *direct sum* (or 1–*sum*) of M and M' is the set–system $(V_1 \cup V_2, \{F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\})$. The direct sum of two delta–matroids is clearly a delta–matroid. We call a proper partition V_1, V_2 of V a *separation* (or 1–*separation*) if M is the direct sum of $M - V_1$ and $M - V_2$. If V_1, V_2 is a separation of M, then it is a separation of every set–system equivalent to M. A set–system with a separation is called *separable* (otherwise *nonseparable*).

Let $(V, \mathcal{F})$ be a matroid. We have already noted that matroids have equicardinal feasible sets, so matroids are not in general preserved under twisting, and hence not closed under the taking of minors. In fact, if $(V, \mathcal{F})$ is nonseparable then the only other matroid equivalent to $(V, \mathcal{F})$ is its dual $(V, \mathcal{F}\Delta V)$. Matroids are however closed under taking rigid minors. (The usual definition of a minor of a matroid (see Oxley [56]) is what we have called a rigid minor.) Actually, if $(V', \mathcal{F}')$ is a nonseparable matroid that is a minor of $(V, \mathcal{F})$, then either $(V', \mathcal{F}')$ or its dual is a rigid minor of $(V, \mathcal{F})$. Therefore our definition of a minor is convenient when studying dual closed families of matroids.

Optimization

Let $(V, \mathcal{F})$ be a delta–matroid, and let $c \in \mathbf{Q}^V$. We wish to find a maximum weight feasible set (that is, a feasible set F maximizing $c(F) = \sum_{v \in F} c_v$). We begin by transforming the problem to one with nonnegative weights. Define V^- to be $\{v \in V : c_v < 0\}$, and define a new cost function c' such that $c'_v = |c_v|$ for all $v \in V$. Then for $X \subseteq V$,

$$\begin{aligned} c(X) &= c'(X \setminus V^-) - c'(X \cap V^-) \\ &= c'(X \setminus V^-) + c'(V^- \setminus X) - c'(V^- \setminus X) - c'(X \cap V^-) \\ &= c'(X\Delta V^-) - c'(V^-). \end{aligned}$$

Hence, F is a maximum weight feasible set of $(V, \mathcal{F})$, with respect to c, if and only if $F\Delta V^-$ is a maximum weight feasible set of $(V, \mathcal{F}\Delta V^-)$, with respect to the nonnegative weights c'. Since c' is nonnegative, to find a maximum weight feasible set we need only optimize over the maximal cardinality feasible sets. This problem reduces to optimizing in matroids, by the following well–known result.

Theorem 3.1 *Let $(V, \mathcal{F})$ be a delta–matroid, and let $\hat{\mathcal{F}}$ consist of the maximal sets in $\mathcal{F}$. Then $(V, \hat{\mathcal{F}})$ is a matroid.*

Proof We begin by proving that the maximal feasible sets are equicardinal. Suppose not; then there exists a maximal feasible set X and a feasible set Y such that $|X| < |Y|$, and,

Input: A delta–matroid $(V, \mathcal{F})$, and $c \in \mathbf{Q}^V$.
Output: A maximum weight feasible set F.
Begin
  Order the elements of V, $\{v_1, \ldots, v_n\}$, such that $|c_{v_1}| \geq |c_{v_2}| \geq \ldots \geq |c_{v_n}|$.
  $F \leftarrow \{v \in V : c_v < 0\}$
  for $i = 1$ to n
    $S_i \leftarrow \{v_{i+1}, \ldots, v_n\}$
    if there exists $F' \in \mathcal{F}$ such that $(F\Delta\{v_i\}) \setminus S_i \subseteq F' \subseteq (F\Delta\{v_i\}) \cup S_i$
      $F \leftarrow F\Delta\{v_i\}$
End.

Figure 3.1: Greedy Algorithm

suppose, $X\Delta Y$ is minimum over all such Y. Since X is maximal, there exists $x \in X \setminus Y$. By the symmetric exchange axiom, there exists $y \in X\Delta Y$ such that $Y\Delta\{x, y\} \in \mathcal{F}$. However, $|Y\Delta\{x, y\}| \geq |Y| > |X|$, and $|X\Delta(Y\Delta\{x, y\})| < |X\Delta Y|$, which contradicts our choice of Y. Therefore the maximal sets are equicardinal.

Let X, Y be maximal feasible sets, and let $x \in Y \setminus X$. By the symmetric exchange axiom, there exists $y \in X\Delta Y$ such that $X\Delta\{x, y\} \in \mathcal{F}$. By the maximality of X, $y \in X \setminus Y$. Now, $|X\Delta\{x, y\}| = |X|$, so $X\Delta\{x, y\}$ is maximal. Therefore, the maximal feasible sets form a matroid. □

Therefore, by the greedy algorithm for optimizing over matroids, we can optimize over delta–matroids. The algorithm is given by Figure 3.1. It appears in Bouchet [4] and Chandrasekaran and Kabadi [17], but many of the ideas are contained in an earlier paper of Dunstan and Welsh [28].

Another way that one might find a minimum weight feasible set is to simply scan the list $\mathcal{F}$. However the number of sets in $\mathcal{F}$ may be exponential in $|V|$, and, for a typical application, the feasible sets may be defined implicitly. (For example, there can be an exponential number of matchable sets of a graph, but they are implicitly captured by the graph.) Therefore, we assume that a delta–matroid is "given" to us by an oracle. Given disjoint subsets X, Y of V, the *separation oracle*, $Sep_{\mathcal{F}}(X, Y)$, of a delta–matroid $(V, \mathcal{F})$, answers the question: *"Does there exist $F \in \mathcal{F}$ such that $X \subseteq F$ and $Y \cap F = \emptyset$?"*. The separation oracle is a natural oracle for the greedy algorithm.

A delta–matroid algorithm is said to be *polynomial* if it is a polynomial algorithm under the assumption that the delta–matroid is represented in space bounded above by a polynomial in $|V|$, and the separation oracle runs in time bounded above by a polynomial in $|V|$ (see Garey and Johnson [37]). The greedy algorithm is an example of a polynomial delta–matroid algorithm.

Let $M = (V, \mathcal{F})$ be a delta–matroid. Given disjoint sets X, Y of V, we define $\rho(X, Y) \in \mathbf{R} \cup \{\infty\}$ by

$$\rho(X, Y) = \max_{F \in \mathcal{F}} |X \cap F| + |Y \setminus F|.$$

Note that $\rho(X,Y)$ can be computed efficiently by the greedy algorithm. Conversely, $Sep_{\mathcal{F}}(X,Y)$ can be easily determined by ρ; indeed $Sep_{\mathcal{F}}(X,Y)$ returns a positive answer if and only if $\rho(X,Y) = |X| + |Y|$. Therefore, in some sense the separation oracle is equivalent to ρ. Cunningham, in Bouchet [9], described the convex hull of a delta–matroid using ρ. For a subset F of V, the *incidence vector* of F is the vector $x \in \mathbf{R}^V$ such that $x_v = 1$ if $v \in F$, and $x_v = 0$ otherwise. Let conv($\mathcal{F}$) denote the convex hull of the incidence vectors of the feasible sets of $\mathcal{F}$. For $x \in \mathbf{R}^V$ and $X \subseteq V$, we denote by $x(X)$ the sum $\sum(x_v : v \in X)$.

Theorem 3.2 *Let $M = (V, \mathcal{F})$ be a delta–matroid. Then conv($\mathcal{F}$) is described by the the following inequalities*

$$x(X) - x(Y) \leq \rho(X,Y) - |Y| \qquad \text{(for all disjoint subsets } X, Y \text{ of } V\text{)};$$

furthermore, this system of inequalities is totally dual integral. □

Negative results

In this section we show that each of the following problems is intractable, that is, there exists no polynomial algorithm that solves the problem.

P_1 Given a delta–matroid M, is M separable?

P_2 Given a delta–matroid M, is M even?

P_3 Given a delta–matroid M and an integer k, does there exist a feasible set of size k?

P_4 Given a delta–matroid $(V, \mathcal{F})$, is there a partition of V into feasible sets?

P_5 Given a delta–matroid $(V, \mathcal{F})$, is there a partition of V into two feasible sets?

We define $\mathcal{P}(V)$ to be the set of all subsets of V, and we denote by $\mathcal{P}^0(V)$ the set of all sets in $\mathcal{P}(V)$ having even cardinality.

Lemma 3.3 *Let $(V, \mathcal{F})$ be a set–system such that $\mathcal{P}^0(V) \subseteq \mathcal{F}$. Then $(V, \mathcal{F})$ is a delta–matroid.*

Proof Suppose $X, Y \in \mathcal{F}$ and $x \in X \Delta Y$. If $|X|$ is odd, or $|X \Delta Y| = 1$, then $X \Delta \{x\} \in \mathcal{F}$, so the symmetric exchange axiom is satisfied. Then we may assume that $|X|$ is even and $|X \Delta Y| \geq 2$. So there exists $y \in (X \Delta Y) \setminus \{x\}$, and $X \Delta \{x, y\} \in \mathcal{F}$, and again the symmetric exchange axiom is satisfied. □

Theorem 3.4 *The problems $P_1, \ldots, P_5$ are intractable.*

Proof Let V be a set of odd cardinality, and let $M = (V, \mathcal{F})$ be a set–system such that $\mathcal{P}^0(V) \subseteq \mathcal{F} \subseteq \mathcal{P}(V)$. By Lemma 3.3, M is a delta–matroid. For $X, Y \subseteq V$, if $X \neq Y$ then $Sep_{\mathcal{F}}(X,Y)$ returns "yes". $Sep_{\mathcal{F}}(X,X)$ indicates whether $X \in \mathcal{F}$.

M is even if and only if $\mathcal{F} = \mathcal{P}_0(V)$. To verify this we need to check that every set of odd cardinality is not feasible; this requires using the separation oracle an exponential number of times. Therefore there is no polynomial algorithm for P_2. A polynomial algorithm for P_3 would imply the existence of a polynomial algorithm for P_2; hence P_3 is also intractable.

Since $|V|$ is odd, any partition of V contains a part of odd cardinality, so if M is even there exists no partition of V into feasible sets. If M is not even, then there exists a feasible set X of odd cardinality, and $X, V \setminus X$ is a partition of V into feasible sets. Hence, there is a partition of V into (two) feasible sets if and only if M is even. It follows that P_4 and P_5 are both intractable.

Now suppose that $|V| > 1$ and $|\mathcal{F}| \geq |\mathcal{P}(V)| - 1$.

Claim *M is separable if and only if $|\mathcal{F}| = |\mathcal{P}(V)|$.*

If $|\mathcal{F}| = |\mathcal{P}(V)|$ then it is clear that M is separable. Suppose then that $|\mathcal{F}| = |\mathcal{P}(V)| - 1$. Twist so that $\mathcal{P}(V) \setminus \mathcal{F} = \{V\}$. For any proper partition V^1, V^2 of V, V^1 and V^2 are feasible, but V is not feasible. Therefore the twisted delta–matroid is not separable, and hence M is not separable. This proves the claim.

Deciding whether $|\mathcal{F}| = |\mathcal{P}(V)|$ is intractable. Therefore, P_1 is intractable. □

For even delta–matroids there are elementary algorithms for solving P_1 and P_3; however the status of P_4 and P_5 is open. For matroids, there exist polynomial algorithms for $P_1, \ldots, P_5$. In fact, P_5 is a special case of the partition problem, which was solved for matroids by Edmonds [30].

Partition problem Given set–systems $M_1 = (V, \mathcal{F}_1)$ and $M_2 = (V, \mathcal{F}_2)$, is there a partition F_1, F_2 of V such that $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$?

The partition problem is intractable for even delta–matroids (see Bouchet [9]). Indeed, given a graph $G = (V, E)$ and a matroid M, let $M_1 = (V, \mathcal{F}_1)$ be the matching delta–matroid of G, and let $M_2 = (V, \mathcal{F}_2)$ be the dual of M. There is a partition F_1, F_2 of V such that $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ if and only if there is a matchable set of G that is a basis of M, Lovasz [49, 50] has shown that the latter problem is intractable. Hence the partition problem is intractable for even delta–matroids.

Even delta–matroids

We have seen that when a delta–matroid has many feasible sets, there is not much structure implied by the symmetric exchange axiom. For even delta–matroids the situation is more promising; by looking at a feasible set, and the feasible sets close to it, we can say quite a bit about the structure of the delta–matroid. Let $(V, \mathcal{F})$ be an even delta–matroid, and let F be a feasible set. Define a graph $G_F = (V, E_F)$, where $E_F = \{vw : F\Delta\{v, w\} \in \mathcal{F}\}$; G_F is called the *fundamental graph* of F. For a graph G and a vertex v of G we denote by $N_G(v)$ the set of neighbours of v in G, that is, the vertices of G that are adjacent to v.

Lemma 3.5 *Let $(V, \mathcal{F})$ be an even delta–matroid, let $F \in \mathcal{F}$, and let $vw \in E_F$. Then, for $x, y \in V \setminus \{v, w\}$,*

(1) *$vx \in E_F$ if and only if $wx \in E_{F\Delta\{v,w\}}$, and*

(2) *if $x, y \notin N_{G_F}(v)$, then $xy \in E_F$ if and only if $xy \in E_{F\Delta\{v,w\}}$.*

Proof (1) is immediate. Suppose $x, y \notin N_{G_F}(v)$, and $xy \in E_{F\Delta\{v,w\}}$. Then $F, F\Delta\{v,w,x,y\} \in \mathcal{F}$, and $w \in F\Delta(F\Delta\{v,w,x,y\})$. Then, by the symmetric exchange axiom, there exists $z \in \{v,w,x,y\}$ such that $F\Delta\{v,w,x,y\}\Delta\{w,z\} \in \mathcal{F}$. Since $(V,\mathcal{F})$ is even, $z \neq w$. Since $vx, vy \notin E_F$, $z \neq y, x$. Hence $z = v$, and $xy \in E_F$, as required. By symmetry, if $xy \in E_F$ then $xy \in E_{F\Delta\{v,w\}}$. So we have proved (2). □

Let $(V,\mathcal{F})$ be an even delta–matroid, and let $F \in \mathcal{F}$. The following observations come as easy corollaries of Lemma 3.5.

(i) If G_F is bipartite with bipartition V^1, V^2, then, for every $F' \in \mathcal{F}$, $G_{F'}$ is bipartite with bipartition $V^1\Delta(F\Delta F'), V^2\Delta(F\Delta F')$, and

(ii) for $X \subseteq V$, if $G_F[X]$ is a component of G_F then, for every $F' \in \mathcal{F}$, $G_{F'}[X]$ is a component of $G_{F'}$.

Theorem 3.6 (Bouchet [9]) *Let F be a feasible set of an even delta–matroid. Then $(V,\mathcal{F})$ is a twisted matroid if and only if G_F is bipartite.*

Proof (Cunningham [20]) Suppose that G_F is bipartite, and let V^1, V^2 be the bipartition. Define $\mathcal{B} = \mathcal{F}\Delta(V^1\Delta F)$. Consider any feasible set $F' \in \mathcal{F}$ and $vw \in E_{F'}$. By observation (i), $G_{F'}$ is bipartite with bipartition $V^1\Delta(F\Delta F'), V^2\Delta(F\Delta F')$, so $|F'\Delta(F\Delta V^1)| = |F'\Delta\{v,w\}\Delta(F\Delta V^1)|$. Therefore all sets in $\mathcal{B}$ are equicardinal, and, hence, $(V,\mathcal{B})$ is a matroid.

The proof of the converse is elementary. □

Theorem 3.6, gives a polynomial algorithm for deciding whether an even delta–matroid is a twisted matroid. We will see that this algorithm can be easily extended to test whether an arbitrary delta–matroid is a twisted matroid. The following theorem shows that there exists a polynomial algorithm that decides whether an even delta–matroid is separable.

Theorem 3.7 (Bouchet [10]) *Let F be a feasible set of an even delta–matroid. For a proper partition V^1, V^2 of V, V^1, V^2 is a separation of $(V,\mathcal{F})$ if and only if $G_F[V^1]$ is a component of G_F.*

Proof (Cunningham [20]) Suppose that $X, V\setminus X$ is a separation of $(V,\mathcal{F})$. Then, since $\mathcal{F}$ is even, for every pair of feasible sets F_1, F_2, $|F_1 \cap X| \equiv |F_2 \cap X|$ modulo 2. Therefore, $G_F[X]$ is a component of G_F.

Now consider the converse. Let X^1, X^2 be a proper partition of V such that $G_F[X^1]$ is a component of G_F, but X^1, X^2 is not a separation of $\mathcal{F}$. Then there exist feasible sets F_1, F_2 such that $(F_1 \cap V^1) \cup (F_2 \cap V^2)$ is not feasible. Suppose that we have chosen such F_1, F_2 with $(F_1\Delta F_2) \cap V^1$ as small as possible. Note that $(F_1\Delta F_2) \cap V_1 \neq \emptyset$, so there exists $x \in (F_1\Delta F_2) \cap V_1$. Then, by the symmetric exchange axiom, there exists $y \in F_1\Delta F_2$ such that $F_2\Delta\{x,y\} \in \mathcal{F}$. By observation (ii), $y \in V_1$. However $|(F_1\Delta(F_2\Delta\{x,y\})) \cap V^1| < |(F_1\Delta F_2) \cap V^1|$, so $(F_1 \cap V^1) \cup ((F_2\Delta\{x,y\}) \cap V^2) = (F_1 \cap V^1) \cup (F_2 \cap V^2)$ is feasible, which is a contradiction. □

Matching and even delta–matroids

Bruäldi [15] proved that matroids satisfy the following property.

Matching property For all $F_1, F_2 \in \mathcal{F}$, $G_{F_1}[F_1 \Delta F_2]$ has a perfect matching.

The matching property implies that, for any feasible set F, $\mathcal{F}\Delta F$ is a subset of the set of matchable sets of G_F.

Theorem 3.8 (Bouchet [9]) *Every even delta–matroid has the matching property.*

Before proving the theorem, we need to state a key lemma. A *hypomatchable* graph is a graph $G = (V, E)$ with the property that, for each $v \in V$, $G - v$ has a perfect matching.

Lemma 3.9 (Gallai [36, 50]) *Let G be a connected graph with the property that, for every vertex v, there is a maximum matching M of G that avoids v (that is, v is not incident with an edge of M). Then G is hypomatchable.* □

Proof of Theorem 3.8. Let $(V, \mathcal{F})$ be an even delta–matroid, and suppose that $(V, \mathcal{F})$ does not have the matching property. Choose feasible sets F_1, F_2 such that

(1) $G_{F_1}[F_1 \Delta F_2]$ has no perfect matching, and

(2) $|F_1 \Delta F_2|$ is minimum with respect to (1).

Suppose that $G_{F_1}[F_1 \Delta F_2]$ is not connected, and let $G_{F_1}[X]$ be a component of $G_{F_1}[F_1 \Delta F_2]$ that has no perfect matching. Consider the minor $\mathcal{F}' = \mathcal{F}\Delta F_1 - (V \setminus (F_1 \Delta F_2))$ of $\mathcal{F}$. By Theorem 3.7, $X, (F_1 \Delta F_2) \setminus X$ is a separation of $\mathcal{F}'$; furthermore $\emptyset, F_1 \Delta F_2 \in \mathcal{F}'$, so $X = ((F_1 \Delta F_2) \cap X) \cup (\emptyset \setminus X) \in \mathcal{F}'$. Then, $X \Delta F_1 \in \mathcal{F}$. However $|X \Delta F_1| < |F_1 \Delta F_2|$, which contradicts (2). Hence $G_{F_1}[F_1 \Delta F_2]$ is connected.

For all $x \in F_1 \Delta F_2$, there exists $y \in F_1 \Delta F_2$ such that $F_2 \Delta \{x, y\} \in \mathcal{F}$. However $|F_1 \Delta (F_2 \Delta \{x, y\})| < |F_1 \Delta F_2|$, so, by (2), there exists a perfect matching M of $G_{F_1}[F_1 \Delta F_2 \Delta \{x, y\}]$. By (1), M is a maximum matching of $G_{F_1}[F_1 \Delta F_2]$ that avoids x. Then, by Lemma 3.9, $G_{F_1}[F_1 \Delta F_2]$ is hypomatchable, so $|F_1 \Delta F_2|$ is odd, a contradiction.□

The previous theorem has a number of interesting applications, which we consider in the remainder of the chapter. In fact, Bouchet originally derived Theorems 3.7 and 3.6 from Theorem 3.8. We state the first corollary without proof; it is a partial converse of Theorem 3.8, that was proved for matroids by Krogdahl [47].

Theorem 3.10 (Bouchet [9]) *Let F be a feasible set of an even delta–matroid $(V, \mathcal{F})$. For $X \subseteq V$, if $G_F[X]$ has a unique perfect matching then $F \Delta X \in \mathcal{F}$.* □

We extend the definition of a fundamental graph to all delta–matroids. Let $(V, \mathcal{F})$ be a delta–matroid. For $F \in \mathcal{F}$, define $G_F = (V, E_F)$ such that $E_F = \{vw : v, w \in V, F\Delta\{v, w\} \in \mathcal{F}\}$. Note that if $(V, \mathcal{F})$ is not even, then G_F may have loops. We now extend Theorem 3.6.

Theorem 3.11 *Let $M = (V, \mathcal{F})$ be a delta–matroid. Then, for $F \in \mathcal{F}$, G_F is bipartite if and only if M is a twisted matroid.*

Proof Suppose that G_F is bipartite. We assume that M is not even, since otherwise the result follows by Theorem 3.6. Let F' be a feasible set such that $|F \Delta F'|$ is odd and as small as possible with this property. If $|F \Delta F'| = 1$ then G_F has a loop, so it is not bipartite. Then assume that $|F \Delta F'| \geq 3$.

For every $x \in F \Delta F'$, there exists $y \in F \Delta F'$ such that $F' \Delta \{x, y\} \in \mathcal{F}$. However, the minor $\mathcal{F} \Delta F - (V \setminus (F \Delta F' \setminus \{x, y\}))$ is even, so, by Theorem 3.8, $G_F[F \Delta F' \setminus \{x, y\}]$ has a perfect matching M. M is a maximum matching of $G_F[F \Delta F']$. Hence, by Lemma 3.9, G_F is hypomatchable, which contradicts that G_F is bipartite.

The converse is implied by Theorem 3.6. □

Brualdi [15] proved that matroids satisfy the following axiom:

Simultaneous exchange axiom: For $X, Y \in \mathcal{F}$, and $x \in X \Delta Y$ there exists $y \in X \Delta Y$ such that $X \Delta \{x, y\}, Y \Delta \{x, y\} \in \mathcal{F}$.

Duchamp generalized Brualdi's result to even delta–matroids; we obtain the result as a corollary of Theorem 3.8. Duchamp's proof is also short, although it requires the introduction of symmetric matroids [4].

Theorem 3.12 (Duchamp [27]) *Even delta–matroids satisfy the simultaneous exchange axiom.*

Proof Let $(V, \mathcal{F})$ be an even delta–matroid. Suppose that $(V, \mathcal{F})$ does not satisfy the simultaneous exchange axiom. Let $F_1, F_2 \in \mathcal{F}$ and $x \in F_1 \Delta F_2$ satisfy

(1) $N_{G_{F_1}}(x) \cap N_{G_{F_2}}(x) \cap (F_1 \Delta F_2)$ is empty, and

(2) $|F_1 \Delta F_2|$ is minimum with respect to (1).

Define $S_i = N_{G_{F_i}}(x) \cap \{F_1 \Delta F_2\}$ for $i = 1, 2$.

Claim *For $i = 1, 2$, if $v, w \in F_1 \Delta F_2$, and $vw \in E_{F_i}$ then $\{v, w\} \cap S_i$ is not empty.*

Suppose the claim is false (that is, $\{v, w\} \cap S_i = \emptyset$), and assume, for convenience, that $i = 1$. Then, by Lemma 3.5, $N_{G_{F_1 \Delta \{v,w\}}}(x) = N_{G_{F_1}}(x)$. Therefore

$$N_{G_{F_1 \Delta \{v,w\}}}(x) \cap N_{G_{F_2}}(x) \cap (F_1 \Delta F_2 \Delta \{v, w\}) = N_{G_{F_1}}(x) \cap N_{G_{F_2}}(x) \cap (F_1 \Delta F_2 \setminus \{v, w\}) = \emptyset.$$

However $|F_1 \Delta \{v, w\} \Delta F_2| < |F_1 \Delta F_2|$, so we have a contradiction to (2). This proves the claim.

By Theorem 3.8, $G_{F_1}[F_1 \Delta F_2]$ has a perfect matching M. However, by the claim, for $v \in S_2 \cup \{x\}$, $N_{G_{F_1}}(v) \cap (F_1 \Delta F_2) \subseteq S_1$, so, $|S_2| + 1 \leq |S_1|$. By similar reasoning, $|S_1| + 1 \leq |S_2|$, which is an absurdity. □

Diameter Problem

Let $(V, \mathcal{F})$ be a delta–matroid. For subsets X_1, X_2 of V, we call $|X_1 \Delta X_2|$ the *distance* between X_1 and X_2. The *diameter* of $\mathcal{F}$, denoted $diam(\mathcal{F})$, is the maximum distance between any two feasible sets. We define $\mathcal{F}^* = \{F_1 \Delta F_2 : F_1, F_2 \in \mathcal{F}\}$; Duchamp [27] proved that $(V, \mathcal{F}^*)$ is a delta matroid. Note that $V \in \mathcal{F}^*$ if and only if there exists a partition F_1, F_2 of V such that $F_1, F_2 \in \mathcal{F}$. We have seen that this problem is intractable, and hence the problem of determining the diameter of a delta–matroid is intractable.

The diameter of a matroid can be computed by the matroid partition algorithm. Let $(V, \mathcal{B})$ be a matroid, the matroid partition problem is to find disjoint independent sets I_1, I_2 such that $|I_1 \cup I_2|$ is maximum, or equivalently, to find bases B_1, B_2 such that $|B_1 \cup B_2|$ is maximum. Since all bases are equicardinal, maximizing $|B_1 \cup B_2|$ is equivalent to maximizing $|B_1 \Delta B_2|$. A min–max formula, and a polynomial algorithm, for the matroid partition problem were given by Edmonds [30].

There is some hope that the diameter problem is solvable for even delta–matroids. Suppose that $(V, \mathcal{F})$ is an even delta–matroid, and let F_1, F_2 be feasible sets. For each $x \in F_1 \Delta F_2$, by the simultaneous exchange axiom, there exists $y \in F_1 \Delta F_2$ such that $F_1 \Delta \{x, y\}, F_2 \Delta \{x, y\} \in \mathcal{F}$. However $F_1 \Delta F_2 = (F_1 \Delta \{x, y\}) \Delta (F_2 \Delta \{x, y\})$. Therefore, for every set $F \in \mathcal{F}^*$ there are a number of pairs F_1, F_2 of feasible sets such that $F = F_1 \Delta F_2$; in particular the diameter is attained by a number of feasible pairs. We present an unpublished conjecture of Bouchet concerning the diameter of an even–delta matroid, and give a new algorithm for computing the diameter of a matroid.

Lemma 3.13 *If F is a feasible set of an even delta–matroid $(V, \mathcal{F})$ then $diam(\mathcal{F}) \leq 2\nu(G_F)$, where $\nu(G_F)$ is the size of a maximum matching in G_F.*

Proof Since $(\mathcal{F} \Delta F)^* = \mathcal{F}^*$, we may assume that $F = \emptyset$. Let F_1, F_2 be feasible sets such that $|F_1 \Delta F_2| = diam(\mathcal{F})$. By Theorem 3.8, $G_F(F_i \Delta F)$ has a perfect matching M_i, for $i = 1, 2$. Let M_i' be the edges of M_i having both ends in $F_1 \Delta F_2$, and let M_i'' be the set of edges in M_i having an end in $F_1 \Delta F_2$ and the other end in $F_1 \cap F_2$. We may assume, by possibly interchanging F_1 and F_2, that $|M_1''| \geq |M_2''|$. $M_1' \cup M_2' \cup M_1''$ is a matching of G_F, with at least $|F_1 \Delta F_2|/2$ edges. □

Conjecture *Let $(V, \mathcal{F})$ be an even delta–matroid. Then $diam(\mathcal{F}) = 2 \min_{F \in \mathcal{F}} \nu(G_F)$.*

An algorithm for computing the diameter of a matroid

Let $(V, \mathcal{B})$ be a matroid, and let B_1 and B_2 be bases such that $B_1 \Delta B_2$ is not a maximum matchable set of G_{B_1}. We describe an algorithm that finds distinct sets S_1, S_2 such that

(i) $G_{B_i}[S_i]$ has a unique perfect matching for $i = 1, 2$, and

(ii) $(S_1 \Delta S_2) \cap (B_1 \Delta B_2) = \emptyset$.

Suppose that we have distinct sets S_1, S_2 satisfying (i) and (ii). Define $B_i' = B_i \Delta S_i$ for $i = 1, 2$. By Theorem 3.10, B_i' is feasible, and, by (ii), $B_1' \Delta B_2' = (B_1 \Delta B_2) \cup (S_1 \Delta S_2)$. Hence $|B_1' \Delta B_2'| > |B_1 \Delta B_2|$. We can iterate the above procedure until we have bases

B_1', B_2' such that $B_1'\Delta B_2'$ is a maximum matchable set of $G_{B_1'}$; then, by Lemma 3.13, $diam(\mathcal{B}) = |B_1'\Delta B_2'|$.

By Theorem 3.8, $B_1\Delta B_2$ is a matchable set of G_{B_1}. However, by assumption, $B_1\Delta B_2$ is not a maximum matchable set of G_{B_1}, so there exists $x, y \in V \setminus (B_1\Delta B_2)$ such that $G_{B_1}[B_1\Delta B_2 \cup \{x,y\}]$ has a perfect matching M_1. By Theorem 3.8, $G_{B_2}[B_1\Delta B_2]$ has a perfect matching M_2. Let $G = (V, E_{B_1} \cup E_{B_2})$. M_1 and M_2 are matchings of G. By considering the edges in $M_1\Delta M_2$, we find an (x,y)–path $P = (x = x_1, y_1, x_2, \ldots, x_k, y_k = y)$ in G such that $x_iy_i \in E_{B_1}$, for $i = 1, \ldots, k$, and $y_ix_{i+1} \in E_{B_2}$, for $i = 1, \ldots, k-1$. By possibly shortcutting, we may assume that P is minimal (that is, there are no edges $x_iy_j \in B_1$ where $1 \leq i < j \leq k$, or $y_ix_j \in B_2$ where $1 \leq i < j-1 \leq k-1$).

Let $S_1 = \{x_1, y_1, x_2, \ldots, x_k, y_k\}$, and $S_2 = \{y_1, x_2, y_2, \ldots, y_{k-1}, x_k\}$. Since $(V, \mathcal{B})$ is a matroid, G_{B_i} is bipartite with bipartition $B_i, V \setminus B_i$ for $i = 1, 2$. Therefore

(1) $G[B_1\Delta B_2]$ is bipartite with bipartition $B_1 \setminus B_2, B_2 \setminus B_1$, and

(2) for $v \in V \setminus (B_1\Delta B_2)$, either $N_{G_1}(v) \cap (B_2 \setminus B_1) = \emptyset$, or $N_{G_1}(v) \cap (B_1 \setminus B_2) = \emptyset$.

By (1) and (2), $G_1[S_1]$ is bipartite with bipartition $\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_k\}$; furthermore, by the minimality of P, $N_{G_1}(x_k) \cap \{y_{i+1}, \ldots, y_k\} = \emptyset$, for $i = 1, \ldots, k$. Therefore, $\{x_iy_i : i = 1, \ldots, k\}$ is a unique perfect matching in $G_{B_1}[S_1]$. Similarly $\{y_ix_{i+1} : i = 1, \ldots, k-1\}$ is a unique perfect matching in G_{B_2}. Therefore S_1, S_2 satisfy conditions (i),(ii), as required.

Chapter 4

Representable delta–matroids

Let A be a V by V matrix over a field $\mathbf{F}$. Recall that $A[X]$ denotes the principal submatrix of A indexed by $X \subseteq V$. Define $M(A) = (V, \mathcal{F}_A)$, where $\mathcal{F}_A = \{S \subseteq V : A[S]$ is nonsingular over $\mathbf{F}\}$. (By convention, we assume that the empty matrix has determinant one.) The following proof requires the pivoting operation introduced in Chapter 2.

Theorem 4.1 (Bouchet [8]) *Let A be a symmetric or skew–symmetric V by V matrix over a field $\mathbf{F}$. Then $M(A)$ is a delta–matroid.*

Proof Suppose $X, Y \in \mathcal{F}_A$ and $x \in X\Delta Y$ such that for all $y \in X\Delta Y$, $X\Delta\{x,y\} \notin \mathcal{F}_A$. Denote by $A' = (a_{ij})$ the matrix $A * X$. By Theorem 2.7, $A'[S]$ is nonsingular if and only if $S\Delta X \in \mathcal{F}_A$. By assumption $X\Delta\{x\} \notin \mathcal{F}_A$, so $a_{xx} = 0$. However, $A'[X\Delta Y]$ is nonsingular, so there exists $y \in X\Delta Y$ such that $a_{xy} \neq 0$. Then, since $a_{xx} = 0$, $A'[\{x,y\}]$ is nonsingular. Therefore, $X\Delta\{x,y\} \in \mathcal{F}_A$, which is a contradiction. □

Delta–matroids arising from symmetric and skew–symmetric matrices are called *representable* (see [8]). For a field of characteristic 2, we use the convention that a skew–symmetric matrix is a symmetric matrix with a zero diagonal; this ensures that all delta–matroids representable by skew–symmetric matrices are even.

We have already seen one interesting example of a representable delta–matroid. Let $(V, \mathcal{M}_G)$ be the matching delta–matroid for a graph $G = (V, E)$. Let $X = \{x_e : e \in E\}$ be a set of algebraically independent indeterminates. Define a skew–symmetric V by V matrix $A = (a_{ij})$, where $a_{ij} = \pm x_{ij}$ if $ij \in E$, and $a_{ij} = 0$ otherwise. Tutte [67] showed that $\mathcal{F}_A = \mathcal{M}_G$. It is not hard to show that there exists $X \in \mathbf{R}^E$, such that $(V, \mathcal{M}_G)$ is representable over $\mathbf{R}$.

We call a delta–matroid *normal* if the empty set is feasible; thus, every representable delta–matroid is normal. Deletion and twisting are both easy to define for representable delta–matroids, however if we twist a representable delta–matroid by a nonfeasible set, then the result cannot be representable. For $X \subseteq V$, the delta–matroid obtained by deleting $V \setminus X$ is $M(A[X])$, and, for $X \in \mathcal{F}_A$, the delta–matroid obtained by twisting X is $M(A * X)$. Therefore if M' is a normal minor of $M(A)$, then M' is representable.

Recall that if A is skew–symmetric then so is $A * X$. Though symmetry is not preserved under pivoting. However, if A is symmetric, then we get a symmetric matrix from $A * X$ by multiplying the columns indexed by X by -1. We redefine the pivoting operation for a symmetric matrix accordingly; this does not alter the validity of Theorem 2.7.

Also, recall that a nonempty set X is called an *elementary set* of A if $A[X]$ is nonsingular but there exists no proper subset X' of X such that $A[X']$ is nonsingular. If A is symmetric or skew–symmetric then all elementary sets have size one or two. We define $V_A^1 = \{v \in V : a_{vv} \neq 0\}$, and $V_A^2 = \{vw : v, w \notin V_A^1, a_{vw} \neq 0\}$. We denote by $A * v$ and $A * vw$ the elementary pivots $A * \{v\}$ and $A * \{v, w\}$ respectively.

Representable matroids

With the exception of matroids of rank zero, matroids are not normal; however, the representable delta–matroids generalize the normal twisted representable matroids. Let $M = (V, \mathcal{B})$ be a matroid representable over a field $\mathbf{F}$, and let B be a representation of M, that is, the columns of B are indexed by V and $F' \in \mathcal{B}$ if and only if F' indexes a basis of the column space of B. Note that the dependence between the columns of B is unaffected by performing elementary row operations and deleting zero rows of B. Therefore, for some $F \in \mathcal{B}$, we may assume that B has the form

$$\begin{array}{cc} & \begin{array}{cc} F & V \setminus F \end{array} \\ F & \begin{pmatrix} I & \quad B' \end{pmatrix} \end{array},$$

where I is the identity matrix. For any $F' \subseteq V$, such that $|F| = |F'|$, $B[F, F']$ is nonsingular if and only if $B'[F \setminus F', F' \setminus F]$ is nonsingular. Now define A to be

$$\begin{array}{cc} & \begin{array}{cc} F & V \setminus F \end{array} \\ \begin{array}{c} F \\ V \setminus F \end{array} & \begin{pmatrix} 0 & B' \\ -B'^T & 0 \end{pmatrix} \end{array}.$$

For $S \subseteq V$, $A[S]$ is nonsingular if and only if $|S \setminus F| = |F \setminus S|$ and $B'[F \setminus S, S \setminus F]$ is nonsingular. Hence, $A[S]$ is nonsingular if and only if $B[F, F \Delta S]$ is a basis of B. Thus $\mathcal{F}_A = \mathcal{B} \Delta F$, and every representable matroid is equivalent under twisting to a representable delta–matroid, as claimed.

Separation for delta–matroid polyhedra

Let $M = (V, \mathcal{F})$ be a delta–matroid, and let conv$(\mathcal{F})$ denote the convex hull of incidence vectors of feasible sets of M. Recall, from Chapter 3, that we have a description of conv$(\mathcal{F})$ by inequalities, and that we can optimize a linear function over conv$(\mathcal{F})$ using the greedy algorithm. Then, by certain results based on the ellipsoid algorithm for linear programming (see Grötschel, Lovász and Schrijver [41]), we can solve the *separation problem* in polynomial time, that is: *Given* $x^* \in \mathbf{R}^V$, *is* x^* *contained in conv*$(\mathcal{F})$*?*

It would be preferable to have a combinatorial algorithm for the separation problem. One special case in which such an algorithm exists is when M is a matroid; see Cunningham [23]. (The algorithm assumes the existence of an efficient subroutine for evaluating the rank function of the matroid.) As a consequence of his separation algorithm for matroids, Cunningham (personal communication) obtained a combinatorial separation algorithm for representable delta–matroids (or more precisely represented delta–matroids).

Suppose that M is represented by a symmetric or skew–symmetric V by V matrix A. We construct a copy $\tilde{V}$ of V, and for any subset X of V, we denote by $\tilde{X}$ the corresponding copy of X. Then we define a matrix B by

$$\begin{array}{cc} & \begin{array}{cc} V & \tilde{V} \end{array} \\ V & \begin{pmatrix} I & A \end{pmatrix}, \end{array}$$

where I denotes an identity matrix. We now define a matroid $M_1 = (V \cup \tilde{V}, \mathcal{B}_1)$, where $X \in \mathcal{B}_1$ if and only if the columns of B indexed by X form a basis of B. The following proposition is the key to the separation algorithm.

Proposition 4.2 *Given $x \in R^V$, let $\tilde{x}$ be the corresponding vector in $\mathbf{R}^{\tilde{V}}$. Now define $y \in \mathbf{R}^{V \cup \tilde{V}}$ such that $y = (1 - x, \tilde{x})$. Then x is in conv($\mathcal{F}$) if and only if y is in conv($\mathcal{B}_1$).*

From Proposition 4.2 it is clear how the separation problem for representable delta–matroids reduces to the separation problem for matroids. The separation algorithm for matroids requires that the rank function of M_1 can be efficiently evaluated. It is easily seen that, for subsets X, Y of V, the rank of $\tilde{X} \cup Y$ in M_1 is $\text{rk}(A[V \setminus Y, X]) + |Y|$. While we have efficient algorithms for computing the rank of a rational matrix, complications arise when A contains indeterminates, like for matching delta–matroids (we will see more on this in Chapter 8).

In order to prove Proposition 4.2 we require the following fundamental theorem of Edmonds [32].

Theorem 4.3 *If $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$ are matroids, then conv($\mathcal{B}_1 \cap \mathcal{B}_2$) = conv($\mathcal{B}_1$) $\cap$ conv($\mathcal{B}_2$).* □

Proof of Proposition 4.2. We first observe that, for a subset F of V, $F \in \mathcal{F}$ if and only if $\tilde{X} \cup (V \setminus F) \in \mathcal{B}_1$. Now suppose that $x \in \text{conv}(\mathcal{F})$, that is, there exists $\lambda \in \mathbf{R}^{\mathcal{F}}$ such that

$$\lambda \geq 0, \sum_{F \in \mathcal{F}} \lambda_F = 1 \text{ and } x = \sum_{F \in \mathcal{F}} \lambda_F \chi^F,$$

where χ^F denotes the incidence vector of $\mathcal{F}$. Then clearly

$$y = \sum_{F \in \mathcal{F}} \lambda_F \chi^{\tilde{F} \cup (V \setminus F)},$$

so $y \in \text{conv}(\mathcal{B}_1)$.

Now, for the converse, suppose that $y \in \text{conv}(\mathcal{B}_1)$. We define a partition matroid $M_2 = (V \cup \tilde{V}, \mathcal{B}_2)$, where $\mathcal{B}_2 = \{X \cup \tilde{X} : X \subseteq V\}$. By the structure of y, we have $y \in \text{conv}(\mathcal{B}_2)$. Therefore, by Theorem 4.3, $y \in \text{conv}(\mathcal{B}_1 \cap \mathcal{B}_2)$. So, there exists $\lambda \in \mathbf{R}^{\mathcal{B}_1 \cap \mathcal{B}_2}$, such that

$$\lambda \geq 0, \sum_{F \in \mathcal{B}_1 \cap \mathcal{B}_2} \lambda_F = 1 \text{ and } y = \sum_{F \in \mathcal{B}_1 \cap \mathcal{B}_2} \lambda_F \chi^F.$$

However,

$$\mathcal{B}_1 \cap \mathcal{B}_2 = \{\tilde{X} \cup (V \setminus X) : X \in \mathcal{F}\}.$$

For $X \in \mathcal{F}$, define μ_X to be $\lambda_{\tilde{X} \cup (V \setminus X)}$. Then

$$y = \sum_{X \in \mathcal{F}} \mu_X \chi^{\tilde{X} \cup (V \setminus X)},$$

and hence $x \in \mathrm{conv}(\mathcal{F})$. □

Even representable delta–matroids and 3–connectivity

Let A be a V by V symmetric or skew–symmetric matrix. We define the *support graph* of A to be $G(A) = (V, E_A)$, where $E_A = \{vw : v \neq w, a_{vw} \neq 0\}$. We refer to the elements of V_A^1 as *loop–vertices*, though they are not in fact distinguished by the support graph. We remark that, if there are no loop–vertices, and the support graph is bipartite, then $M(A)$ is a twisted matroid.

Let $M = (V, \mathcal{F})$ be a delta–matroid represented over a field $\mathbf{F}$ by a matrix $A = (a_{ij})$. If A is skew–symmetric then M is even. A partial converse also holds: if M is even, then M is representable over $\mathbf{F}$ by a skew–symmetric matrix. Indeed, suppose that A is symmetric and M is even. Since M is even, A has a zero diagonal. We assume that $G(A)$ is not bipartite, since otherwise we could make A skew–symmetric by multiplying some columns of A by -1. Let $x_1, x_2, \ldots, x_k, x_1$ be an odd circuit of $G(A)$. Then

$$\det(A[\{x_1, \ldots, x_k\}]) = \pm 2 a_{x_1x_2} a_{x_2x_3} \ldots a_{x_{k-1}x_k} a_{x_kx_1}.$$

However, since M is even, $\det(A[\{x_1, \ldots, x_k\}]) = 0$. Therefore $\mathbf{F}$ has characteristic 2, so A is skew–symmetric.

Pfaffians

Pfaffians are a powerful tool for studying skew–symmetric matrices. For example, for even representable delta–matroids, Theorems 3.8 and 3.10 follow easily from the definition of the pfaffian. We now review some basic results about pfaffians; we use the definition of Stembridge [64].

Let $A = (a_{ij})$ be a V by V skew–symmetric matrix, let $\mathcal{M}_A$ denote the set of perfect matchings of $G(A)$, and let $\prec$ be a linear order of V. A pair of edges u_1v_1, u_2v_2 of $G(A)$, where $u_1 \prec v_1$ and $u_2 \prec v_2$, is said to *cross* if $u_1 \prec u_2 \prec v_1 \prec v_2$ or $u_2 \prec u_1 \prec v_2 \prec v_1$. (If we place u_1, u_2, v_1, v_2 on a circle, according to the linear order, then u_1v_1 crosses u_2v_2 if and only if the chords u_1v_1 and u_2v_2 cross.) The *sign* of a perfect matching M of $G(A)$, denoted σ_M, is $(-1)^k$ where k is the number of pairs of crossing edges in M. The *pfaffian* of A, denoted $pf(A)$, is defined as follows:

$$pf(A) = \sum_{M \in \mathcal{M}_A} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv}. \tag{4.1}$$

Surprisingly $pf(A)$ is independent of the linear order; this is reflected by the fundamental identity $\det(A) = pf(A)^2$. Like determinants, pfaffians can be calculated by "row expansion" [39]:

$$pf(A) = \sum_{k=2}^{n}(-1)^{k+1}a_{v_1v_k}pf(A[V \setminus \{v_1, v_k\}]), \tag{4.2}$$

where $V = \{v_1, v_2, \ldots, v_n\}$ and $v_i \prec v_{i+1}$, for $i = 1, 2, \ldots, n-1$.

Connectivity

Let A_1 and A_2 be skew–symmetric matrices. Define $A' = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$. It is obvious that $M(A')$ is the 1–sum of $M(A_1)$ and $M(A_2)$. (This also holds when A_1 and A_2 are symmetric.) It is more interesting that we can describe the "2–sum" of $M(A_1)$ and $M(A_2)$.

Let $M_1 = (V_1, \mathcal{F}_1)$ and $M_2 = (V_2, \mathcal{F}_2)$ be set–systems. We define the *composition of* M_1 *and* M_2 to be the set–system $M = (V, \mathcal{F})$ where $V = V_1 \Delta V_2$ and $\mathcal{F} = (F_1 \Delta F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, F_1 \cap V_2 = F_2 \cap V_1\}$. Bouchet and Cunningham [13] proved that the composition of two delta–matroids is a delta–matroid. If V_1 and V_2 are disjoint, then the composition is just the 1–sum. If $|V_1 \cap V_2| = 1$, then we call M the *2–sum* of M_1 and M_2; see Bouchet [10]. If $|V_1 \setminus V_2|, |V_2 \setminus V_1| \geq 2$ and $|V_1 \cap V_2| \geq 1$, then the partition $V_1 \setminus V_2, V_2 \setminus V_1$ of V is called a *2–separation of* M. A set–system without 1– or 2–separations is *3–connected.*

Suppose that A_i is a V_i by V_i matrix, for $i = 1, 2$, where $V_1 \cap V_2 = \{v\}$. Define $V = V_1 \Delta V_2$, and construct a V by V matrix

$$A = \left(\begin{array}{c|c} A_1 - v & \chi\psi^T \\ \hline -\psi\chi^T & A_2 - v \end{array}\right),$$

where $A_i - v$ denotes $A_i[V_i \setminus \{v\}]$, χ is the submatrix $A_1[V_1 - v, \{v\}]$, and ψ is the submatrix $A_2[\{v\}, V_2 - v]$. A is the *composition* of A_1 and A_2.

Let A be a V by V skew–symmetric matrix. A partition V_1, V_2 of V is a *k–separation* of A if $|V_1|, |V_2| \geq k$, and $A[V_1, V_2]$ has rank at most $k-1$. Note that A has a 2–separation if and only if it is the composition of two smaller matrices.

Lemma 4.4 *Let $A_i = (a^i_{vw})$ be a V_i by V_i skew–symmetric matrix, for $i = 1, 2$, where $V_1 \cap V_2 = \{v\}$. Let $A = (a_{ij})$ be the composition of A_1 and A_2. Then*

$$pf(A) = pf(A_1 - v)pf(A_2 - v) - pf(A_1)pf(A_2).$$

Proof Let $X = V_1 - v$, $Y = V_2 - v$, and $V = X \cup Y$. Suppose $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_l\}$. Define a linear order $\prec$ such that

$$x_k \prec x_{k-1} \prec \ldots \prec x_1 \prec v \prec y_1 \prec y_2 \prec \ldots \prec y_l.$$

For $S \subseteq E_A$, let $S[X, Y]$ denote the edge set $S \cap \{xy : x \in X, y \in Y\}$. Now let $\mathcal{M}_A^{(i)} = \{M \in \mathcal{M}_A : |M[X, Y]| = i\}$; then, by (4.1),

$$pf(A) = \sum_{i \geq 0} \sum_{M \in \mathcal{M}_A^{(i)}} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv}. \tag{4.3}$$

Claim *For $i \geq 2$,*

$$\sum_{M \in \mathcal{M}_A^{(i)}} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv} = 0.$$

For each matching $M \in \mathcal{M}_A^{(i)}$, we define another matching M' as follows: choose edges $x_{i_1}y_{j_1}$ and $x_{i_2}y_{j_2}$, where $i_1 < i_2$, such that

$$M[\{x_1, x_2, \ldots, x_{i_2}\}, Y] = \{x_{i_1}y_{j_1}, x_{i_2}y_{j_2}\};$$

then define

$$M' = M\Delta\{x_{i_1}y_{j_1}, x_{i_2}y_{j_2}, x_{i_1}y_{j_2}, x_{i_2}y_{j_1}\}.$$

Note that $M = (M')'$, and

$$\sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv} = -\sigma'_M \prod_{\substack{uv \in M' \\ u \prec v}} a_{uv};$$

this proves the claim.

Every matching in $\mathcal{M}_A^{(0)}$ can be expressed as the union of a matching in $\mathcal{M}_{A[X]}$ with a matching in $\mathcal{M}_{A[Y]}$. Therefore

$$\begin{aligned}
\sum_{M \in \mathcal{M}_A^{(0)}} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv} &= \sum_{M_X \in \mathcal{M}_A[X]} \sum_{M_Y \in \mathcal{M}_A[Y]} \sigma_{M_X \cup M_Y} \prod_{\substack{uv \in M_X \cup M_Y \\ u \prec v}} a_{uv} \\
&= \left(\sum_{M_X \in \mathcal{M}_A[X]} \sigma_{M_X} \prod_{\substack{uv \in M_X \\ u \prec v}} a_{uv} \right) \left(\sum_{M_X \in \mathcal{M}_A[X]} \sigma_{M_X} \prod_{\substack{uv \in M_X \\ u \prec v}} a_{uv} \right) \\
&= pf(A[X])pf(A[Y]), \\
&= pf(A_1 - v)pf(A_2 - v). \qquad (4.4)
\end{aligned}$$

Every matching $M \in \mathcal{M}_A^{(1)}$ can be expressed as $M_1 \cup M_2 \cup \{x_iy_j\}$, where $M_1 \in \mathcal{M}_{A[X-x_i]}$ and $M_2 \in \mathcal{M}_{A[Y-y_i]}$. The set of edges of M that cross x_iy_j is

$$M_1[\{x_1, \ldots, x_{i-1}\}, \{x_{i+1}, \ldots, x_k\}] \cup M_2[\{y_1, \ldots, y_{j-1}\}, \{y_{j+1}, \ldots, y_l\}];$$

furthermore

$$\begin{aligned}
|M_1[\{x_1, \ldots, x_{i-1}\}, \{x_{i+1}, \ldots, x_k\}]| &\equiv i - 1 \pmod 2 \text{ and} \\
|M_2[\{y_1, \ldots, y_{j-1}\}, \{y_{j+1}, \ldots, y_l\}]| &\equiv j - 1 \pmod 2.
\end{aligned}$$

Therefore $\sigma_M = ((-1)^{i-1}\sigma_{M_1})((-1)^{j-1}\sigma_{M_2})$, and

$$\begin{aligned}
\sum_{M \in \mathcal{M}_A^{(1)}} \sigma_M \prod_{\substack{uv \in M \\ u \prec v}} a_{uv} &= \sum_{i=1}^{k} \sum_{j=1}^{l} \sum_{M_1 \in \mathcal{M}_{A[X-x_i]}} \sum_{M_2 \in \mathcal{M}_{A[Y-y_i]}} ((-1)^{i-1}\sigma_{M_1})((-1)^{j-1}\sigma_{M_2}) \\
&\quad a_{x_iy_j} \left(\prod_{\substack{uv \in M_1 \\ u \prec v}} a_{uv} \right) \left(\prod_{\substack{uv \in M_2 \\ u \prec v}} a_{uv} \right)
\end{aligned}$$

$$= \left(\sum_{i=1}^{k}(-1)^{i+1}a^1_{vx_i} \sum_{M_1\in\mathcal{M}_{A[X-x_i]}} \sigma_{M_1} \prod_{\substack{uv\in M_1\\ u\prec v}} a_{uv}\right)$$
$$\left(\sum_{j=1}^{l}(-1)^{j+1}a^2_{vy_j} \sum_{M_2\in\mathcal{M}_{A[Y-y_j]}} \sigma_{M_2} \prod_{\substack{uv\in M_2\\ u\prec v}} a_{uv}\right).$$

Now, applying equations (4.1) and (4.2),

$$\begin{aligned}\sum_{M\in\mathcal{M}^{(1)}_A} \sigma_M \prod_{\substack{uv\in M\\ u\prec v}} a_{uv} &= \left(\sum_{i=1}^{k}(-1)^{i+1}a^1_{vx_i} pf(A[X-x_i])\right) \\ &\quad \left(\sum_{j=1}^{l}(-1)^{j+1}a^2_{vy_j} pf(A[Y-y_j])\right), \\ &= -pf(A_1)pf(A_2). \end{aligned} \tag{4.5}$$

The result follows by combining equations (4.3), (4.4), and (4.5), with the claim. □

Theorem 4.5 *Let A be a V by V skew–symmetric matrix, and let V_1, V_2 be a partition of V. If V_1, V_2 is a 2–separation of A, then V_1, V_2 is a 2–separation of $M(A)$.*

Proof Suppose that V_1, V_2 is a 2–separation of A. Then A is the composition of skew–symmetric matrices A_1, A_2, where A_i is $V_i \cup \{v\}$ by $V_i \cup \{v\}$. For any subset X of V, let X_i denote $(X \cap V_i) \cup \{v\}$, for $i = 1, 2$. By Lemma 4.4,

$$pf(A[X]) = pf(A_1[X_1 - v])pf(A_2[X_2 - v]) - pf(A_1[X_1])pf(A_2[X_2]).$$

Every skew–symmetric matrix of odd size is singular; hence either

$$pf(A_1[X_1 - v])pf(A_2[X_2 - v]) = 0, \text{ or } pf(A_1[X_1])pf(A_2[X_2]) = 0.$$

Therefore $X \in \mathcal{F}_A$ if and only if either $X_1 - v \in \mathcal{F}_{A_1}$ and $X_2 - v \in \mathcal{F}_{A_2}$, or $X_1 \in \mathcal{F}_{A_1}$ and $X_2 \in \mathcal{F}_{A_2}$; and hence V_1, V_2 is a 2–separation of $M(A)$. □

The converse of the previous theorem does not hold in general; however, if an even representable delta–matroid has a 2–separation, then it can be represented by a matrix with a 2–separation. Indeed, such a representation can be found by decomposing across the 2–separation, then composing representations of the two delta–matroids got from the decomposition.

Corollary 4.6 *For any field $\mathbf{F}$, the family of even $\mathbf{F}$–representable delta–matroids is closed under 1– and 2–sums.* □

Binary delta–matroids

Let $M = (V, \mathcal{F})$ be a binary delta–matroid (that is, a delta–matroid representable over $GF(2)$), and let $A = (a_{ij})$ be a representation of M. An interesting feature of binary delta–matroids is that the representation is uniquely determined by the feasible sets of size 1 and 2. For $v \in V$, $a_{vv} = 1$ if and only if $\{v\} \in \mathcal{F}$. For $v, w \in V$, $a_{vw} = 1$ if and only if either

- $\{v\}, \{w\} \in \mathcal{F}$, and $\{v, w\} \notin \mathcal{F}$, or
- $\{v, w\} \in \mathcal{F}$, and at most one of $\{v\}$ and $\{w\}$ is feasible.

This unique representability enabled Bouchet and Duchamp [14] to characterize the binary delta–matroids; their result generalizes Tutte's characterization of binary matroids [68].

Theorem 4.7 (Bouchet and Duchamp [14]) *Let M be a delta–matroid. Then M is binary if and only if M does not have a minor isomorphic to one of the following delta–matroids.*

1. $(V_3, \{\emptyset, 12, 23, 13, 123\})$,
2. $(V_3, \{\emptyset, 1, 2, 3, 12, 23, 13\})$,
3. $(V_3, \{\emptyset, 2, 3, 12, 13, 123\})$,
4. $(V_4, \{\emptyset, 12, 13, 14, 23, 24, 34\})$,
5. $(V_4, \{\emptyset, 12, 23, 34, 41, 1234\})$,

where V_i denotes $\{1, \ldots, i\}$. □

Binary pivoting

We note that a binary matrix A is uniquely described by V_A^1 and $G(A)$. Then, since binary delta–matroids have a unique representation, pivoting in a binary matrix is essentially a graphic operation. We denote by $A \times X$ the pivot $A * X$ performed over $GF(2)$; we refer to this as a *binary pivot*. We now describe the elementary binary pivots graphically.

Let $A = (a_{ij})$ be a V by V symmetric binary matrix. For a loop–vertex v of A, we have

$$A \times v = \left(\begin{array}{c|c} a_{vv} & \chi_v^T \\ \hline \chi_v & A[V - v] - a_{vv}\chi_v\chi_v^T \end{array} \right),$$

where χ_v is the submatrix of A indexed by rows $V - v$ and column v. Let v be a vertex of a graph G. We define a graph $G \times v$ by replacing the induced subgraph $G[N_G(v)]$ by its complement; that is, $E_G \Delta E_{G\times v} = \{uv : u, w \in N_G(v)\}$. The operation that changes G to $G \times v$ is called *local complementation*. The following proposition is immediate from the definitions.

Proposition 4.8 *If v is a loop–vertex of a symmetric binary matrix A, then $G(A \times v) = G(A) \times v$, and $V_{A\times v}^1 = V_A^1 \Delta N_{G(A)}(v)$.* □

Let $uw \in V_A^2$. Define vectors χ_u and χ_w, so that

$$A = \left(\begin{array}{c|c|c} 0 & a_{uw} & \chi_u^T \\ \hline a_{uw} & 0 & \chi_w^T \\ \hline \chi_u & \chi_w & A[V-u-w] \end{array}\right),$$

where the first and second rows are indexed by u and w respectively. Then

$$A \times uw = \left(\begin{array}{c|c|c} 0 & a_{uw} & \chi_w^T \\ \hline a_{uw} & 0 & \chi_u^T \\ \hline \chi_w & \chi_u & A[V-u-w] - a_{uw}(\chi_w\chi_u^T + \chi_u\chi_w^T) \end{array}\right).$$

Graphically explaining the binary pivot in this case is more awkward. For a pair of disjoint subsets S, S' of V we define $[S, S'] = \{ss' : s \in S, s' \in S'\}$. Let uw be an edge of a graph G, we define sets $S_u = (N_G(u) - w) \setminus N_G(w)$, $S_w = (N_G(w) - u) \setminus N_G(u)$, and $S_{vw} = N_G(u) \cap N_G(w)$. Now define an intermediate graph G' such that

$$E_G \Delta E_{G'} = [S_v, S_w] \cup [S_v, S_{vw}] \cup [S_w, S_{vw}].$$

$G \times uw$ is obtained from G' by switching the vertex labels u and w. We call the operation that converts G to $G \times uw$ a *pivot*. (Curiously $G \times uw = G \times u \times w \times u$.) The following proposition follows from these definitions.

Proposition 4.9 *Let A be a symmetric binary matrix. Then, for $uw \in V_A^2$, $G(A \times uw) = G(A) \times uw$, and $V^1_{A\times uw} = V^1_A$.* □

Splits and prime graphs

We begin by proving the converse of Theorem 4.5, for even binary delta–matroids.

Theorem 4.10 *Let $A = (a_{ij})$ be a V by V skew–symmetric binary matrix, and let X, Y be a partition of V. Then X, Y is a 2–separation of A if and only if X, Y is a 2–separation of $M(A)$.*

Proof If either $|X| < 2$ or $|Y| < 2$, then the result is immediate; we assume that $|X|, |Y| \geq 2$.

Suppose that X, Y is not a 2–separation of A. Then there exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $a_{x_1y_1} = a_{x_2y_2} = 1$, and $a_{x_1y_2} = 0$. Therefore, $\{x_1, y_1\}, \{x_2, y_2\} \in \mathcal{F}_A$. Note that

$$\{x_1, y_2\} = (X \cap \{x_1, y_1\}) \cup (Y \cap \{x_2, y_2\}).$$

However, $\{x_1, y_2\} \notin \mathcal{F}_A$, so X, Y is not a 2–separation of $M(A)$.

The converse is given by Theorem 4.5. □

We now describe the 2–separations of a binary matrix graphically; first, we introduce some more notation. The *adjacency matrix* of a graph $G = (V, E)$ is the V by V symmetric $(0,1)$–matrix that has a 1 in entry i, j if and only if $ij \in E$. We use the following notation. Let $G = (V, E)$ be a graph, and let X, Y be disjoint subsets of V. We denote by $[X]$ the set of all distinct pairs of vertices in X, and we denote by $[X, Y]$ the set of all pairs of vertices containing an element of X and an element of Y. We denote by $E[X]$ and $E[X,Y]$ the edge sets $E \cap [X]$ and $E \cap [X,Y]$ respectively. The set $E[X,Y]$ is referred to as a *cut* of G. The graph *induced* by X, denoted $G[X]$, is the graph $(X, E[X])$. For a graph G' we denote by $V_{G'}$ and $E_{G'}$ its vertex–set and edge–set.

A *split* of G is a partition (X,Y) of V such that $|X|, |Y| \geq 2$, and the cut $E[X,Y]$ induces a complete bipartite graph. (For a connected graph G, (X,Y) is a split of G if and only if X, Y is a 2–separation of the adjacency matrix of G.) A *prime* graph is a connected graph without any splits.

Let X, Y be a partition of the vertices of G. We denote by $G \circ X$ the graph obtained from G by shrinking X to a single vertex, which we label X, and then removing multiple edges. If (X,Y) is a split of G, then we can decompose G into $G \circ X$ and $G \circ Y$; this decomposition was introduced by Cunningham [21]. It is easy to verify that the adjacency matrix of G is the 2–sum of the adjacency matrices of $G \circ X$ and $G \circ Y$ (when we associate the vertex labels X and Y).

The following lemmas are implied by the fact that 2–separations of binary matrices are preserved under (elementary) pivoting.

Lemma 4.11 (Bouchet [3]) *Let X, Y be a partition of the vertices of a graph $G = (V, E)$. For any vertex v, (X,Y) is a split of G if and only if (X,Y) is a split of $G \times v$.* □

Lemma 4.12 (Bouchet [3]) *Let X, Y be a partition of the vertices of a graph $G = (V, E)$. For any edge vw, (X,Y) is a split of G if and only if (X,Y) is a split of $G \times vw$.* □

Regular delta–matroids

Recall that a matroid that is representable by a totally unimodular matrix is called regular [56]. We call a delta–matroid *regular* if it is representable by a skew–symmetric principally unimodular matrix. Analogous to regular matroids, regular delta–matroids are precisely the even delta–matroids representable over every field.

Theorem 4.13 *Let $M = (V, \mathcal{F})$ be an even delta–matroid. The following are equivalent*

(i) M is regular,

(ii) M is representable over every field, and

(iii) M is representable over both $GF(2)$ and $GF(3)$.

Proof That (i) implies (ii), and that (ii) implies (iii) are both easy. So it suffices to prove that (iii) implies (i). Let $A^{(2)}$ and $A^{(3)}$ be skew–symmetric representations of M over $GF(2)$ and $GF(3)$ respectively. Therefore $A^{(2)}$ and $A^{(3)}$ have the same support (that is, nonzero elements), so there exists a real $(0, \pm 1)$–matrix $A = (a_{ij})$ that is equivalent to $A^{(3)}$ modulo 3, and to $A^{(2)}$ modulo 2. We claim that A is PU. Suppose not, and let $S \subseteq V$ be minimal such that $A[S]$ is not unimodular.

Claim *We may assume that $|S| = 4$.*

Suppose the assumption is not satisfied. Then there exists $S' \subseteq S$ such that $|S'| = |S|-4$, and $A[S']$ is nonsingular. Then $A[S']$ is unimodular, so, by Theorem 2.7, for $X \subseteq V$, $\det(A * S'[X]) = \pm\det(A[X\Delta S'])$. Hence, $A * S'$ is a $(0, \pm 1)$–matrix that represents the delta–matroid $(V, \mathcal{F}\Delta S')$ over $GF(2)$ and $GF(3)$, and $A * S'[S \setminus S']$ is minimally non–unimodular. Now replace S by $S \setminus S'$, A by $A * S'$, and M by $(V, \mathcal{F}\Delta S')$. This proves that claim.

By the claim,

$$pf(A[S]) \quad = \quad a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Therefore, $|pf(A[S])| \leq 3$.

Let k be the $0, \pm 1$ value equivalent to $pf(A[S])$ modulo 3. Note that $pf(A[S]) \equiv pf(A^{(2)}[S]) \equiv k$ modulo 2, and hence $pf(A[S]) \equiv k$ modulo 6. However $|pf(A[S])| \leq 3$, so $pf(A[S]) = k$, contradicting our choice of S. □

Note that every principal submatrix of a PU–matrix is PU. Furthermore, by Theorem 2.7, pivoting preserves principal unimodularity. Therefore, we get the following elementary result.

Lemma 4.14 *If M is a regular delta–matroid, then every normal minor of M is regular.* □

Ideally, we would like to generalize Tutte's famous excluded minor characterization of regular matroids [68]. Unfortunately, this problem remains open.

The following lemma is due to Bouchet and Cunningham, personal communication; their proof was based on pivoting.

Lemma 4.15 *The class of regular delta–matroids is closed under* 1– *and* 2–*sums.*

Proof It is sufficient to show that the composition of two skew–symmetric PU–matrices is PU. This follows from Lemma 4.4, and the fact that skew–symmetric matrices of odd size have zero pfaffian. □

We discuss regular delta–matroids further in Chapter 6.

Eulerian delta–matroids

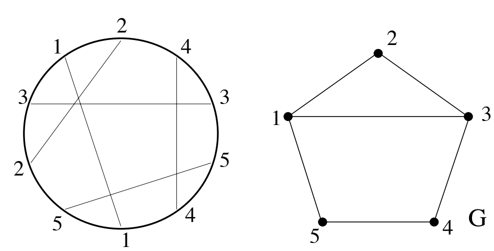

Figure 4.1: Circle graphs.

A *circle graph* is the intersection graph of a finite set of chords of a circle. (See Figure 4.1.) The representation of the circle graph is called a *diagram.* The binary delta–matroids that are represented by the adjacency matrices of circle graphs are called *Eulerian delta–matroids*. (The term *Eulerian* comes from an interesting relationship between the feasible sets and Euler tours of a 4–regular graph [12].) The interest in Eulerian delta–matroids arises through the following theorem.

Theorem 4.16 (De Fraysseix [26]) *Let F be a feasible set of a matroid M. Then, $M\Delta F$ is Eulerian if and only if M is a planar matroid (that is, the forest matroid of a planar graph).* □

Bouchet [7, 11] introduced the notion of principal unimodularity with regard to circle graphs. It is well known that graphic matroids are regular (see Oxley [56]), and thus planar matroids are regular. This generalizes to Eulerian delta–matroids.

Theorem 4.17 (Bouchet [7, 11]) *Eulerian delta–matroids are regular.* □

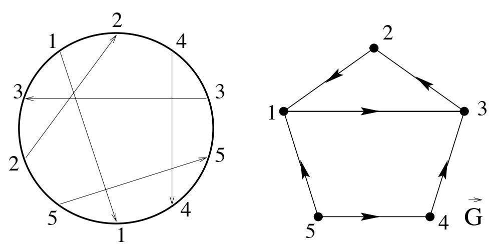

Figure 4.2: Orienting circle graphs.

We briefly describe how to construct a PU–matrix from a circle graph. Let $G = (V, E)$ be a circle graph represented by a set V of chords of a circle. By possibly perturbing the diagram, we may assume that no two chords intersect on the circle. Given an arbitrary orientation to the chords, we define an orientation $\vec{G}$ of G. Namely, an edge uv of G is oriented with v as its head if and only if the chord v crosses u from left to right (that is, the tail of v is encountered before the head of u when the circle is traversed in the clockwise direction from the tail of u). Figure 4.2 depicts an arbitrary orientation of the diagram in Figure 4.1, and the corresponding orientation of the circle graph. Now construct an adjacency matrix $A = (a_{ij})$ for the directed graph $\vec{G}$, that is, A is a skew–symmetric V by V $(0, \pm 1)$–matrix such that $a_{ij} = 1$ if ij is an arc of $\vec{G}$. Then A is principally unimodular. (See [7, 11].)

A characterization of circle graphs

An interesting open problem is the excluded minor characterization of Eulerian delta–matroids. By Theorem 4.16, a special case of this problem is the excluded minor characterization of planar matroids.

Theorem 4.18 (Tutte [69]) *Let M be a binary matroid. M is planar if and only if M does not have a minor isomorphic to one of $M(B_1), M(B_2), M(B_3)$, where B_1, B_2, B_3 are depicted graphically in Figure 4.3.* □

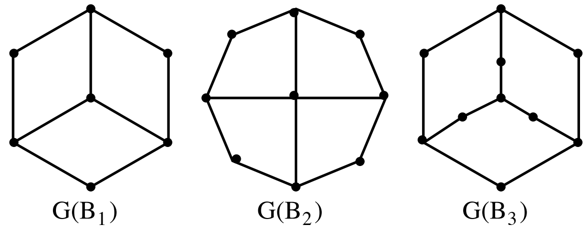

Figure 4.3: Fundamental graphs of non–planar matroids.

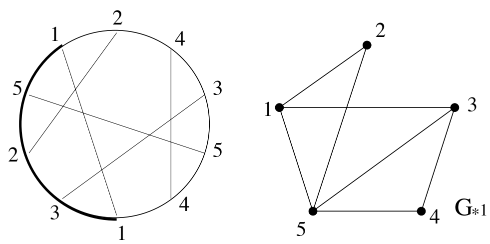

Figure 4.4: Local complementation.

$M(B_1)$ is the twisted Fano matroid, which is not a regular matroid, and hence not graphic. $M(B_2)$ is the twisted graphic matroid of $K_{3,3}$, and $M(B_2)$ is the twisted graphic matroid of K_5.

Kotzig [46] noted that G is a circle graph if and only if $G \times v$ is a circle graph. Figure 4.4 demonstrates local complementation on the graph in Figure 4.1, and the new diagram. (In general, if G is a circle graph, then a diagram of $G \times v$ can be obtained from a diagram of G by reversing the order in which chords are encountered while traversing the circle in a clockwise direction from one end of v to the other.) We say a graph G' is *locally equivalent* to G if G' can be obtained from G by a sequence of local complementations. An *l–reduction* of G is an induced subgraph of any graph locally equivalent to G. Bouchet proved the following deep analogue to Theorem 4.18.

Theorem 4.19 (Bouchet [12]) *Let G be a graph. Then, G is not a circle graph if and only if G has an l–reduction that is isomorphic to one of the graphs depicted in Figure 4.5.* □

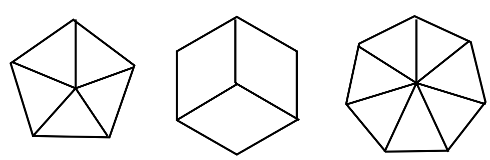

Figure 4.5: Minimal non–circle graphs

The binary delta–matroids represented by the adjacency matrices of the graphs in Figure 4.5 are not regular. Thus, we get the following consequence of Theorem 4.19.

Corollary 4.20 (Bouchet, personal communication) *Let M be a binary delta–matroid represented by the adjacency matrix of a graph G. Then M is Eulerian if and only if, for every graph G' locally equivalent to M, the binary delta–matroid represented by the adjacency matrix of G' is regular.* □

2–separations

The following lemma implies that the family of Eulerian delta–matroids is closed under taking 2–sums. It is independently due to Bouchet [6], Naji [53] and Gabor, Hsu and Supowit [35].

Lemma 4.21 *If (X, Y) is a split in a graph G, then G is a circle graph if and only if $G \circ X$ and $G \circ Y$ are both circle graphs.*

Proof $G \circ X$ and $G \circ Y$ are both induced subgraphs of G, so if G is a circle graph, then $G \circ X$ and $G \circ Y$ are both circle graphs. The converse is demonstrated in Figure 4.6. □

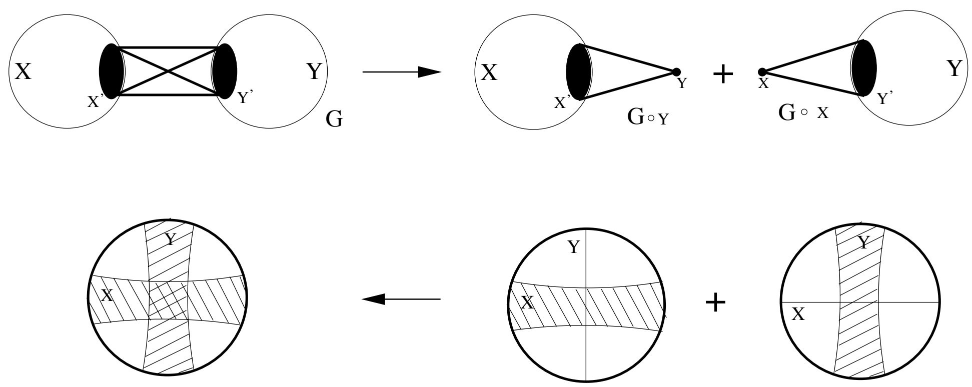

Figure 4.6: Circle graph diagram and splits

Consider a diagram of a circle graph G. We may assume that no two chords of the diagram intersect on the perimeter of the circle, since otherwise we could perturb the diagram. Now we can combinatorially encode the diagram by traversing the perimeter of the circle once, while recording the labels of the chords as they are passed. In such an encoding every chord is recorded exactly twice; we call the encoding a *double occurrence word*. (For example, the diagram in Figure 4.1 is encoded by the double occurrence word 1243541523.) A diagram has many encodings as a double occurrence word; they depend upon where we start on the perimeter of the circle, and the direction in which we choose to traverse the circuit. Thus, we call two double occurrence words *equivalent* if they are equivalent up to cyclic shifting and/or reversing. Two diagrams of a circle graph are considered equivalent if they are encoded by equivalent double occurrence words. The following lemma generalizes a theorem of Whitney [73], that a 3–connected planar graph has a unique embedding. It is independently due to Bouchet [6], Naji [53] and Gabor, Hsu and Supowit [35].

Lemma 4.22 *Let G be a prime circle graph. Then there exists a unique diagram that represents G.* □

Chapter 5

Decomposing 3–connected even binary delta–matroids

In this chapter we develop decompositions for 3–connected even binary delta–matroids (or prime graphs). The first decomposition unifies the ideas in the circle graph recognition algorithms of Gabor, Hsu and Supowit [35], Spinrad [63], and Bouchet [6]. We then develop a more refined decomposition, which allows us to strengthen a theorem of Allys, and Theorem 4.19. Finally, we prove an unpublished theorem of Bouchet that extends Seymour's splitter theorem [61] to even binary delta–matroids.

Blocking sequences

A *subsplit* of G is a pair (X, Y) of disjoint subsets of V such that (X, Y) is a split in $G[X \cup Y]$, and the cut $E_G[X, Y]$ is nonempty. A *blocking sequence* for the subsplit (X, Y) is a sequence $v_1, \ldots, v_p$ of vertices in $V \setminus (X \cup Y)$ satisfying the following conditions:

1. **(a)** $(X, Y \cup \{v_1\})$ is *not* a subsplit of G,
(b) for all $i < p$, $(X \cup \{v_i\}, Y \cup \{v_{i+1}\})$ is *not* a subsplit of G, and
(c) $(X \cup \{v_p\}, Y)$ is *not* a subsplit of G, and

2. no proper subsequence of $v_1, \ldots, v_p$ satisfies 1.

We remark that the problem of finding a blocking sequence for (X, Y), if one exists, can be solved by finding a shortest directed path in a certain digraph. Indeed, we construct a digraph $\vec{G}$ with vertices $V \setminus (X \cup Y) \cup \{X, Y\}$ and arcs $\vec{E}$, where, for $v, w \in V \setminus (X \cup Y)$, $Xv \in \vec{E}$ if and only if $(X, Y \cup \{v\})$ is not a subsplit, $vY \in \vec{E}$ if and only if $(X \cup \{v\}, Y)$ is not a subsplit, and $vw \in \vec{E}$ if and only if $(X \cup \{v\}, Y \cup \{w\})$ is not a subsplit. The blocking sequences for (X, Y) are in one to one correspondence with the minimal (X, Y)–dipaths in $\vec{G}$.

Lemma 5.1 *Let (X, Y) be a subsplit of G. There exists a blocking sequence for (X, Y) in G if and only if there exists no split (X', Y') of G with $X \subseteq X'$ and $Y \subseteq Y'$.*

Proof If there exists a split (X',Y') of G with $X \subseteq X'$ and $Y \subseteq Y'$, then for every $x \in X' \setminus X$ and $y \in Y' \setminus Y$, $(X \cup \{x\}, Y \cup \{y\})$ is a subsplit; therefore no blocking sequence exists. Conversely, if no blocking sequence exists then there exists a partition (X',Y') of V such that for every $x \in X'$ and $y \in Y'$, $(X \cup \{x\}, Y \cup \{y\})$ is a subsplit; then (X',Y') is a split of G. □

We now consider blocking sequences more carefully; they have a surprisingly simple structure. Let $v_1, \ldots, v_p$ be a blocking sequence for a subsplit (X,Y) in G. Define $X' = N_G(Y) \cap X$, and $Y' = N_G(X) \cap Y$. For $v \in V \setminus (X \cup Y)$, $(X, Y \cup \{v\})$ is a subsplit if and only if $N_G(v) \cap X$ equals $\emptyset$ or X'. Therefore, $N_G(v_i) \cap X$ is equal to $\emptyset$ or X', if and only if $i \neq 1$; for $i > 1$, we define $x_i = 0$ (1) when $N_G(v_i) \cap X$ is equal to $\emptyset$ (X'). Similarly, $N_G(v_i) \cap Y$ is equal to $\emptyset$ or Y', if and only if $i \neq p$; for $i < p$, we define $y_i = 0$ (1) when $N_G(v_i) \cap Y$ is equal to $\emptyset$ (Y'). Now consider v_i, v_j, where $i < j$. Define z_{ij} to be 1 if $v_i v_j \in E$, and otherwise to be 0. $(X \cup \{v_i\}, Y \cup \{v_j\})$ is a subsplit if and only if $i \neq j - 1$. So it is easy to verify that $i = j - 1$ if and only if

$$y_i x_j + z_{ij} \equiv 1 \pmod{2}.$$

Lemma 5.2 *Let $v_1, \ldots, v_p$ be a blocking sequence for a subsplit (X,Y) in G. If (X,Y) is the unique split in $G[X \cup Y]$, then $G[X \cup Y \cup \{v_1, \ldots, v_p\}]$ is prime.*

Proof Suppose not; then there exists a split (X',Y') in $G[X \cup Y \cup \{v_1, \ldots, v_p\}]$. Therefore $E_G((X \cup Y) \cap X', (X \cup Y) \cap Y')$ induces a complete bipartite graph. $((X \cup Y) \cap X', (X \cup Y) \cap Y')$ cannot be a split of $G[X \cup Y]$, since (X,Y) is the unique split of $G[X \cup Y]$ and, by Lemma 5.1, (X,Y) cannot be extended to a split in $G[X \cup Y \cup \{v_1, \ldots, v_p\}]$. Therefore either $|(X \cup Y) \cap X'| \leq 1$ or $|(X \cup Y) \cap Y'| \leq 1$. We assume with no loss of generality that $|(X \cup Y) \cap X'| \leq 1$. We complete the proof by considering two cases.

Case 1: $|(X \cup Y) \cap X'| = 0$. Thus $X' \subseteq \{v_1, \ldots, v_p\}$ and $|X'| \geq 2$. Let i be minimum such that $v_i \in X'$ and let j be maximum such that $v_j \in X'$. Since $v_1, \ldots, v_p$ is a blocking sequence for (X,Y), $N(v_i) \cap Y' \neq \emptyset$ and $N(v_j) \cap Y' \neq \emptyset$. Therefore, since (X',Y') is a subsplit, $N(v_i) \cap Y' = N(v_j) \cap Y'$. This contradicts that $v_i, v_{i+1}, \ldots, v_j$ is a blocking sequence for the subsplit $(X \cup \{v_1, \ldots, v_{i-1}\}, Y \cup \{v_{j+1}, \ldots, v_p\})$.

Case 2: $|(X \cup Y) \cap X'| = 1$. Define x so that $(X \cup Y) \cap X' = \{x\}$, and assume without loss of generality that $x \in X$. Let i be maximum such that $v_i \in X'$. We have that $N(x) \cap Y' \neq \emptyset$ and $N(v_i) \cap Y' \neq \emptyset$. Therefore, since (X',Y') is a subsplit, $N(x) \cap Y' = N(v_i) \cap Y'$. Consequently $N(x) \cap (Y \cup \{v_{i+1} \ldots v_p\}) = N(v_i) \cap (Y \cup \{v_{i+1} \ldots v_p\})$, contradicting that $v_1, \ldots, v_i$ is a blocking sequence for the subsplit $(X, Y \cup \{v_{i+1}, \ldots, v_p\})$. □

Decomposing prime graphs

Let v, w be vertices of a graph G. We call v *pendent* if v has exactly one neighbour, and we call v, w *twins* if $N(v) - w = N(w) - v$. A prime graph with at least four vertices contains neither pendent vertices, nor twins.

Lemma 5.3 *Let G be a connected graph with at least five vertices, and let v be a vertex of G such that $G - v$ is prime. Then v is pendent, v has a twin, or G is prime. Furthermore, if G is not prime, then G has a unique split.*

Proof Suppose that G is not prime, and let (X, Y) be a split such that v is in X. Then the cut $E_G[X - v, Y]$ induces a complete bipartite graph. However, since $G - v$ is prime, $(X - v, Y)$ is not a subsplit. Therefore $|X| = 2$; so either v is pendent, or the two vertices in X are twins.

We have shown that, for any split (X', Y') in G, the side of the split that contains v has exactly two elements. It is easy to verify that, if $(\{v, x_1\}, Y_1)$ and $(\{v, x_2\}, Y_2)$ are distinct splits in G, then $(\{v, x_1, x_2\}, Y_1 \cap Y_2)$ is also a split in G, which is a contradiction. Hence, G has a unique split. □

We now describe a decomposition of a prime graph $G = (V, E)$. The decomposition finds a sequence $G_0, \ldots, G_l = G$, where G_i is an induced subgraph of G_{i+1} and the primeness of G_i implies the primeness of the G_{i+1}. Thus the sequence certifies that G is prime. G_0 is chosen to be an induced path of length two (that is, a path with two edges), which is prime; furthermore, every prime graph with at least four vertices contains such an induced subgraph. In a general step of the decomposition, G_i is constructed from G_{i-1} by adding a sequence of vertices $v_0, \ldots, v_p$, where either $p = 0$, or $G[V_{G_{i-1}} \cup \{v_0\}]$ has a unique split, say (X, Y), and $v_1, \ldots, v_p$ is a blocking sequence for the subsplit (X, Y) in G. Therefore, by Lemma 5.2, G_i is prime. All that remains to prove is, given the prime induced subgraph G_{i-1}, we can find a vertex v_0 of G such that $G[V_{G_{i-1}} \cup \{v_0\}]$ is either prime, or has a unique split.

If $i \geq 2$ then G_{i-1} has at least four vertices. Hence, by Lemma 5.3, for any vertex $v \in N(V_{G_{i-1}})$, $G[V_{G_{i-1}} \cup \{v\}]$ is either prime, or contains a unique split. So we now consider the particular case that $i = 1$. G_0 is an induced path of length two; let x_1, x_2, x_3 be the vertices of this path. Since G is prime, $N(x_1) \neq N(x_3)$. By possibly swapping x_1 and x_3, we assume there exists $v \in N(x_3) \setminus N(x_1)$. Then $(\{x_1, x_2\}, \{x_3, v\})$ is the unique split in $G[\{x_1, x_2, x_3, v\}]$. This completes the description of the decomposition.

We remark that the decomposition can only be found for prime graphs, so we have an algorithm that finds a split in a graph, or declares that the graph is prime. The problem of recognizing prime graphs was originally solved by Cunningham [22]. The fastest algorithm is due to Ma and Spinrad [51]; it runs in $O(n^2)$ time, where n is the number vertices of the graph. In fact, the algorithms of Cunningham, and Ma and Spinrad are more general; they decompose a graph into prime graphs.

Recognizing circle graphs

Consider the problem of deciding whether a binary matroid is a planar matroid. This problem was solved by Tutte [70], and others, who actually solved the more general problem of deciding which binary matroids are graphic. An alternative solution comes by means of Theorem 4.16. It suffices to be able to check which binary delta–matroids are Eulerian; that is, to be able to recognize circle graphs. The problem of circle graph recognition was solved independently by Naji [53], Gabor, Hsu and Supowit [35], and Bouchet [6]. Spinrad [63] refined the algorithm of Gabor *et al.* to recognize circle graphs in $O(n^2)$ time.

With the exception of Naji's algorithm, the circle graph recognition algorithms involve the decomposition of prime graphs. Bouchet's decomposition uses local complementation and pivoting, but gives a conceptually simple algorithm. Unfortunately, Gabor *et al.*, and Spinrad do not cleanly separate the problem of decomposing prime graphs from the construction of diagrams, which makes their circle graph recognition algorithms appear complicated. We describe an algorithm that, while being less efficient than that of Spinrad, is simple.

Remark: Seymour [62] generalized the result of Tutte, by giving an efficient algorithm to test whether any given matroid is graphic. This leaves open an interesting question for delta–matroids: *Is there an efficient algorithm that, given an arbitrary even delta–matroid, determines whether it is Eulerian?*

We now begin the description of the recognition algorithm. We are given a graph $G = (V, E)$, and we are asked whether G is a circle graph. By Lemma 4.21, we may assume that G is prime. Also, we assume that G is a circle graph, and we algorithmically construct its diagram. If our assumption fails then so must our algorithm, and hence we can decide if G is a circle graph. We begin by finding the nested sequence of prime graphs $G_0, \ldots, G_l$, as described above. Trivially, we can find a diagram for G_0; furthermore the diagram is unique. We assume that we have found a diagram for G_{i-1}. By Lemma 4.22, the diagram of G_{i-1} is unique. Thus, we can extend this diagram to a diagram for G_i.

Recall that G_i is constructed from G_{i-1} by adding a sequence of vertices $v_0, \ldots, v_p$, where $G[V_{G_{i-1}} \cup \{v_0\}]$ is either prime (and $p = 0$), or has a unique split, say (X, Y), and $v_1, \ldots, v_p$ is a blocking sequence for the subsplit (X, Y).

Consider the case that $p = 0$. We want to add a single chord v_0 to the diagram of G_{i-1}. Let k be the number of vertices of G_{i-1}. Then, in the diagram of G_{i-1}, there are $2k$ intervals on the circle in which we might attach an end of the chord v_0. We can test all pairs of such intervals to find the diagram for G_i.

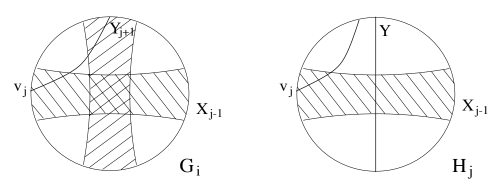

Figure 5.1: Circle graph diagram and blocking sequences

Now, we consider the more general case when $p > 0$. Let H denote the graph $G[V_{G_{i-1}} \cup \{v_0\}]$, and let (X, Y) be the split in H. We define $X_j = X \cup \{v_1, \ldots, v_j\}$ and $Y_j = Y \cup \{v_j, \ldots, v_p\}$, and we define $H_j = G[X_j \cup Y] \circ Y$. Initially, we have a unique diagram for H_0 (since $H_0 = H \circ Y$). We add the chords $v_1, \ldots, v_p$ in sequence; we assume that we have a "unique" diagram for H_{j-1}, and we find a "unique" diagram for H_j. In general, H_j is not prime, so it does not necessarily have a unique representation, but it has a unique representation that extends to a diagram of G_i. From the definition of a blocking sequence, we have that (X_{i-1}, Y_{i+1}) is a subsplit, and v_i is a blocking sequence for (X_{i-1}, Y_{i+1}). Then, a diagram of G_i must have the general form depicted in Figure 5.1. Hence, when we add the chord v_j to the diagram of H_{j-1}, one end of the chord v_j must be placed adjacent, on the circle, to an end of the chord Y. Furthermore, from Figure 5.1, it is clear that any diagram of H_j with the property that an end of chord v_j is adjacent, on the circle, to an end of the chord Y, extends to a diagram of G_i. However, since G_i is prime, it has a unique representation. Hence, there is a unique way to extend the diagram of H_{j-1} to a diagram of H_j, with the required property.

So we can construct a diagram for H_p. Finally, from the diagram of H_p and $H \circ X$, we can construct the diagram of G_i. This completes the description of the algorithm.

A refined decomposition

We now describe a second decomposition of a prime graph. Like the previous decomposition, it constructs a nested sequence $H_1, \ldots, H_l$ of prime induced subgraphs. However, the present decomposition resembles the classical "ear decomposition" of a graph. It also differs from the above decomposition in its use of isomorphism. To begin the decomposition we require an induced prime subgraph. We could use the previous decomposition to find such a graph; however, Gabor, Hsu, and Supowit [35] have a far more elegant solution, given in the following theorem. Their proof is long and technical; they first search for an induced path of length three and then use this path to find a nice induced prime subgraph. We include a simpler proof based on blocking sequences.

Theorem 5.4 (Gabor, Hsu,Supowit [35]) *Let G be a prime graph with at least four vertices. Then G has an induced (prime) subgraph that isomorphic to either H_1, H_2, H_3 (defined in Figure 5.2) or a circuit with at least five vertices.*

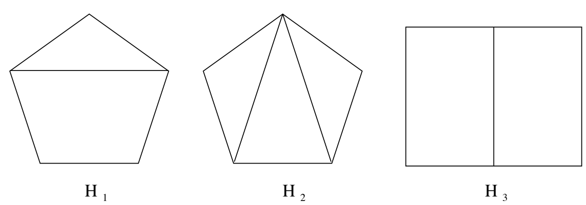

Figure 5.2: Small prime graphs

Proof Let x_1x_2 be an edge of G. Since G is prime, $N(x_1) - x_2 \neq N(x_2) - x_1$. We assume, by possibly interchanging x_1 and x_2, that there exists $y_1 \in (N(x_1) - x_2) \setminus (N(x_2) - x_1)$. Since G is prime, $N(x_2) \neq N(y_1)$; we assume, by possibly interchanging y_1 and x_2, that there exists $y_2 \in N(y_1) \setminus N(x_2)$. We define $X = \{x_1, x_2\}$, and $Y = \{y_1, y_2\}$. The graph $G[\{x_1, x_2, y_1, y_2\}]$, is depicted in Figure 5.3. Note that (X, Y) is the unique split in $G[\{x_1, x_2, y_1, y_2\}]$; let $v_1, \ldots, v_p$ be a blocking sequence for this subsplit of G. Let H denote the graph $G[\{x_1, x_2, y_1, y_2, v_1, \ldots, v_p\}]$. We claim that H contains, as an induced subgraph, either a circuit with at least five vertices, or a graph isomorphic to one of H_1, H_2, H_3. The proof is inductive on the length of the blocking sequence; we consider two separate cases.
Case 1: x_1y_2 is an edge. For a vertex v_i, $(X, Y \cup \{v_i\})$ is a subsplit if and only if $i \geq 2$. Hence $x_2v_i \in E$ if and only if $v_i = 1$. Similarly, $(X \cup \{v_i\}, Y)$ is a subsplit if and only if $i < p$. Hence, $|N(v_i) \cap Y| = 1$ if and only if $i = p$; we assume, by possibly interchanging y_1 and y_2, that y_2v_p is an edge. If $p = 1$ then H is isomorphic to either H_1 or H_2; so we assume that $p > 1$. It is easy to verify that $v_1, \ldots, v_{p-1}$ is a blocking sequence for the

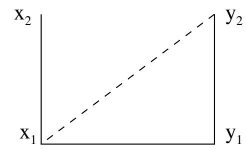

Figure 5.3: $G[\{x_1, x_2, y_1, y_2\}]$

subsplit $(X, \{y_2, v_p\})$; so, by induction, $H - y_1$ contains either a circuit with at least five vertices, or a graph isomorphic to one of H_1, H_2, H_3 as an induced subgraph.
Case 2: x_1y_2 is not an edge. For a vertex v_i, $(X, Y \cup \{v_i\})$ is a subsplit if and only if $i \geq 2$. Hence $x_2v_i \in E$ if and only if $i = 1$. Similarly, $(X \cup \{v_i\}, Y)$ is a subsplit if and only if $i \leq p - 1$. Hence $y_2v_i \in E$ if and only if $i = p$. Suppose, for some $i \in \{2, \ldots, p\}$, that v_iy_1 is an edge. Then, it is easy to verify that $v_1, \ldots, v_{i-1}$ is a blocking sequence for the subsplit $(X, \{y_1, v_i\})$; so, by induction, $G[\{x_1, x_2, y_1, v_1, \ldots, v_i\}]$ contains either a circuit with at least five vertices, or a graph isomorphic to one of H_1, H_2, H_3 as an induced subgraph. Therefore, we may assume that, for $i = 2, \ldots, p$, v_iy_1 is not an edge. Similarly, we may assume, for $i = 1, \ldots, p - 1$, that v_ix_1 is not an edge. The graph H is depicted

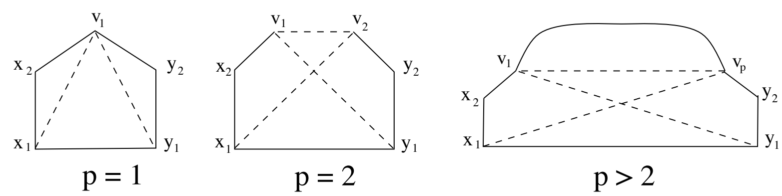

Figure 5.4: H

in Figure 5.4. If $p = 1$ then H is isomorphic to H_1, H_2, or C_5; so we may assume that $p \geq 2$. Suppose that $p = 2$. $(\{x_1, x_2, v_1\}, \{y_1, y_2, v_2\})$ is not a split in H, so v_1v_2 is not an edge if and only if v_1y_1 and v_2x_1 are both edges. In any case, H is isomorphic to H_3 or C_6. Thus, we may assume that $p \geq 3$. Consider v_i, v_j, such that $i < j$. By the definition of a blocking sequence, $(X \cup \{v_i\}, Y \cup \{v_j\})$ is a subsplit if and only if $i < j - 1$. Hence, if $i \neq 1$ or $j \neq p$ then v_iv_j is an edge if and only if $i = j - 1$. By the definition of a blocking sequence, $(X \cup \{v_1\}, Y \cup \{v_p\})$ is a subsplit. Hence, v_1v_p is an edge if and only if v_1y_1 and v_px_1 are both edges. If v_1v_p is not an edge then H contains an induced circuit of length at least 5. If v_1v_p is an edge, then x_1v_p is also an edge, and $H - y_1 - y_2$ is either isomorphic to H_2 or H_3, or H contains an induced circuit of length at least 5. □

The decomposition

Let H be a prime induced subgraph of a graph G. A prime graph H' containing H as an induced subgraph is called a *k-element prime extension* of H, where $k = |V_{H'} \setminus V_H|$. A path $v_0, \ldots, v_{p+1}$, of length at least three, is called a *handle of H* if $\{v_0, \ldots, v_{p+1}\} \cap V_H = \{v_0, v_{p+1}\}$, and the vertices $v_1, \ldots, v_p$ all have degree two in $G[V_H \cup \{v_1, \ldots, v_p\}]$.

Proposition 5.5 *Let $G = (V, E)$ be a graph, H be an induced prime subgraph of H with at least four vertices, and $v_0, \ldots, v_{p+1}$ be a handle of H. Then, $G[V_H \cup \{v_1, \ldots, v_p\}]$ is a prime extension of H.*

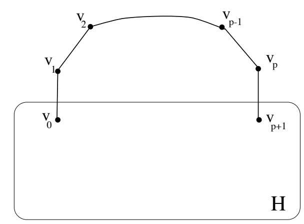

Figure 5.5: A handle of H

Proof The result follows from Lemma 5.2, since $v_2, \ldots, v_p$ is a blocking sequence for the unique split $(\{v_0, v_1\}, V_H - v_0)$ in $G[V_H \cup \{v_1\}]$. □

The decomposition is basically described by the following theorem.

Theorem 5.6 *Let $G = (V, E)$ be a prime graph, and let H be an induced prime subgraph of G such that $4 \leq |V_H| < |V|$. Then, there exists a subgraph H' of G such that H' is isomorphic to H, and either there exists a 1– or 2–element prime extension of H', or there exists a handle for H'.*

Proof Suppose that there does not exist a 1– or 2–element prime extension of H, or of any induced subgraph H' of G isomorphic to H. We construct a family $\mathcal{S} = (S_v : v \in V_H)$ of disjoint sets. Initially, $S_v = \{v\}$ for each $v \in V_H$. At any stage $\mathcal{S}$ satisfies the following properties.

(i) $S_v \cap V_H = \{v\}$, for each $v \in V_H$, and

(ii) for any two distinct vertices v, w of H, either $E[S_v, S_w] = \emptyset$, or $E[S_v, S_w] = [S_v, S_w]$.

We define X to be $\cup_{v \in V_H} S_v$. A subset W of X, is called a *transversal of* $\mathcal{S}$ if $|S_v \cap W| = 1$, for each $v \in V_H$. For any transversal W of $\mathcal{S}$, $G[W]$ is isomorphic to H.

Suppose there exists a vertex $x \in V \setminus X$ that has neighbours in at least two distinct sets of $\mathcal{S}$. Let W be a transversal of $\mathcal{S}$ such that x has at least two neighbours in W. By possibly relabeling, we may assume that $W = V_H$. Since H has no 1–element prime extension, then, by Lemma 5.3, x has a twin x' in $G[V_H \cup \{x\}]$. We construct a family $\mathcal{S}' = (S'_v : v \in V_H)$ from $\mathcal{S}$ by adding x to $S_{x'}$. $\mathcal{S}'$ satisfies (i); we claim that (ii) is also satisfied by $\mathcal{S}'$. Suppose not; then there exists $y' \in V_H \setminus \{x'\}$, and $y \in S'_{y'}$ such that $E[\{x, x'\}, \{y, y'\}]$ is neither complete nor empty. Then, it is easy to verify that y is a blocking sequence for the unique split $(\{x, x'\}, V_H - x')$ in $G[V_H \cup \{x\}]$. Hence, by Lemma 5.2, $G[V_H \cup \{x, y\}]$ is a 2–element prime extension of H, a contradiction. So, (ii) is satisfied by $\mathcal{S}'$ as claimed.

We continue the construction of $\mathcal{S}$ until each vertex in $V \setminus X$ has neighbours in at most one set of $\mathcal{S}$. Since G is prime, and H is a proper induced subgraph of G, $X \neq V$. Therefore, there exists a vertex v of X such that $N(S_v) \setminus X$ is non–empty. Let Y be the set of vertices in $V \setminus X$ that are in the same component as a vertex of S_v in the graph $G[V \setminus (X \setminus S_v)]$. G is prime, so $(Y \cup S_v, V \setminus (Y \cup S_v))$ is not a split. Hence, Y has neighbours in $X \setminus S_v$. Therefore, there exists a path of length at least two in G from S_v to $X \setminus S_v$, such that the internal vertices are in $V \setminus X$; let $v_0, \ldots, v_{p+1}$ be a shortest such path. Now let W be a transversal of $\mathcal{S}$ that contains v_0 and v_{p+1}. Then $v_0, \ldots, v_{p+1}$ is a handle for $G[W]$. □

Theorem 5.6 can be restated in the following useful form.

Theorem 5.7 *Let $G = (V, E)$ be a prime graph, and let H be an induced prime subgraph of G with at least four vertices. Then, there exists a sequence of induced prime subgraphs $H_1, \ldots, H_k$ such that H_1 is isomorphic to H, $H_k = G$ and, for $i > 1$, either H_i is a 1– or 2–element prime extension of H_{i-1}, or H_i is got from H_{i-1} by adding a handle.* □

We now consider the case where there exists a 2–element prime extension of an induced prime subgraph, but no 1–element prime extensions.

Lemma 5.8 *Let v_1, v_2 be vertices of a graph $G = (V, E)$ such that, G and $G - v_1 - v_2$ are both prime, and have at least four vertices. Then either there exists a 1–element prime extension of an isomorphic copy of $G - v_1 - v_2$ in G, or there exist a sequence $x_1, x_2, \ldots, x_{2k}$ of distinct vertices such that*

1. *for $j < 2k$, $G - x_j - x_{j+1}$ is isomorphic to $G - v_1 - v_2$,*
2. *x_2 is pendent in $G - x_1$ and, for even i, $N(x_i) \setminus \{x_1, \ldots, x_{2k}\} = N(x_2)$,*
3. *x_{2k-1} is pendent in $G - x_{2k}$ and, for odd i, $N(x_i) \setminus \{x_1, \ldots, x_{2k}\} = N(x_{2k-1})$, and*
4. *for $i < j$, $x_i x_j$ is an edge if and only if i is odd and j is even.*

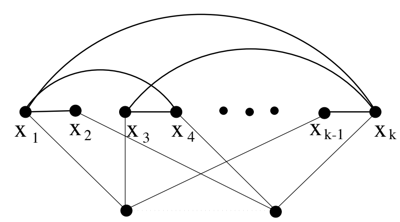

Figure 5.6: Demonstrating the lemma.

Proof We assume that there exists no 1–element prime extension of an isomorphism of $G - v_1 - v_2$ in G.

We extend the ordered pair v_1, v_2 to a sequence $v_1, v_2, v_3, \ldots$ as follows. At a general stage we have a finite sequence $x_1, \ldots, x_s$ such that, for $i > 1$, v_{i-1}, v_{i+1} are twins in $G - v_i$. If v_{s-1} is pendent in $G - v_s$, then we stop with the sequence $x_1, \ldots, x_s$; so assume otherwise. By the initial assumption, $G - v_s$ is not prime. Therefore, by Lemma 5.3, v_{s-1} has a unique twin, say v_{s+1}, in $G - v_s$. We add v_{s+1} to our sequence and continue.

We claim that the vertices $v_1, v_2, v_3, \ldots$ are distinct. Suppose not, and let $p < q$ be such that $v_p = v_q$; furthermore, assume that $q - p$ is minimum with this property. By Lemma 5.3, for $i \geq 1$, there exists a unique split in $G - v_i$, so $G - v_i$ has either a pendent vertex or a (unique) pair of twins. It is not possible that $G - v_p$ contain a pendent vertex, since otherwise the sequence would stop at v_p. Now, v_{p-1} and v_{p+1} are the unique twins in $G - v_p$, so $\{v_{p-1}, v_{p+1}\} = \{v_{q-1}, v_{q+1}\}$. Furthermore, by our choice of p and q, we must have $v_{p-1} = v_{q-1}$ and $v_{p+1} = v_{q+1}$. Iteratively, replacing p, q by $p + 1, q + 1$, we see that the sequence $v_1, v_2, \ldots$ is infinite and periodic. Let $v_1, \ldots, v_l$ be the first period. For $i > 1$,

v_{i-1}, v_{i+1} are twins in $G - v_i$, but not in G (since G is prime). Therefore v_i is adjacent to exactly one of v_{i-1} and v_{i+1}. Therefore, we may assume, by possible shifting the period, that $v_{i-1}v_i$ is an edge if and only if i is even. Hence p, the length of the period, must be even. Note that $v_{l+1} = v_1$, so v_1v_l is not an edge, while v_1v_2 is an edge. Therefore, there exists some (odd) i, less than p, such that v_1v_{i+1} is not an edge, but v_1v_{i-1} is an edge, contradicting that v_{i-1} and v_{i+1} are twins in $G - v_i$. Hence $v_1, v_2, \ldots$ is distinct as claimed, and the sequence must be finite.

Let $x_1 = v_s$, and $x_2 = v_{s-1}$. Then, as above, we can extend the pair x_1, x_2 to a sequence $x_1, x_2, x_3, \ldots, x_r$ of distinct vertices such that, for $i > 1$, x_{i-1}, x_{i+1} are twins in $G - x_i$, x_{r-1} is pendent in $G - x_r$ and x_2 is pendent in $G - x_1$. For $1 < i < r$, since x_{i-1} and x_{i+1} are twins in $G - x_i$, $G - x_{i-1} - x_i$ is isomorphic to $G - x_i - x_{i+1}$. Thus $x_1, \ldots, x_r$ satisfies 1. Also, since G is prime, x_{i-1} and x_{i+1} are not twins in G. Hence x_i is adjacent to exactly one of x_{i-1} and x_{i+1}. Now, since G is prime and x_2 is pendent in $G - x_1$, x_1x_2 is an edge of G. Similarly $x_{r-1}x_r$ is an edge of G. Therefore, for $j < r$, x_jx_{j+1} is an edge of G if and only if j is odd, and, since $x_{r-1}x_r$ is an edge of G, r is even.

Let X denote $\{x_1, \ldots, x_r\}$. For any p, q equivalent modulo 2, $N_G(x_p) \setminus X = N_G(x_q) \setminus X$, since x_{i-1} and x_{i+1} are twins in $G - x_i$ for $1 < i < r$. In particular, $N_G(x_i) \setminus X = N_G(x_2) \setminus X$, for odd i, and $N_G(x_i) \setminus X = N_G(x_{r-1}) \setminus X$, for even i.

Given $i_1 < j_1$ and $i_2 < j_2$ such that i_1 and i_2 are equivalent, modulo 2, and j_1 and j_2 are equivalent, modulo 2, then, we claim that $x_{i_1}x_{j_1}$ is an edge if and only if $x_{i_2}x_{j_2}$ is an edge. This follows easily from the fact that, for $1 < i < r$, x_{i-1} and x_{i+1} are twins in $G - x_i$. Therefore, if $|j_1 - i_1|$ is odd, then $x_{i_1}x_{j_1}$ is an edge if and only if $x_{i_1}x_{i_1+1}$ is an edge (which is the case exactly when i_1 is odd). Now we suppose that $|j_1 - i_1|$ is even. Suppose, by way of contradiction, that $x_{i_1}x_{j_1}$ is an edge. We assume that i_1 is even, since otherwise we could reverse the labeling on the sequence $x_1, \ldots, x_r$. Then, since $x_{i_1}x_{j_1}$ is an edge, x_2x_4 is an edge. Note that x_2 has degree 2, and is also adjacent to x_1. Therefore, it must be the case that $r = 4$. However, G has at least six vertices, and $N_G(X) = N_G(x_{r-1}) \setminus X$, so $(X, V \setminus X)$ is a split in G, contradicting that G is prime. Thus $x_1, \ldots, x_r$ satisfies conditions 2, 3 and 4, as required. □

Local complementation

In this section, we use local complementation to further refine the decomposition of prime graphs. Recall the definition of local complementation: For a vertex v of a graph G, $G \times v$ is got from G by replacing $G[N(v)]$ by its complement. Two graphs are *locally equivalent* if they differ by a sequence of local complementations, and an *l–reduction of* G is an induced subgraph of a graph locally equivalent to G.

Lemma 5.9 *Let G be a prime graph, and H be a proper induced prime subgraph of G having at least four vertices. Then either there exists a vertex v of G such that $G - v$ is prime and $G - v$ has an l–reduction isomorphic to H, or there exists a degree–two vertex v of G such that $G \times v - v$ is prime and $G \times v - v$ has an l–reduction that is isomorphic to H.*

Proof By Theorem 5.7, we can find a nested sequence of induced prime subgraphs $H_1, \ldots, H_k$, where H_1 is isomorphic to H, $G = H_k$, and, for $i > 1$, either H_i is a 1– or 2–element prime extension of H_{i-1}, or H_i is got from H_{i-1} by adding a handle. Since H is isomorphic to an l–reduction of H_{k-1}, we may assume that $H = H_{k-1}$. Thus, either G is a 1– or 2–element prime extension of H, or G is got from H by adding a handle.

Let $X = V_G \setminus V_H$. If G is a 1–element prime extension of H, then we are done. If $|X| \geq 3$, then G is got from H by adding a handle; so for any $x \in X$, $G \times x - x$ is got from H by adding a shorter handle. Thus, $G \times x - x$ is prime, and we are done. Therefore, we may assume that X has two elements, say x_1 and x_2. Furthermore, by Lemma 5.8, we may assume that x_1 is pendent in $G - x_2$; let y_1 be the neighbour of x_1 in $G - x_2$. Note that $H = G - x_1 - x_2$ and $H = G \times x_1 - x_1 - x_2$, so if $G - x_1$ or $G \times x_1 - x_1$ is prime, then we are done; assume otherwise.

Case 1: x_2 is pendent in $G - x_1$. Let y_2 be the neighbour of x_2 in $G - x_1$. Now, x_2 is not pendent in $G \times x_1 - x_1$. Therefore, by Lemma 5.3, there exists a twin v of x_2 in $G \times x_1 - x_1$. Note that $N(v) = \{y_1, y_2\}$, and $G \times x_1 - x_1 - v$ is isomorphic to H. Therefore, if $G - v$ is prime then we are done; we assume otherwise. By Lemma 5.3, there exists a twin v' of x_1 in $G - v$. Since v' must be adjacent to x_2, we have that $v' = y_2$. Therefore, $N(y_2) = \{v, y_1, x_2\}$; but then $N(\{x_1, x_2, y_2, v\}) = \{y_1\}$. This is a contradiction, since G is a prime graph with at least six vertices, and thus cannot have a cut–vertex.

Case 2: x_2 is pendent in $G \times x_1 - x_1$. This case is similar to Case 1.

Case 3: x_2 is pendent in neither $G - x_1$, nor $G \times x_1 - x_1$. Then, by Lemma 5.3, x_2 has a twin y_2 in $G - x_1$, and a twin y_2' in $G \times x_1 - x_1$. Now, $G - x_1 - y_2$ is isomorphic to H, and x_1 is not pendent in $G - y_2$. If $G - y_2$ is prime, then we are done; so assume otherwise. Then, by Lemma 5.3, there exists a twin v of x_1 in $G - y_2$, so $N_G(v) = \{y_1, y_2, x_2\}$. Then, since v is a neighbour of y_2 but not a neighbour of y_2', either $v = y_2'$ or $v = y_1$, both of which yield contradictions. □

The following corollary is a strengthening of a theorem of Allys [1], who showed that, if G is prime, then there exists a vertex v such that either $G - v$ or $G \times v - v$ is prime. Allys' theorem implies that there exists a graph G' that is locally equivalent to G, and an ordering $x_1, \ldots, x_n$ of V such that, for $n \geq 5$, $G'[\{x_1, \ldots, x_i\}]$ is prime. Testing whether G' (and hence, also G) is a circle graph is then easy. This is essentially Bouchet's algorithm for circle graph recognition [6], although Allys' theorem is cleaner than the original decomposition used by Bouchet. (Bouchet's theorem requires a third possibility that $G \times vw - v$ is prime, where w is any neighbour of v.)

Corollary 5.10 *Let G be a prime graph with at least six vertices. Either there exists a vertex v such that $G - v$ is prime, or there exists a vertex w, of degree two, such that $G \times w - w$ is prime.*

Proof Let H be an induced prime subgraph of G of the type guaranteed by Lemma 5.4. If $H = G$, then the result follows easily. Otherwise, the result follows by Lemma 5.9. □

The following theorem can be viewed as a "splitter" theorem for l–reductions. An l–reduction of G is called *elementary* if it has one fewer vertex than G.

Corollary 5.11 *Let G be a prime graph, and let H be an l–reduction of G that is prime and has at least four vertices. Then there exists an elementary l–reduction of G that is prime, and has an l–reduction that is isomorphic to H.*

Proof Immediate, by Lemma 5.9. □

Circle graphs

We now derive the following results from Corollary 5.11. The graphs W_5, W_7, F_7 and Q_3, are defined in Figure 5.7.

Proposition 5.12 *Let G be a prime graph having W_7 as an l–reduction. Then, either G is locally equivalent to W_7, or G has an l–reduction that is isomorphic to W_5.*

Proposition 5.13 *Let G be a prime graph having F_7 as an l–reduction. Then, either G is locally equivalent to F_7, G is isomorphic to a graph locally equivalent to Q_3, or G has an l–reduction that is isomorphic to W_5.*

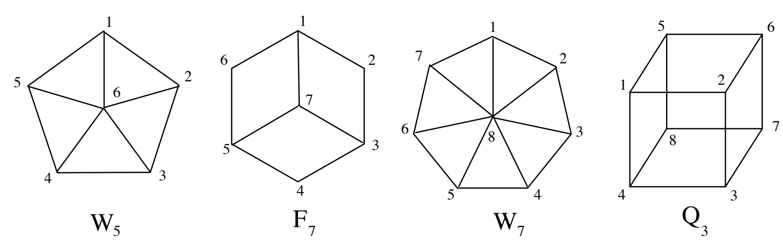

Figure 5.7:

As an immediate corollary of Propositions 5.12 and 5.13, and Theorem 4.19, we get the following strengthening of Theorem 4.19. The theorem is analogous to the well–known fact that, if G is a 3–connected graph, then G is nonplanar if and only if either G is isomorphic to K_5, or G contains a minor isomorphic to $K_{3,3}$.

Theorem 5.14 *Let G be a prime graph. Then G is not a circle graph if and only if either G is locally equivalent to an isomorphism of W_7, F_7 or Q_3, or G has an l–reduction that is isomorphic to W_5.* □

For $n \geq 3$ we define a simple graph W_n, the *n–wheel*, with vertices $1, 2, \ldots, n+1$, where $1, 2, \ldots n$ defines an induced circuit, and $n+1$, the *hub* of W_n, is adjacent to all other vertices. A *partial wheel*, with *hub* v, is a graph G, such that v is a vertex of degree at least three in G, and $G - v$ is an induced circuit. We require the following elementary result of Bouchet [12].

Proposition 5.15 *Let G be a partial wheel with hub v. Then G is a circle graph if and only if $N(v)$ can be partitioned into two disjoint sets X_1, X_2, each having at most two elements, such that, for $i = 1, 2$, if X_i contains two vertices then they are adjacent.* □

Proposition 5.16 *Let W be a partial wheel, with hub x, that is not a circle graph and is not isomorphic to F_7 or W_7. Then W has an l–reduction that is isomorphic to W_5.*

Proof Suppose that y is a vertex of W that has degree two. Then, $W \times y - y$ is a partial wheel, whose hub, x, has the same degree as the hub of W. Therefore, we may assume that, for any degree two vertex y of W, either $W \times y - y$ is not a circle graph, or $W \times y - y$ is isomorphic to W_7 or F_7. Let k be the degree of x.

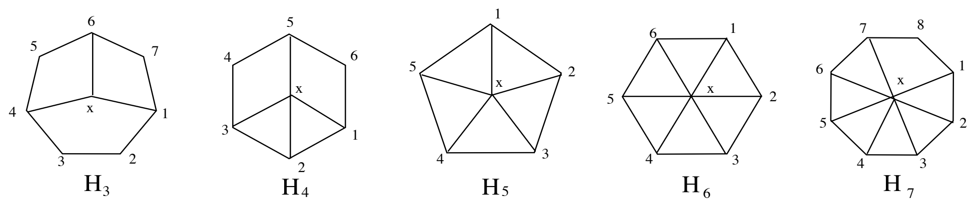

Figure 5.8: Partial wheels

Consider the partial wheels $H_3, \ldots, H_7$, depicted in Figure 5.8. These represent the different possibilities for W, when $k = 3, 4, 5, 6, 7$. Following are the promised l–reductions that are isomorphic to W_5: $H_3 \times 1 \times 5 - 1 - 5$, $H_4 \times 5 - 5$, H_5, $H_6 \times x \times 1 - x$, and $H_7 \times 7 \times 8 \times 1 - 1 - 7 - 8$.

Now, suppose that $k \geq 8$. Then, by our assumption, W is a k–wheel. Therefore, for any $y \in V_W - x$, $W \times y - y$ is a partial wheel, whose hub, x, has degree $k - 3$. Therefore, by Proposition 5.15, $W \times y - y$ is not a circle graph. Furthermore, $W \times y - y$ is isomorphic to neither F_7 nor W_7. So, inductively, $W \times y - y$ contains an l–reduction that is isomorphic to W_5. □

Proof of Proposition 5.12. We consider, up to isomorphism, all 1–element prime extensions of W_7. (There are an embarrassing 29 such extensions.) In each case we show, by Proposition 5.16, that the graph obtained contains an l–reduction that is isomorphic to W_5. Then, by Corollary 5.11, every prime graph that contains W_7 as a proper l–reduction, also contains an isomorphism of W_5 as an l–reduction. The case analysis is summarized in Table 5.1. Each entry in the table contains the neighbours of the new vertex, x, in a 1–element prime extension G of W_7, and an l–reduction of G to a partial wheel that is not a circle graph, and is isomorphic to neither W_7 nor F_7. □

Proof of Proposition 5.13. Recall the definition of a *pivot* in a graph. For an edge vw, $G \times vw = G \times v \times w \times v$. For any edge vw of F_7, $F_7 \times vw$ is isomorphic to F_7; by using such pivotings, many prime extensions of F_7 are locally equivalent. We use the notation $G \rightarrow (v_1, \ldots, v_r)$ to indicate that G is isomorphic to the graph obtained by adding a vertex to F_7 and joining it to the vertices $v_1, \ldots, v_r$. Table 5.2 contains all 1–element prime extensions of F_7 and indicates that, with the exception of one graph that is isomorphic to Q_3, all extensions contain an l–reduction that is isomorphic to W_5.

Note that, for every vertex v of Q_3, $Q_3 - v$ is isomorphic to F_7. It is easy to show that every 1–element prime extension of Q_3 contains an induced subgraph, different from Q_3, that is a 1–element prime extension of F_7. Hence, any 1–element prime extension of Q_3 contains an l–reduction that is isomorphic to W_5. □

neighbours of x	l–reduction	neighbours of x	l–reduction
$1,2$	$G \times x$	$1,3$	$G-2$
$1,4$	$G-2-3$	$1,2,3$	$G-2$
$1,2,4$	$G-2-3$	$1,2,5$	$G \times 2-3-4$
$1,3,5$	$G-8$	$1,2,3,4$	$G-2-3$
$1,2,3,5$	$G-8$	$1,2,4,5$	$G-8$
$1,2,4,6$	$G-8$	$1,2,3,4,5$	$G-8$
$1,2,3,4,6$	$G-8$	$1,2,3,5,6$	$G-8$
$1,2,3,4,5,6$	$G-8$	$1,8$	$G \times x-x$
$1,2,8$	$G \times x$	$1,4,8$	$G-2-3$
$1,2,4,8$	$G-2-3$	$1,2,5,8$	$G-2-3-4$
$1,3,5,8$	$G-8$	$1,2,3,4,8$	$G-2-3$
$1,2,3,5,8$	$G-8$	$1,2,4,5,8$	$G-2-3-4$
$1,2,4,6,8$	$G-8$	$1,2,3,4,5,8$	$G-8$
$1,2,3,4,6,8$	$G-8$	$1,2,3,5,6,8$	$G-8$
$1,2,3,4,5,6,8$	$G-8$		

Table 5.1: 1–element prime extensions of W_7.

neighbours of x	l–reduction	neighbours of x	l–reduction
$1,2$	$G \times x$	$2,4$	$G \times 56 \to (2,7)$
$1,4$	$G \times x-x$	$1,2,4$	$G \times 56 \to (1,2,7)$
$2,4,6$	$G \cong Q_3$	$1,2,5$	$G \times 34 \to (1,3,7)$
$1,2,3,5$	$G-7$	$1,2,6$	$G \times 34 \to (1,2,7)$
$1,2,4,6$	$G-7$	$1,2,3,4$	$G \times 56 \to (1,2,3,7)$
$1,2,3,4,5$	$G-7$	$2,3,5,6$	$G \times 56 \to (1,2,3,4,5,6,7)$
$2,3,4,5,6$	$G-7$	$1,2,4,7$	$G \times 23 \to (1,2)$
$1,2,3,4,5,6$	$G-7$	$1,2,5,7$	$G \times 56 \to (1,2,4)$
$1,7$	$G \times 1-1$	$2,4,6,7$	$G \times 56 \to (2,4)$
$2,7$	$G \times x-x$	$1,2,3,4,7$	$G \times 23 \to (1,2)$
$1,2,7$	$G \times x$	$2,3,5,6,7$	$G \times 56 \to (1,2,3,4,5,7)$
$1,4,7$	$G \times x-x$	$1,3,7$	$G-2$
$1,2,3,5,7$	$G-7$	$1,2,3,7$	$G-2$
$1,2,4,6,7$	$G-7$	$1,2,3,4,5,7$	$G-7$
$1,2,3,5,6,7$	$G-7$	$1,2,3,4,5,6,7$	$G-7$

Table 5.2: 1–element prime extensions of F_7.

The splitter theorem

In this section, we extend Seymour's splitter theorem [61] to even binary delta–matroids. Seymour's theorem is a fundamental step in the proof of his decomposition theorem for regular matroids.

Consider the wheel W_n. The edges incident with the hub of W_n form a spanning tree T of W_n. Then the fundamental graph of W_n, with respect to T, is a circuit of length $2n$. (The *fundamental graph* of a graph $G = (V, E)$ with respect to a spanning tree T is a bipartite graph $F = (E, S)$ with edges xy, where $x \in T$, $y \in E \setminus T$ and x is in the unique circuit of $T + y$.) We generalize "wheels" to delta–matroids. An even binary delta–matroid is called a *wheel* if it has an induced circuit as a fundamental graph. The main result of this section is the following unpublished theorem of Bouchet. An *elementary minor* of a delta–matroid M is a minor of M with precisely one fewer elements than M.

Theorem 5.17 (Bouchet) *Let M be a 3–connected, even, binary delta–matroid, and let N be a 3–connected minor of M having at least four elements. Then either M is a wheel, M is equivalent (under twisting) to N, or there exists a 3–connected elementary minor M' of M that contains a minor isomorphic to N.*

As a corollary of Theorem 5.17, we get the following result of Allys [1], which is an extension of Tutte's wheels and whirls theorem [72]. Bouchet's proof of Theorem 5.17 and Allys' proof of Corollary 5.18 are algebraic, using isotropic systems of Bouchet [5], whereas our proof is mostly graphical. Apart from the overhead of introducing isotropic systems, Bouchet's proof is shorter.

Corollary 5.18 (Allys [1]) *Let M be a 3–connected, even, binary delta–matroid with at least four elements. Then either M is a wheel, or there exists a 3–connected elementary minor of M.*

Proof Immediate by Theorem 5.4, and Theorem 5.17. □

Let Z_n be a graph with vertices $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ and edges $\{x_i y_j : i \leq j\}$. Construct Z'_n from Z_n by adding a single vertex z such that $N_{Z'_n}(z) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$. Now construct a second graph Z''_n from Z_n by adding two vertices x, y such that $N_{Z''_n}(x) = \{x_1, \ldots, x_n\}$, and $N_{Z''_n}(y) = \{y_1, \ldots, y_n\}$. (For example, Z_4, Z'_4 annd Z''_4 are depicted in Figure 5.9.)

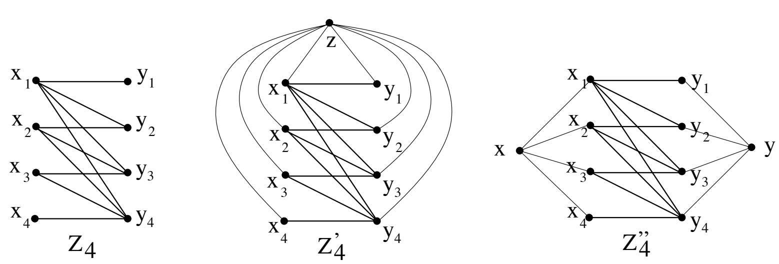

Figure 5.9:

Let $A(G) = (a_{ij})$ denote the adjacency matrix of the graph $G = (V, E)$, that is, A is the V by V symmetric binary matrix, where $a_{ij} = 1$, if and only if $ij \in E$.

Proposition 5.19 *For $n \geq 2$, the delta–matroids $M(A(Z'_n))$, and $M(A(Z''_n))$ are both wheels.*

Proof Note that, for all k, Z_k has a unique perfect matching; so, by Lemma 3.10, V_{Z_n} is a feasible set of $M(A(Z'_n))$ and $M(A(Z''_n))$. Let G, G' and G'' be, respectively, the fundamental graphs of $M(A(Z_n))$, $M(A(Z'_n))$, and $M(A(Z''_n))$ with respect to the feasible set V_{Z_n}.

Recall that $vw \in E_G$ if and only if $A(G[V_{Z_n}\Delta\{v,w\}])$ is nonsingular; that is $A(G-v-w)$ is nonsingular. We claim that G is the induced path $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. Note that, if vertices v, w are twins in a graph H, and vw is not an edge, then rows v and w are identical in $A(H)$, and hence $A(H)$ is singular. For $i > 1$, y_i and y_{i-1} are nonadjacent twins in $G - x_i$. Hence, $A(G - x_i - y_j)$ is singular unless $j = i$ or $j = i - 1$. If $j = i$ or $j = i - 1$, then $G - x_i - y_j$ is isomorphic to Z_{n-1}, so $A(G - x_i - y_j)$ is nonsingular. Therefore, for $i \geq 2$, $N_G(x_i) = \{y_i, y_{i-1}\}$. Note that y_1 is an isolated vertex in $G - x_1$, so $A(G - x_1 - y_j)$ is singular unless $j = 1$, and $G - x_1 - y_1$ is isomorphic to Z_{n-1}. Therefore $N_G(x_1) = \{y_1\}$. Similarly, for $j \leq n-1$, $N_G(y_j) = \{y_i, y_{i-1}\}$, and $N_G(y_1) = \{x_1\}$. So, G is the induced path $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, as claimed.

We now prove that G'' is the induced circuit $x_1, y_1, \ldots, x_n, y_n, x, y$. Note that $G'' - x - y = G$, so we need only find the neighbours of x and y. Note that x and y_n are twins in $Z''_n - y$, so, for all i, $A(Z''_n - y - x_i)$ is singular, and $A(Z''_n - y - y_i)$ is singular unless $i = n$. However, $Z''_n - y - y_n$ is isomorphic to Z_n, so $A(Z''_n - y - y_n)$ is nonsingular. Therefore, $N_{G''-y}(x) = \{y_n\}$. Similarly $N_{G''-x}(y) = \{x_1\}$. Note that Z''_n is prime, and G'' is obtained from Z''_n by pivoting, so G'' is prime. Thus, xy is an edge of G'', and G'' is an induced circuit. So $M(A(Z''_n))$ is a wheel.

We now prove that G' is the induced circuit $x_1, y_1, \ldots, x_n, y_n, z$. Note that $G' - z = G$, so we need only find the neighbours of z. For all i, zx_i is an edge of G' if and only if $A(Z'_n - x_i)$ is nonsingular. The set $y_1, \ldots, y_n$ is a stable set of size n, in $Z'_n - x_i$, and $Z'_n - x_i$ has $2n$ vertices. Therefore, every edge of a perfect matching of $Z'_n - x_i$ is incident with some y_i. So, no edge zx_j is contained in a perfect matching of $Z'_n - x_i$. Let H be the graph got from Z'_n by deleting the edges zx_j, for $j = 1, \ldots, n$. By the definition of the pfaffian, $A(H - x_i)$ is nonsingular if and only if $A(Z'_n - x_i)$ is nonsingular. However, z and x_1 are twins in H, so $A(H - x_i)$ is singular unless $i = 1$. $H - x_1$ is isomorphic to Z_n, so $A(H - x_1)$ is nonsingular. Therefore, $N_{G'}(z) \cap \{x_1, \ldots, x_n\} = \{x_1\}$. Similarly, $N_{G'}(z) \cap \{y_1, \ldots, y_n\} = \{y_n\}$. Therefore, G' is an induced circuit, and $M(A(Z'_n))$ is a wheel. □

Proof of Theorem 5.17. By twisting, we may assume that N is normal. Since N is a minor of M, there exist subsets X_1, X_2 of V, such that $N = M\Delta X_1 - X_2$. By twisting M we may assume that X_1 is empty. Let G be the fundamental graph of M with respect to the empty set. Let H denote $G[V \setminus X_2]$. Then H is the fundamental graph of N with respect to the empty set. Furthermore, since M and N are both 3–connected, G and H are both prime. By Theorem 5.7, there exists a nested sequence $H_1, \ldots, H_k$ of induced prime subgraphs of G, such that H_1 is isomorphic to H, $H_k = G$, and either H_{i+1} is a 1– or 2–element prime extension of H_i, or H_{i+1} is got from H_i by adding a handle. We assume that $k = 2$, since otherwise we can replace N, by $M - (V \setminus V_{H_{k-1}})$. Therefore, either G

is a 1– or 2–element prime extension of H, or G is got from H by adding a handle. We assume that G is not a 1–element prime extension of H or any graph isomorphic to H, since otherwise we are done.

Now suppose that G is not a 2–element prime extension of H. Thus, G is obtained by adding a handle $v_0, \ldots, v_{p+1}$ to H, where $p \geq 3$. We claim that $G \times v_0v_1 - v_0$ is prime, and contains an induced subgraph that is isomorphic to H. (That is, $M\Delta\{v_0, v_1\} - \{v_0\}$ is a 3–connected elementary minor of M that contains a minor isomorphic to N.) Note that $N_G(v_1) = \{v_0, v_2\}$. Let G' denote $G \times v_0v_1$. Therefore, G' is got from G by adding the edges $[\{v_2\}, N_G(v_0) - v_1]$, and then exchanging the labels v_0, v_1. Let H' denote $G'[V_H\Delta\{v_0, v_1\}]$. Then H' is isomorphic to H. So it only remains to show that $G' - v_0$ is prime. Note that $(\{v_1, v_2\}, V_{H'} \setminus \{v_1\})$ is a subsplit in G', and $v_3, \ldots, v_p$ is a blocking sequence for this subsplit. Hence, by Lemma 5.2, $G - v_0$ is indeed prime.

Now consider the case that G is a 2–element prime extension of H. Let $x_1, \ldots, x_{2k}$ be the sequence offered by Lemma 5.8, and let $X = \{x_1, \ldots, x_{2k}\}$. Let y_1 be the unique neighbour of x_2 in $V \setminus X$ and y_2 be the unique neighbour of x_{2k-1} in $V \setminus X$. If $y_1 = y_2$ then $V = X \cup \{y_1\}$ (since otherwise $(X, V \setminus X)$ would be a split in G); hence, G is isomorphic to Z_k', and M is a wheel. So we assume that $y_1 \neq y_2$. Suppose that $V = X \cup \{y_1, y_2\}$. Then, y_1y_2 is not an edge (since otherwise y_2 and x_{2k} are twins in G); so G is isomorphic to Z_k'', and M is a wheel. So,we assume that $V \neq X \cup \{y_1, y_2\}$.

Since $(X \cup \{y_2\}, V \setminus (X \cup \{y_2\}))$ is not a split in G, y_2 has a neighbour, say w, in $V \setminus (X \cup \{y_2\})$. Let G' denote $G \times wy_2$. Now, $x_1, \ldots, x_{2k}$ is a sequence of distinct vertices in G', such that, for $1 < i < 2k$, x_{i-1} and x_{i+1} are twins in $G' - x_i$, for $j < 2k$, $G' - x_j - x_{j+1}$ is isomorphic to $H \times wy_2$, and x_2 is pendent in $G - x_1$. However, x_{2k-1} has degree at least two in $G' - x_{2k}$. Therefore, by the proof of Lemma 5.8, we can extend $x_1, \ldots, x_k$ to a longer sequence $x_1, \ldots, x_{k'}$ satisfying the conclusions of Lemma 5.8. The result then follows inductively. □

Chapter 6

Regular delta–matroids

An important open problem for delta–matroids is the characterization of regular delta–matroids by excluded minors. From matroid theory we learn that a fundamental step in proving excluded minor characterizations is proving some kind of uniqueness theorem concerning representation. Indeed, this is certainly the case for regular, graphic and GF(2)– and GF(3)–representable matroids. Kahn [45] showed that 3–connected GF(4)–representable matroids have "unique" representations; however, 3–connectivity is not very tangible and, consequently, an excluded minor characterization has not been found. In this chapter, we consider representations of regular delta–matroids; the situation is remarkably similar to that of GF(4)–representable matroids. The results in this chapter were found in collaboration with Bouchet and Cunningham. We begin by recalling the situation for regular matroids.

Theorem 6.1 (Camion [16]) *If a $(0,1)$–matrix can be signed to be totally unimodular, then the signing is unique up to multiplication of certain rows and columns by -1.* □

Let A be a V by V *PU*–matrix. We can construct other PU–matrices from A; for instance, $-A$ is PU (we call this construction *negation*). Also, for $X \subseteq V$, the matrix $\left(\begin{array}{c|c} A[X] & -A[X, V\setminus X] \\ \hline -A[V\setminus X, X] & A[V\setminus X] \end{array}\right)$ is PU; this operation is called *cut–switching*. Collectively, we refer to negation and cut–switching as *switching*. Note that switching preserves symmetric and skew–symmetric matrices. We say that a regular delta–matroid M is *uniquely representable* if every two skew–symmetric PU–matrices that represent M are equivalent up to switching. The main result of this chapter is the following.

Theorem 6.2 *Every 3–connected regular delta–matroid is uniquely representable.*

We now show that the assumption of 3–connectivity in Theorem 6.2 is necessary.

Lemma 6.3 *Let A_1, A_2 be skew–symmetric PU–matrices. Then the composition of A_1 and A_2 is PU.*

Proof Immediate by Lemma 4.4, and the fact that skew–symmetric matrices of odd size have zero pfaffian. □

Let A be a V by V skew–symmetric matrix that is the composition of PU–matrices A_1 and A_2. The composition of $-A_1$ and A_2 need not be equivalent up to switching to A. Therefore a regular delta–matroid that contains a 2–separation may not be uniquely representable.

The proof of Theorem 6.2 is constructive; it provides an efficient algorithm for the following problem: *Given a binary representation A of a 3–connected regular delta–matroid, find a skew–symmetric PU–matrix A' that represents $M(A)$.* Consequently, we can efficiently recognize PU–matrices if and only if we can efficiently recognize regular delta–matroids. Indeed, suppose we have a binary matrix A, and we want to know if $M(A)$ is regular. We may assume that $M(A)$ is 3–connected. Then, by our algorithm, we can construct a real matrix A' that is PU if and only if $M(A)$ is regular. Conversely, given an integral skew–symmetric matrix B, suppose that we want to know if B is PU. Again, we may assume that $M(B)$ is 3–connected. If B is PU, then the binary matrix A equivalent to B modulo 2, is a binary representation of $M(B)$. Now we test if $M(A)$ is regular; if not, then B is not PU. So, suppose that $M(A)$ is regular. Then, by our algorithm, we construct a real matrix A' that is PU. By Theorem 6.2, B is PU if and only if A' and B are equivalent up to switching, which is easy to check.

Support graphs

The arguments in the proof of Theorem 6.2 are mainly graph theoretic, so we begin by restating the problem in terms of support graphs. The *adjacency matrix* of an oriented graph $\vec{G} = (V, \vec{E})$ is the V by V skew–symmetric $(0, \pm 1)$–matrix that has a 1 in entry i, j if and only if $ij \in \vec{E}$. A digraph $\vec{G}$ is called an *orientation* of a graph G if, for every edge vw of G, exactly one of vw and wv is an arc of $\vec{G}$, and, for nonadjacent vertices v, w of G, neither vw nor wv is an arc of $\vec{G}$. A **PU–orientation** of G is an orientation of G whose adjacency matrix is principally unimodular. For an orientation $\vec{G}$ of G, we define the operations of *negation*, *cut–switching* and *switching* for $\vec{G}$ as the result of applying the corresponding operations to the adjacency matrix of $\vec{G}$.

Counting PU–orientations

Let $G = (V, E)$ be a graph with a PU–orientation, and define $\alpha(G)$ to be the number of PU–orientations of G distinct up to cut–switching. By Theorem 6.1, if G is bipartite then $\alpha(G) = 1$; Theorem 6.2 implies that if G is prime, but not bipartite, then $\alpha(G) = 2$. In this section we describe how $\alpha(G)$ can be computed by a canonical decomposition of graphs into graphs that are either prime, bipartite, or complete.

Let $\vec{G}$ be an orientation of G, and let C be an even circuit of G. We say that $\vec{G}$ is *even* (*odd*) on C if, while traversing C in an arbitrary direction, the number of edges of C that are oriented in the forward direction by $\vec{G}$ is even (odd). Because C has an even number of edges this definition is independent of the direction in which we traverse C.

Lemma 6.4 *Let C be the circuit x_1, x_2, x_3, x_4, x_1 of a graph G, and let $\vec{G}$ be a PU–orientation of G that is odd on C. Then $G[\{x_1, x_2, x_3, x_4\}]$ is a complete graph and $\vec{G}$ is even on the circuit x_1, x_2, x_4, x_3, x_1.*

Proof This follows by an easy pfaffian calculation, which is left to the reader. □

Let (X_1, X_2) and (Y_1, Y_2) be splits of G. We say that (X_1, X_2) and (Y_1, Y_2) *cross* if $X_i \cap Y_j \neq \emptyset$ for each i, j; we call the cut (X_1, X_2) *good* if there are no cuts of G that cross (X, Y). We recursively define a *decomposition* of a graph G as follows.

- $D = \{H : H \text{ a connected component of } G\}$ is a decomposition of G,
- If D is a decomposition of G and $H \in D$ has a good split (X, Y) then $(D \setminus H) \cup \{H \circ X, H \circ Y\}$ is a decomposition of G.

We call the elements of a decomposition D the D*–components*.

Theorem 6.5 *If D is a decomposition of G then $\alpha(G) = \prod_{H \in D} \alpha(H)$.*

Proof It is clear that $\alpha(G)$ is the product, taken over all connected components H of G, of $\alpha(H)$. Thus, it is sufficient to prove that if (X, Y) is a good split of G then $\alpha(G) = \alpha(G \circ X)\alpha(G \circ Y)$. By the composition of PU–orientations of $G \circ X$ and $G \circ Y$, we have that $\alpha(G) \geq \alpha(G \circ X)\alpha(G \circ Y)$. Therefore, it suffices to show that every PU–orientation $\vec{G}$ of G is a composition of PU–orientations of $G \circ X$ and $G \circ Y$. Suppose, by way of contradiction, that $\vec{G}$ is a PU–orientation of G, and that $\vec{G}$ is not the composition of PU–orientations of $G \circ X$ and $G \circ Y$.

Let $X' = N_G(Y)$ and $Y' = N_G(X)$. Choose $x_1 \in X'$ and $y_1 \in Y'$. Then, for all $y \in Y'$ and $x \in X'$, use cut–switching so that the edge $x_1 y$ is oriented with x_1 as the tail, and the edge xy_1 is oriented with y_1 as the head in $\vec{G}$. Since $\vec{G}$ is not the composition of PU–orientations of $G \circ X$ and $G \circ Y$, there exists an edge $x_2 y_2$ of G, where $x_2 \in X'$ and $y_2 \in Y'$, that is oriented with x_2 as its head. Partition X' into sets X_1, X_2 such that $x \in X_1$ if and only if the edge xy_2 has y_2 as its head; similarly, partition Y' into sets Y_1, Y_2 such that $y \in Y_1$ if and only if the edge $x_2 y$ has y as its head.

For any $x'_i \in X_i$ and $y'_i \in Y_i$ $(i = 1, 2)$, $\vec{G}$ is odd on the circuit $x'_1, y'_1, x'_2, y'_2, x'_1$, so, by Lemma 6.4, $G[\{x'_1, x'_2, y'_1, y'_2\}]$ is a complete graph. Hence $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a split of $G[X_1 \cup X_2 \cup Y_1 \cup Y_2]$. However, since (X, Y) is a good split, there cannot exist a split (X', Y') with $X_1, Y_1 \subseteq X'$ and $X_2, Y_2 \subseteq Y'$. Then, there exists a chordless path $v_1, \ldots, v_p$ in $V \setminus (X' \cup Y')$ such that $N_G(v_i) \cap (X_1 \cup Y_1) \neq \emptyset$ if and only if $i = 1$, and $N_G(v_j) \cap (X_2 \cup Y_2) \neq \emptyset$ if and only if $j = p$. Since (X, Y) is a split in G, $\{v_1, \ldots, v_p\}$ is a subset of either X or Y; we assume, by possibly exchanging the roles of X and Y, that $\{v_1, \ldots, v_p\}$ is a subset of Y. Choose $y'_1 \in Y_1$ adjacent to v_1, and choose $y'_2 \in Y_2$ adjacent to v_p. Let H be the graph induced by $\{x_1, x_2, y'_1, y'_2, v_1, \ldots, v_p\}$; this is depicted by Figure 6.1.

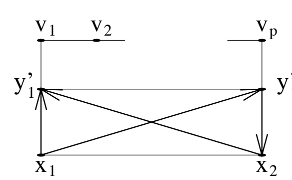

Figure 6.1: H

We assume that $p = 1$ or 2, since otherwise we shorten the path $y'_1, v_1, v_2, \ldots, v_p, y'_2$ by pivoting on $v_1 v_2$, and then deleting v_1 and v_2 from G. If $p = 1$ then $\vec{G}$ is odd on exactly one of the circuits $v_1, y'_1, x_1, y'_2, v_1$ and $v_1, y'_1, x_1, y'_2, v_1$, which, by Lemma 6.4, contradicts that v_1 is adjacent to neither x_1 nor x_2. If $p = 2$ then pivoting on $v_1 v_2$ deletes the edge $y'_1 y'_2$ while leaving $\vec{G}$ odd on the circuit $x_1, y'_1, x_2, y'_2, x_1$, contradicting Lemma 6.4. $\square$

Lemma 6.6 *For every integer n, $\alpha(K_n) = (n-1)!$, where K_n is the complete graph on n vertices.*

Proof Let $\vec{K}_n$ be a PU–orientation of K_n, and let v be any vertex of K_n. There exists a unique orientation equivalent under cut–switching to $\vec{K}_n$ with the property that every edge incident with v has v as its tail; we assume that $\vec{K}_n$ has this property.

Suppose that $\vec{K}_n$ has a directed circuit, and let $\vec{C}$ be a shortest directed circuit. $\vec{C}$ must have length 3, since otherwise there exists a chord e of $\vec{C}$ and $\vec{C} + e$ contains a directed circuit shorter than $\vec{C}$. Let X be the vertex set of $\vec{C}$. $\vec{K}_n$ is odd on every circuit of length four in $K_n[X + v]$, which contradicts Lemma 6.4. Hence $\vec{K}_n$ contains no directed circuits. We call such an orientation *transitive*.

There are $(n-1)!$ transitive orientations of $K_n - v$; thus, $\alpha(K_n) \leq (n-1)!$, with equality only if every transitive orientation of K_n is PU. Every two transitive orientations are isomorphic, so we may assume that $V_{K_n} = \{1, \ldots, n\}$, and for $1 \leq i < j \leq n$, the edge i, j is oriented with j as its head in $\vec{K}_n$. We have that $\vec{K}_3$ is PU; and, for $n > 3$, K_n is the composition of transitive orientations of two smaller complete graphs. Therefore, by Theorem 6.3 and induction, $\vec{K}_n$ is PU. □

A decomposition D is called a *total decomposition* if no D–component has a good split. A *star graph* with n vertices is a graph containing a vertex that is adjacent to n-1 vertices of degree 1. Total decompositions were introduced in [22], though our definition of the term *decomposition* differs slightly from the original.

Theorem 6.7 (Cunningham [22]) *Let G be a graph. Then*

- *All total decompositions of G are essentially the same; specifically, if D_1 and D_2 are total decompositions of G, then there exists a bijection $\pi : D_1 \to D_2$ such that, for each D_1–component H, H and $\phi(H)$ are isomorphic.*
- *If D is the total decomposition of G then every D–component is a complete graph, a star graph, or a prime graph.*
- *The total decomposition can be found in polynomial time.* □

Let D be the total decomposition of a graph G. By Theorem 6.7, every D–component H is either complete, prime or bipartite; so, assuming that G has a PU–orientation, we know $\alpha(H)$. Therefore, by Theorem 6.5, we know $\alpha(G)$ explicitly.

PU–orientations of prime graphs

In this section, we focus on proving Theorem 6.2.

Let A be a V by V binary skew–symmetric matrix. For $vw \in E_A$, let A' be obtained from $A \times vw$, by switching the labels v and w; we refer to this variation of pivoting as *partial pivoting*. Denote by $G = (V, E)$ and $G' = (V, E')$ the graphs $G(A)$ and $G(A')$.

For a pair S, S' of subsets of V, if S and S' are disjoint we have defined $[S, S'] = \{ss' : s \in S, s' \in S'\}$, for intersecting sets S, S' we define

$$[S, S'] = [S \setminus S', S' \setminus S] \cup [S \setminus S', S \cap S'] \cup [S' \setminus S, S \cap S'].$$

Then, we have

$$E' = E\Delta[N_G(u) - w, N_G(w) - u].$$

We say that G' is obtained from G by a *partial pivot* on vw.

Let $\vec{G}$ be a PU–orientation of G. A consequence of Theorem 2.7 is that partial pivoting, over the reals, on the adjacency matrix of $\vec{G}$ yields a $(0, \pm 1)$-matrix A''. Let $\vec{G}'$ be the directed graph having A'' as its adjacency matrix. Note that $\vec{G}'$ is a PU–orientation of G'. The orientation of uw is reversed by this partial pivot. The only other common edges of G and G' that may be oriented differently in $\vec{G}$ and $\vec{G}'$ are edges whose ends are both common neighbours of u and w.

Following are some results that relate pivoting operations with blocking sequences.

Lemma 6.8 *Let (X, Y) be a subsplit of G and let G' be a graph obtained by performing a pivot (or partial pivot) on an edge of $G[X]$. A sequence $v_1, \ldots, v_p$ is a blocking sequence of (X, Y) in G if and only if it is a blocking sequence of (X, Y) in G'.*

Proof Let X', Y' be disjoint subsets of V with $X \subseteq X'$ and $Y \subseteq Y'$. By Lemma 4.12, (X', Y') is a subsplit of G' if and only if it is a subsplit of G. The result follows by considering the definition of a blocking sequence. □

Lemma 6.9 *Let $v_1, \ldots, v_p$ be a blocking sequence for a subsplit (X, Y) of G, let $x \in X \cap N_G(v_1)$ and let G' be the graph obtained by performing a partial pivot on the edge xv_1 in G. Suppose that $N_G(x) \cap X \neq \emptyset$ and $N_G(x) \cap X \neq N_G(Y) \cap X$. Then*

(i) if $p = 1$, (X, Y) is not a subsplit in G', and

(ii) if $p > 1$, $v_2, \ldots, v_p$ is a blocking sequence for (X, Y) in G'.

Proof **(i)** Suppose $p = 1$. Let $X' = N_G(Y) \cap X$ and $Y' = N_G(X) \cap Y$. Then, since (X, Y) is a subsplit, $E_G[X, Y] = [X', Y']$. Therefore

$$\begin{aligned} E_{G'}[X, Y] &= (E_G \Delta[N_G(v_1) - x, N_G(x) - v_1]) \cap [X, Y] \\ &= [X', Y']\Delta[(N_G(v_1) \setminus \{x\}) \cap X, N_G(x) \cap Y]\Delta[N_G(x) \cap X, N_G(v_1) \cap Y]. \end{aligned}$$

We consider two cases; in each case we use the following fact:

> Suppose $E_{G'}[X, Y] = [X_1, Y_1]\Delta[X_2, Y_2]$ where X_1 and X_2 are distinct nonempty subsets of X, and Y_1 and Y_2 are distinct, nonempty subsets of Y. Then (X, Y) is not a subsplit in G'.

Case 1: $x \notin X'$. Then $N_G(x) \cap Y = \emptyset$, so

$$E_{G'}(X, Y) = [X', Y']\Delta[N_G(x) \cap X, N_G(v_1) \cap Y].$$

Furthermore, by the conditions of the lemma, X', $N_G(x) \cap X$ are distinct, nonempty subsets of X, and, by the definition of a blocking sequence, Y', $N_G(v_1) \cap Y$ are distinct, nonempty subsets of Y. So (X, Y) is not a subsplit in G'.

Case 2: $x \in X'$. Then $N_G(x) \cap Y = Y'$. Note that, for any sets $A \subseteq Y$, $B_1, B_2 \subseteq X$, $[A, B_1]\Delta[A, B_2] = [A, B_1 \Delta B_2]$, so

$$E_{G'}[X, Y] = [X'\Delta((N_G(v_1) \setminus \{x\}) \cap X), Y']\Delta[N_G(x) \cap X, N_G(v_1) \cap Y].$$

Now $x \in X'\Delta((N_G(v_1) \setminus \{x\}) \cap X)$, but $x \notin N_G(x) \cap X$, so $X'\Delta((N_G(v_1) \setminus \{x\}) \cap X)$, $N_G(x) \cap X$ are distinct, nonempty subsets of X. Furthermore, by the definition of a blocking sequence, Y', $N_G(v_1) \cap Y$ are distinct nonempty subsets of Y. Hence (X, Y) is not a subsplit in G'.

(ii) Suppose $p > 1$. By the minimality of a blocking sequence we have that $(X, Y \cup \{v_2\})$ is a subsplit in G. Note that v_1 is a blocking sequence for the subsplit $(X, Y \cup \{v_2\})$ in G. By part (i) of the lemma, $(X, Y \cup \{v_2\})$ is not a subsplit in G'. Also note that $(X \cup \{v_1\}, Y)$ is a subsplit in G and that $v_2, \ldots, v_p$ is a blocking sequence for $(X \cup \{v_1\}, Y)$ in G. By Lemma 6.8, $v_2, \ldots, v_p$ is also a blocking sequence for $(X \cup \{v_1\}, Y)$ in G', and, since $(X, Y \cup \{v_2\})$ is not a subsplit in G', $v_2, \ldots, v_p$ is also a blocking sequence for (X, Y) in G'. □

Sign–fixed circuits

Let C be a circuit in a graph G. We say that C is *sign–fixed* with respect to G if any two PU–orientations of G differ on an even number of edges of C. For subgraphs H_1, H_2 of G, we denote by $H_1 \Delta H_2$ the subgraph of G induced by the edges $E_{H_1} \Delta E_{H_2}$.

Lemma 6.10 *Let C be a circuit of a graph G. If there exist sign–fixed circuits $C_1, \ldots, C_k$ of G such that $C = C_1 \Delta C_2 \Delta \ldots \Delta C_k$, then C is sign–fixed in G.*

Proof Let $\vec{G}_1, \vec{G}_2$ be any pair of PU–orientations of G. Let S be the set of edges of G in which the orientations $\vec{G}_1$ and $\vec{G}_2$ differ. For each sign–fixed circuit C_i, $|C_i \cap S|$ is even. Now

$$\begin{aligned} C \cap S &= (C_1 \Delta \ldots \Delta C_k) \cap S \\ &= (C_1 \cap S)\Delta \ldots \Delta(C_k \cap S). \end{aligned}$$

Since $C \cap S$ can be represented as the symmetric difference of even sets, $C \cap S$ has even cardinality. Hence C is sign–fixed in G. □

The following lemma is attributed to Bondy in [42]; it can be proved using Menger's theorem.

Lemma 6.11 *Let H be an Eulerian subgraph of a 2–vertex–connected graph G. If H has an even number of edges, then there exist even circuits $C_1, \ldots, C_k$ of G such that $H = C_1 \Delta C_2 \Delta \ldots \Delta C_k$.* □

Lemma 6.12 *If G is prime and every even circuit of G is sign–fixed, then all PU–orientations of G are switching–equivalent.*

Proof Trivially we may assume that G has at least 4 vertices. Note that every prime graph with at least 4 vertices is 2–vertex–connected. Let $\vec{G}_1, \vec{G}_2$ be PU–orientations of G.
Claim *We may assume, without loss of generality, that for every circuit C' of G the orientations $\vec{G}_1$ and $\vec{G}_2$ differ on an even number of edges of C'.*

Proof of claim By the premise of the lemma, the claim is true for even circuits. Let C be an odd circuit of G. We may assume that the orientations $\vec{G}_1$ and $\vec{G}_2$ differ on an even number of edges of C; otherwise we reverse the orientation $\vec{G}_2$.

Consider any odd circuit C' of G. By Lemma 6.11, there exist even circuits $C_1, \ldots, C_k$ such that $C' \Delta C = C_1 \Delta \ldots \Delta C_k$, therefore $C' = C \Delta C_1 \Delta \ldots \Delta C_k$. It follows similarly to the the proof of Lemma 6.10, that the orientations $\vec{G}_1$ and $\vec{G}_2$ differ on an even number of edges of C'. Which proves the claim.

Let S be the set of edges upon which the orientations $\vec{G}_1$ and $\vec{G}_2$ differ. It follows from the claim that if we contract each of the edges in $E_G \setminus S$, then we obtain a bipartite graph. Therefore the edges S form a cut in G. Hence $\vec{G}_1$ and $\vec{G}_2$ are equivalent under cut–switching. □

Lemma 6.12 generalizes the ideas used in Seymour's proof of Theorem 6.1. Following is a summary of Seymour's proof. Suppose C is a circuit of a bipartite graph G. If C is chordless then it is easy to show that C is sign–fixed. Otherwise, if C has a chord, then C can be expressed as the symmetric difference of two shorter circuits, so inductively we can prove that C is sign–fixed. Then, by Lemma 6.12, all PU–orientations of G are switching–equivalent.

Decomposition of circuits

In this section we show that some even circuits can be expressed as the symmetric difference of shorter even circuits.

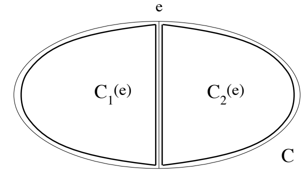

Figure 6.2: $C + e$

Let C be an even circuit and let e be a chord of C. C can be expressed as the symmetric difference of two shorter circuits (see Figure 6.2) denoted $C_1(e), C_2(e)$ (in no particular order). Since C is even, $C_1(e)$ and $C_2(e)$ are either both even or both odd. We say that e is an *even* (*odd*) chord of C if $C_1(e)$ and $C_2(e)$ are both even (odd).

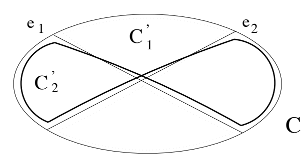

Figure 6.3: Decomposition of $C + e_1 + e_2$

Let e_1 and e_2 be odd chords of an even circuit C. We say that e_1 and e_2 *cross* if e_1 and e_2 have disjoint ends and e_2 has exactly one end in $C_1(e_1)$. If e_1 and e_2 are crossing then define $C_1' = C_1(e_1)\Delta C_1(e_2)$, and $C_2' = C_1(e_1)\Delta C_2(e_2)$; see Figure 6.3. C_1' and C_2' are both even circuits and

$$\begin{aligned} C_1'\Delta C_2' &= (C_1(e_1)\Delta C_1(e_1))\Delta(C_1(e_1)\Delta C_2(e_2)) \\ &= C_1(e_2)\Delta C_2(e_2) \\ &= C. \end{aligned}$$

If either C_1' or C_2' has length 4 then the other has the same length as C; otherwise both C_1' and C_2' are shorter than C. We say that e_1 and e_2 are *tight* crossing chords if either C_1' or C_2' has length 4.

Note that it is not possible to have three odd chords of a circuit such that each pair is a tight crossing pair, so if we have any three mutually crossing odd chords of a circuit C, we can apply one of the above decompositions to express C as the symmetric difference of two shorter even circuits.

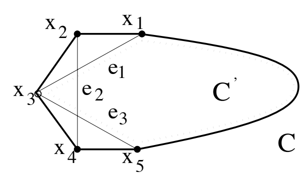

Figure 6.4: $C + e_1 + e_2 + e_3$

In the third decomposition we have three odd chords e_1, e_2 and e_3 of an even circuit C such that $\{e_1, e_2\}$ and $\{e_2, e_3\}$ are pairs of tight crossing chords, and e_1 and e_3 do not cross. In this situation there are consecutive vertices $x_1, \dots, x_5$ in C such that e_1, e_2 and e_3 have ends $\{x_1, x_3\}$, $\{x_2, x_4\}$ and $\{x_3, x_5\}$ respectively, as depicted in Figure 6.4. Also depicted in Figure 6.4 is an even circuit C'; C is the symmetric difference of C' and the two circuits x_1, x_2, x_4, x_3, x_1 and x_5, x_4, x_2, x_3, x_5. Furthermore, each of these circuits is even and shorter than C.

A circuit is said to be *decomposable* (otherwise *indecomposable*) if by one of the above decompositions we can express C as the symmetric difference of shorter even circuits. More rigorously, an even circuit C is indecomposable if the chords of C are all odd, each chord crosses at most one other chord and all crossings are tight.

PU–orientations of prime graphs

We now prove the main result of the chapter.

Proof of Theorem 6.2. By Lemma 6.12, it suffices to show that in a prime graph all even circuits are sign-fixed. We prove this by induction on the length of an even circuit. Let $k \geq 4$ be an even integer. We assume that in every prime graph every even circuit of length less than k is sign-fixed.

Let C' be a circuit of length k in a prime graph G'. If C' can be expressed as the symmetric difference of sign-fixed circuits in G' then, by Lemma 6.12, C' is sign-fixed. In particular, if C' is decomposable then C' is sign-fixed.

Claim 1 *Let C be a circuit of length k in a prime graph G. If there exists a vertex that has degree 2 in $G[V_C]$ then C is sign-fixed.*

Proof of claim In the case that C has length 4, the claim follows from Lemma 6.4. Now suppose that $k > 4$ and that C is indecomposable. Let v be a vertex of degree 2 in $G[V_C]$, let u, w be the neighbours of v in $G[V_C]$ and let G' be the graph obtained be performing a partial pivot on vw in G.

Let $u'u$ and ww' be the edges other than uv and uw incident to u and w respectively in C. Note that u' is not adjacent to w in G since such an edge would be an even chord of C, and similarly u is not adjacent to w'. We have that $N_{G[V_C]}(v) \setminus \{w\} = \{v\}$, so

$$E_{G'}[V_C] = E_G[V_C]\Delta[\{u\}, N_{G[V_C]}(w) \setminus \{u, v\}].$$

Therefore the partial pivot affects only edges incident with u. But the edges uu' and uv are unaffected by the partial pivot, so C is a circuit in G'. Furthermore, if the partial pivot were performed on any orientation of G, then exactly one edge of C, namely vw, will be reoriented. So C is sign-fixed in G if and only if C is sign-fixed in G'. Now uw' is an edge of G', so C has an even chord in G'. Hence C is sign-fixed in G'. This proves Claim 1.

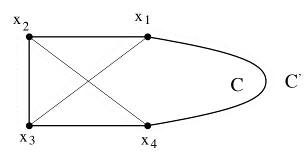

Figure 6.5: Circuits in Claim 2.

Claim 2 *Let C be a circuit of length k in a prime graph G, and suppose $x_1, \ldots, x_4$ are consecutive vertices of C such that x_1x_3 and x_2x_4 are chords of C. Finally let C' be the symmetric difference of C and the circuit x_1, x_3, x_4, x_2, x_1 (see Figure 6.5). Then at least one of C and C' is sign-fixed.*

Proof of claim The claim is trivially true when C is decomposable, so suppose that C is indecomposable. Let $X = \{x_2, x_3\}$ and $Y = V_C \setminus X$, and let e_1 and e_2 be the edges x_1x_3 and x_2x_4 respectively. Note that e_1 and e_2 are crossing chords of C, so there are no other chords which cross either e_1 or e_2. Hence (X, Y) is a subsplit of G; let $v_1, \ldots, v_p$ be a blocking sequence for this subsplit. We prove the claim by induction on the length of the blocking sequence.

Case 1: $p = 1$. v_1 is a blocking sequence for the subsplit (X, Y) in G. Then v_1 is adjacent to exactly one of x_2 and x_3. Assume with no loss of generality that v_1 is adjacent to x_2. v_1 must also be adjacent to some vertex in Y. This gives rise to two subcases.

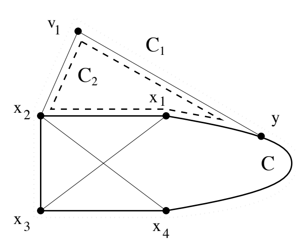

Figure 6.6: Decomposition in Case 1.1.

Case 1.1: v_1 is adjacent to a vertex y in $Y \setminus \{x_1, x_4\}$. We assume that x_2 and y are an even distance apart in C (otherwise x_2 and y are an even distance apart in C' and we can interchange the roles of C and C'). Consider the circuits C_1 and C_2 defined by Figure 6.6. C_1 and C_2 are both even and have length at most k. x_3 and x_2 have degree 2 in $G[V_{C_1}]$ and $G[V_{C_2}]$ respectively, so by Claim 1 C_1 and C_2 are both sign-fixed. Furthermore C is the symmetric difference of C_1 and C_2 so C is also sign-fixed. Thus proving Claim 2 in Case 1.1.

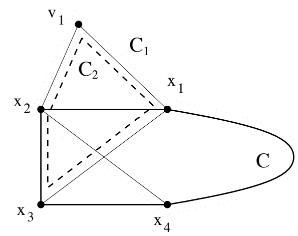

Figure 6.7: Decomposition in Case 1.2.

Case 1.2: v_1 is not adjacent to any vertices in $Y \setminus \{x_1, x_4\}$. In this Case v_1 cannot be adjacent to both v_1 and v_4 since otherwise $(X \cup \{v_1\}, Y)$ would be a subsplit, contradicting Lemma 5.1. So v_1 is adjacent to exactly one of x_1 and x_4; we assume that v_1 is adjacent to x_1 (the other case is equivalent under interchanging the roles of C and C' and changing labels). Consider the even circuits C_1 and C_2 defined by Figure 6.7. v_1 has degree 2 in both $G[V_{C_1}]$ and $G[V_{C_2}]$, so by Claim 1, C_1 and C_2 are both sign-fixed. C' is the symmetric difference of C_1 and C_2 so C' is also sign-fixed. This completes the proof of Claim 2 in Case 1.

Case 2: $p > 1$. As with Case 1, v_1 is adjacent to exactly one of x_2 and x_3, and we assume with no loss of generality that x_2 and v_1 are adjacent. $(X \cup \{v_1\}, Y)$ is a subsplit, so either $N_G(v_1) \cap Y = \emptyset$ or $N_G(v_1) \cap Y = N_G(X) \cap Y = \{x_1, x_4\}$. This gives two subcases.

Case 2.1: $N_G(v_1) \cap Y = \emptyset$. Let G' be the graph defined by performing a partial pivot on the edge x_2v_1. Note that $N_G(v_1) \cap V_C = \{x_2\}$, so $G[V_C] = G'[V_C]$. Then C and C' are circuits in G' and, by considering the effect of this partial pivot on an orientation of G, C and C' are sign-fixed in G if and only if they are sign-fixed in G'. Now, by Lemma 6.9, $v_2, \ldots, v_p$ is a blocking sequence for the subsplit (X, Y) in G'; so, by the induction hypothesis of the claim, one of C and C' is sign-fixed in G'.

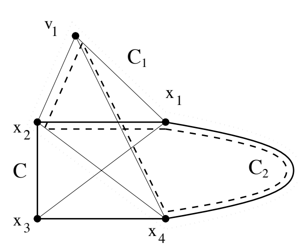

Figure 6.8: Decomposition in Case 2.2.

Case 2.2: $N_G(v_1) \cap Y = \{x_1, x_4\}$. We have that $v_2, \ldots, v_p$ is a blocking sequence for the subsplit $(X \cup \{v_1\}, Y)$. Furthermore, for $i > 1$, $(X, Y \cup \{v_i\})$ is a subsplit; it follows that v_i is adjacent with x_2 if and only if v_i is adjacent with x_3. Consequently $v_2, \ldots, v_p$ is a blocking sequence for the subsplit $(\{x_2, v_1\}, Y)$. Now, by the induction hypothesis of the claim, one of the circuits C_1 or C_2, defined in Figure 6.8, is sign-fixed. Let C_1' and C_2' be the circuits v_1, x_1, x_3, x_2, v_1 and v_1, x_4, x_3, x_2, v_1 respectively. C_1' and C_2' are both sign-fixed by Claim 1. If C_1 is sign-fixed then C', which is the symmetric difference of C_1 and C_1', is sign-fixed. Otherwise C_2 is sign-fixed; then C, which is the symmetric difference of C_2 and C_2', is sign-fixed. In either case we have proved Claim 2.

The proof is now completed by settling two final cases.

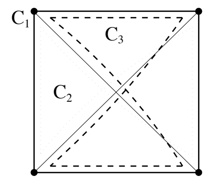

Figure 6.9: Decomposition when $k = 4$.

Case 1: $k = 4$. Let C_1 be a circuit of length 4 in a prime graph G. If $G[V_{C_1}]$ is not complete then $G[V_{C_1}]$ contains a vertex of degree 2; so, by Claim 1, C_1 is sign-fixed. So suppose that $G[V_{C_1}]$ is complete. Let C_2 and C_3 be defined by Figure 6.9. By Claim 2, one of C_1 and C_2 are sign-fixed. If C_1 is sign-fixed we are done, so suppose C_2 is sign-fixed. Similarly one of C_1 and C_3 are sign-fixed, so suppose C_3 is sign-fixed. However C_1 is the symmetric difference of C_2 and C_3, so C_1 is sign-fixed.

Case 2: $k > 4$. Let C be a circuit of length k in a prime graph G. If C is decomposable or if $G[V_C]$ contains a vertex of degree 2 then C is sign-fixed. Suppose then that C is indecomposable and that every vertex in $G[V_C]$ has degree at least 3. Let e be a chord of C such that the distance in C between the ends of e is minimum among all chords of C. Let $y_1, \ldots, y_r$ be the internal vertices of a shortest path in C between the ends of e. Since each vertex in V_C has degree at least 3 in $G[V_C]$, each y_i must subtend at least one chord of C; let e_i be a chord having y_i as an end. The distance in C between the ends of e_i is at least the distance between the ends of e in C, so e_i must cross e. Since C is indecomposable, there is at most one chord crossing e; therefore $r = 1$. Furthermore e_1 and e must be a tight crossing pair, so the other end of e_1 must also be adjacent to an end of e in C. Therefore there are consecutive vertices x_1, x_2, x_3, x_4 of C such that x_1 and x_3 are the ends of e, and x_2 and x_4 are the ends of e_1. Let C' be the circuit x_1, x_2, x_4, x_3, x_1; C' is sign-fixed since it has length 4. By Claim 2 at least one of C and $C \Delta C'$ is sign-fixed. If C is sign-fixed we are done. Otherwise $C \Delta C'$ is sign-fixed, so C (which is the symmetric difference of $C \Delta C'$ and C') is also sign-fixed. This completes the proof. □

Partial results

One of the more important open problems for delta–matroids is to characterize regular delta–matroids by excluded minors; this would generalize Tutte's characterization of regular matroids.

Theorem 6.13 (Tutte [71]) *Let M be a binary matroid, and let F_7 be the binary matroid $M(A)$, where A is depicted graphically in figure 6.10. Then M is regular if and only if M does not have a minor isomorphic to F_7.* □

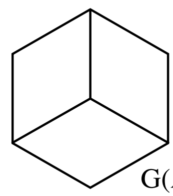

Figure 6.10: Fano matroid

We call a binary delta–matroid M an *obstruction* if M is minimally non–regular with respect to taking normal minors. Since the family of twisted matroids is closed under taking minors, F_7 is an obstruction. We have seen two other obstructions in relation to circle graphs, namely, $M(A_1)$ and $M(A_2)$, where A_1 and A_2 are depicted graphically in Figure 6.11. We obtain other excluded minors by the following proposition.

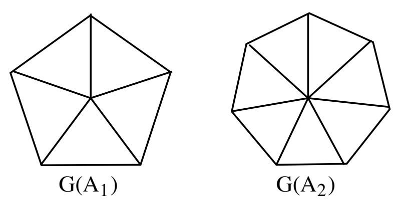

Figure 6.11: Obstructions

Proposition 6.14 *Let A be a V by V skew–symmetric matrix, and X be a subset of V, such that $A[X]$ is identically zero. Then define*

$$B = \left(\begin{array}{c|c} 0 & A[X, V \setminus X] \\ \hline A[V \setminus X, X] & 0 \end{array}\right).$$

If A is PU then B is PU.

Proof Suppose that B is not PU; then there exists $S \subseteq V$ such that $pf(B[S]) \neq 0, \pm 1$. In particular $pf(B[S]) \neq 0$, so $G(B[S])$ has a perfect matching; hence, $|X \cap S| = |S \setminus X|$. Then, since $S \cap X$ is a stable set of $G(A[S])$ and $G(B[S])$, $G(A[S])$ and $G(B[S])$ share the same set of perfect matchings. Consequently, $pf(A[S]) = pf(B[S])$, so A is not PU. □

By Proposition 6.14, the delta–matroids $M(A_3), \ldots, M(A_6)$, where A_3, A_4, A_5, A_6 are depicted in Figure 6.12, are not regular; they are, in fact, obstructions. Furthermore, they are the only obstructions that arise from Proposition 6.14. We pray that $M(A_1), \ldots, M(A_6)$ and F_7 are the only obstructions for the class of regular delta–matroids.

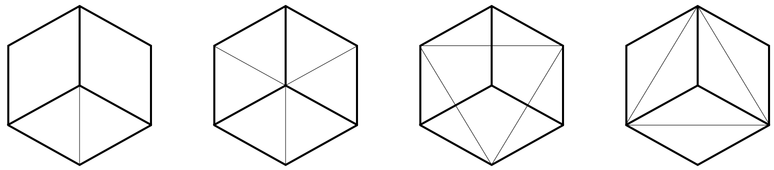

Figure 6.12: More obstructions

Seymour's decomposition

Seymour [61, 56] proved that every regular matroid could be obtained from a natural class of regular matroids by 1– 2– and 3–sums. The natural class of regular matroids consists of graphic matroids and R_{10}, and the duals of such matroids. Here R_{10} is the matroid whose fundamental graph is depicted in Figure 6.13.

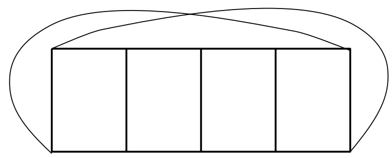

Figure 6.13: R_{10}

Perhaps Seymour's decomposition extends to regular delta–matroids; we have many key ingredients in obtaining such a decomposition. We have seen that regular delta–matroids are closed under 1– and 2–sums, and under taking minors. (The situation is not yet clear regarding "3-sums".) We also have a "splitter theorem" for binary delta–matroids (Theorem 5.17), which is fundamental in the proof of Seymour's decomposition. Also, we have seen a nice class of regular delta–matroids, namely the Eulerian delta–matroids; we even have a recognition algorithm for the class. It is perhaps a little discouraging that the class of Eulerian delta–matroids does not contain the normal twisted graphic (or cographic) matroids. Also, it would be helpful to have excluded minor characterizations of regular and Eulerian delta–matroids. However, the greatest obstacle is a class of regular delta–matroids obtained by the following theorem; this class contains highly connected members, and is not closed under pivoting.

Theorem 6.15 *Let G be a bipartite graph, and let x_1, x_2 be vertices of G that are in different colour classes of G. Now define a graph G' by shrinking x_1 and x_2 to a single vertex x. Then, G' has a PU–orientation if and only if both $G - x_1$ and $G - x_2$ have PU–orientations.*

Proof Suppose that $G - x_1$ and $G - x_2$ have PU–orientations. By Theorem 6.1, these orientations are equivalent up to switching, on $G - x_1 - x_2$. So there exists an orientation $\vec{G}$ of G such that both $\vec{G} - x_1$ and $\vec{G} - x_2$ are PU. Let $\vec{G}'$ be the orientation of G' obtained by identifying x_1 and x_2 in $\vec{G}$. We claim that $\vec{G}'$ is a PU–orientation. For $X \subseteq V$, if $x \notin X$ then it is clear that the adjacency matrix of $\vec{G}'[X]$ is unimodular. We assume that $x \in X$, we also assume that X has even cardinality, since otherwise the adjacency matrix of $\vec{G}'[X]$ is singular. Let $\mathcal{M}'$ be the set of perfect matchings of $G'[X]$, and let $\mathcal{M}$ be the corresponding matchings in G. Let X' be the larger colour class of the bipartite graph $G'[X]-x$, since $|X|$ is even, $|X'| \geq |X|/2$. Hence either $\mathcal{M}$ is the set of perfect matchings of $G[X\Delta\{x, x_1\}]$, or $\mathcal{M}$ is the set of perfect matchings of $G[X\Delta\{x, x_2\}]$. By considering the pfaffian of the adjacency matrix of $\vec{G}'[X]$ we find that $\vec{G}'$ is a PU–orientation.

The converse follows from Proposition 6.14. □

Chapter 7

Equable delta–matroids

We call a delta–matroid *equable* if it is representable by a symmetric $(0, \pm 1)$ PU–matrix. Analogous to regular matroids and regular delta–matroids, equable delta–matroids are precisely the delta–matroids representable over every field by a symmetric matrix.

Theorem 7.1 *Let $M = (V, \mathcal{F})$ be a delta–matroid. The following are equivalent.*

(i) M is equable,

(ii) M can be represented over every field by a symmetric matrix, and

(iii) M can be represented over both $GF(2)$ and $GF(3)$ by a symmetric matrix.

Proof That (i) implies (ii), and that (ii) implies (iii) are both easy. So it suffices to prove that (iii) implies (i). Let $A^{(2)}$ and $A^{(3)}$ be representations of M over $GF(2)$ and $GF(3)$ respectively. Therefore $A^{(2)}$ and $A^{(3)}$ have the same support (that is, nonzero elements), so there exists a real $(0, \pm 1)$–matrix $A = (a_{ij})$ that is equivalent to $A^{(3)}$ modulo 3, and to $A^{(2)}$ modulo 2. We claim that A is PU. Suppose not, and let $S \subseteq V$ be minimal such that $A[S]$ is not unimodular.

Claim *We may assume that $|S| = 3$, or $|S| = 4$ and $A[S]$ has a zero diagonal.*

Suppose the assumption is not satisfied. Then there exists $S' \subseteq S$ such that $0 < |S'| \leq |S| - 3$, and $A[S']$ is nonsingular. Then $A[S']$ is unimodular, so, by Theorem 2.7, for $X \subseteq V$, $\det(A * S'[X]) = \pm \det(A[X \Delta S'])$. Hence, $A * S'$ is a $(0, \pm 1)$–matrix that represents the delta–matroid $(V, \mathcal{F} \Delta S')$ over $GF(2)$ and $GF(3)$, and $A * S'[S \setminus S']$ is minimally non–unimodular. Now replace S by $S \setminus S'$, A by $A * S'$, and M by $(V, \mathcal{F} \Delta S')$. Inductively we will satisfy the claim.

Let k be the $0, \pm 1$ value equivalent to $\det(A[S])$ modulo 3. Note that $\det(A[S]) \equiv \det(A^{(2)}[S]) \equiv k$ modulo 2, and hence $\det(A[S]) \equiv k$ modulo 6. However $\det(A[S]) \neq k$, so $|\det(A[S])| \geq 5$.

Suppose that $|S| = 3$. We may assume that $\det(A[S]) \geq 0$, since otherwise we replace A by $-A$. Now

$$\det(A[S]) = a_{11}a_{22}a_{33} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 + 2a_{12}a_{13}a_{23}.$$

Hence, since $\det(A[S]) \geq 5$, no element of A can be zero. Then

$$\det(A[S]) = a_{11}a_{22}a_{33} - a_{11} - a_{22} - a_{33} + 2a_{12}a_{13}a_{23},$$

so, since $\det(A[S]) \geq 5$, $a_{ii} = -1$, for $i = 1, 2, 3$. Then

$$\det(A[S]) = 2a_{12}a_{13}a_{23} + 2 < 5,$$

which is a contradiction.

Therefore $|S| = 4$ and $A[S]$ has a zero diagonal. Then

$$\begin{aligned} \det(A[S]) &= (a_{12}a_{34})^2 + (a_{13}a_{24})^2 + (a_{14}a_{23})^2 \\ &\quad -2(a_{12}a_{23}a_{34}a_{14} + a_{12}a_{24}a_{34}a_{13} + a_{13}a_{23}a_{24}a_{14}). \end{aligned}$$

Therefore, since $|\det(A[S])| \geq 5$, $a_{ij} \neq 0$ for $1 \leq i < j \leq 4$. However, this implies that $\det(A[\{1, 2, 3\}]) = 2a_{12}a_{23}a_{13} = \pm 2$, contradicting the minimality of S. □

The main result of this chapter is the generalization of Tutte's excluded minor characterization of regular matroids [68].

Theorem 7.2 *Let M be a binary delta–matroid. Then M is equable if and only if M does not have a minor isomorphic to one of the following binary delta–matroids $M(B_1), \ldots, M(B_5)$, where $B_1, \ldots, B_5$ are defined in Figure 7.1.*

$$\underset{B_1}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \quad \underset{B_2}{\left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right)} \quad \underset{B_3}{\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)} \quad \underset{B_4}{\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)}$$

$$\underset{B_5}{\left(\begin{array}{ccc|cccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)}$$

Figure 7.1: Excluded minors

Figure 7.2 depicts the matrices $B_1, \ldots, B_5$ graphically; we have depicted the loop–vertices in bold, though they are not distinguished by the support graph. Note that, with Theorem 4.7, we have a complete excluded minor characterization of equable delta–matroids. Equable delta–matroids are preserved under deletion and, by Theorem 2.7, twisting by a feasible set. Therefore, proving that $M(B_1), \ldots, M(B_5)$ are not equable proves Theorem 7.2, in the easy direction; this is left to the reader.

Recall that twisted matroids are preserved under taking minors. Therefore, as a corollary of Theorem 7.2, we obtain Tutte's excluded minor characterization of totally unimodular matrices.

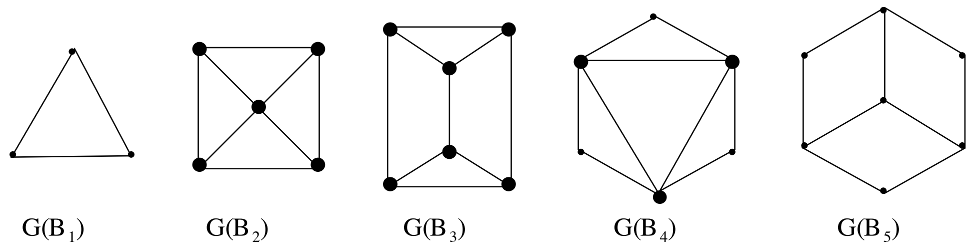

Figure 7.2: Support graphs

Corollary 7.3 (Tutte [68, 71]) *Let M be a binary delta–matroid. Then, M is regular if and only if M does not contain a minor isomorphic to $M(B_5)$.* □

To prove our result, we consider the class of binary delta–matroids that do not contain $M(B_1)$ as a minor, and then we use a theorem of Truemper [65] on beta–balanced matrices which gives us the general form of the matrices that do not admit PU–signings. Our original proof of Theorem 7.2 generalized Gerards' short proof [38] of Tutte's theorem. By using Truemper's theorem we simplify the final case analysis.

We restate the problem directly in terms of matrices. Let A be a V by V symmetric binary matrix. A V by V symmetric $(0, \pm 1)$–matrix A' is a referred to as a *signing* of A if A and A' have the same support. A signing that is PU is referred to as a *PU–signing.* Thus, $M(A)$ is equable if and only if A admits a PU–signing. Given symmetric binary matrices A and B, we say that A *reduces to* B if $M(B)$ is a minor of $M(A)$; that is, B is a principal submatrix of a matrix equivalent to A under binary pivoting.

We use the following notation. For a graph $G = (V, E)$ we denote by $G - v$ the graph $G[V \setminus \{v\}]$. Similarly, for a V by V matrix A, we denote by $A - v$ the matrix $A[V \setminus \{v\}]$.

Beta–balancedness

Let G be a graph. A *signing* of G is an assignment of ± 1 to the edges of G. Suppose that, for every chordless circuit C of G, we assign a $\{0, 1\}$ value β_C to C. A *β–balanced signing* of G is a signing with the property that, for every chordless circuit C, the number of edges of C signed $+1$ is equivalent to β_C modulo 2.

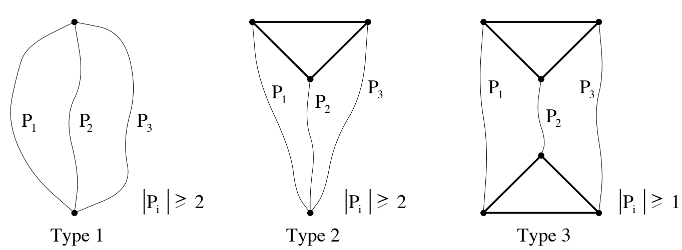

Figure 7.3: Three–path configurations

We now define two interesting classes of graphs. A *three–path configuration* is a graph of the form described in Figure 7.3, where P_i is a path of length $|P_i|$, $i = 1, 2, 3$. The second class of graphs consists of the partial wheels; a graph G is a *partial wheel* with *hub* v if v is a vertex of G and $G - v$ is a circuit. We call a partial wheel *proper* if the hub has degree at least 3. The following remarkable result is due to Truemper [65].

Theorem 7.4 *Let G be a graph with a $\{0,1\}$ value β_C assigned to every induced circuit C of G. If G has no β–balanced signing then G contains an induced subgraph that is either a proper partial wheel or a three–path configuration, and which has no β–balanced signing.* □

Loop–balanced signings

In this section we show that to find a PU–signing of a matrix, we can sign the diagonal without knowing the signs of the nondiagonal entries. Let A be a symmetric binary matrix. For a path P of $G(A)$ we denote by $\kappa_A(P)$ the number of nonloop–vertices of P. A signing $A' = (a'_{ij})$ of A is called *loop–balanced* if, for every pair of loop–vertices v, w and every chordless (v,w)–path P, $a'_{vv} = (-1)^{\kappa_A(P)} a'_{ww}$. If $G(A)$ is connected then any two loop–balanced signings of A sign the loop–vertices equivalently under negation.

Lemma 7.5 *Let A be a symmetric $(0, \pm 1)$–matrix such that $G(A)$ is a path. A is PU if and only if A is loop–balanced.*

Proof If A has a zero diagonal then by an elementary determinant calculation we find that A is PU. Let v be a loop–vertex of A. If $A * v$ is not a $(0, \pm 1)$–matrix then A is neither loop–balanced nor PU. If $A * v$ is a $(0, \pm 1)$–matrix then $G(A * v) - v$ is a path; furthermore $A * v - v$ is loop-balanced if and only if A is loop–balanced. Hence the result follows inductively. □

The following lemma is an immediate consequence of Lemma 7.5.

Lemma 7.6 *Every PU–signing of a symmetric binary matrix is loop–balanced.* □

Lemma 7.7 *Let A be a symmetric binary matrix. If A has no loop–balanced signing then A reduces to B_1.*

Proof Suppose A has no loop–balanced signing. We begin by proving the result in the special case that $G(A)$ is a circuit.

Claim *If $G(A)$ is a circuit then A can be reduced to B_1.*

Let $G(A)$ be a circuit. Then A has *no* loop–balanced signing if and only if the following conditions are satisfied:

(i) A has an odd number of nonloop–vertices, and

(ii) there exist two loop–vertices that are not adjacent in $G(A)$.

We prove the result by induction on the size of A. By (ii), if A has size 3 then A has a loop–balanced signing. Suppose that A has size 4. By (i) and (ii), A has exactly three loop–vertices; let v be a loop–vertex whose neighbours in $G(A)$ are both loop–vertices. Then $(A \times v) - v$ is isomorphic to B_1.

Now suppose that A has size at least 5. By (ii), there exist two loop–vertices that are not adjacent in $G(A)$, and, by (i), A has at least one nonloop–vertex. Then, since A has size at least 5, there exist vertices v, v', w such that v, w are loop–vertices that are not adjacent in $G(A)$, and v' is a nonloop–vertex that is adjacent in $G(A)$ to v but not w. Note that $G(A \times v) - v$ is a circuit, and $A \times v - v$ has an odd number of nonloop–vertices. Furthermore, v', w are loop–vertices of $A * v$ that are not adjacent in $G(A \times v) - v$; hence $(A \times v) - v$ has no loop–balanced signing. Then, by induction, $(A \times v) - v$ reduces to B_1, so A reduces to B_1, which proves the claim.

We now suppose that there exist loop–vertices v, w and a pair of chordless (v, w)–paths, $P_1 = v, x_1, \ldots, x_a, w$, and $P_2 = v, y_1, \ldots, y_b, w$, of $G(A)$ such that $\kappa_A(P_1) + \kappa_A(P_2)$ is odd. Furthermore, we suppose that the paths P_1 and P_2 are chosen so that $|V(P_1) \cup V(P_2)|$ is as small as possible.

Note that in $G(A \times v)$, P_1 and P_2 are chordless (v, w)–paths, and $\kappa_{A\times v}(P_1) + \kappa_{A\times v}(P_2)$ is odd. Hence $A \times v$ is not loop–balanceable. Similarly, $A \times w$ is not loop–balanceable.

Suppose that $x_1 = y_1$. We may assume, in this case, that x_1 is a loop–vertex, for otherwise we can pivot on v. Now define $P_1' = x_1, \ldots, x_a, w$ and $P_2' = y_1, \ldots, y_b, w$; P_1' and P_2' are chordless (x_1, w)–paths such that $\kappa_A(P_1') + \kappa_A(P_2')$ is odd, and $\left|V_{P_1'} \cup V_{P_2'}\right| < |V_{P_1} \cup V_{P_2}|$, which is a contradiction. Hence, we may assume that $x_1 \neq y_1$; similarly we may assume that $x_a \neq y_b$. We may also assume that $x_1 y_1$ is not an edge, since otherwise pivoting on v would remove it. Similarly, we may assume that $x_a y_b$ is not an edge.

If $v, x_1, x_2, \ldots, x_a, w, y_b, y_{b-1}, \ldots, y_1$ is a chordless circuit then, by the claim, we can reduce A to B_1. Hence we may assume that there exists an edge $x_i y_j$ in $G(A)$. Let i be minimum such that x_i is adjacent to some y_j, and let j be maximum such that y_j is adjacent to x_i. Let P be the path $v, x_1, \ldots, x_i, y_j, \ldots, y_b, w$; note that P is chordless. Now let P' be one of P_1, P_2 such that $\kappa_A(P') \not\equiv \kappa_A(P)$ modulo 2. However, $|V(P) \cup V(P')| < |V(P_1) \cup V(P_2)|$. Hence we have a contradiction to the choice of P_1, P_2.

Therefore, for every pair of loop–vertices v, w, and every pair of chordless (v, w)–paths P_1, P_2, we have $\kappa_A(P_1) \equiv \kappa_A(P_2)$ modulo 2; denote by $\kappa(v, w)$ the value $\kappa_A(P_1)$. We may assume that $G(A)$ is connected, so $\kappa(v, w)$ is well defined modulo 2, for every pair v, w of loop–vertices. Let x_1 be a loop–vertex of A. Define a signing $A' = (a_{ij})$ of A such that $a'_{x_1 x_1} = +1$ and, for every other loop–vertex v of A, $a'_{vv} = (-1)^{\kappa(v, x_1)}$. Since A has no loop–balanced signing, A' is not loop–balanced, so there exist loop–vertices x_2, x_3 such that $a'_{x_2 x_2} \neq (-1)^{\kappa(x_2, x_3)} a'_{x_3 x_3}$. Therefore $\kappa(x_2, x_3) + \kappa(x_1, x_3) + \kappa(x_1, x_2)$ is odd.

Let X be a minimal subset of V containing x_1, x_2, x_3, such that $G(A[X])$ is connected. For each i, j, let P_{ij} be a chordless (x_i, x_j)–path in $G(A[X])$. The union of any two of the paths P_{12}, P_{23}, P_{13} yields a connected graph containing the vertices x_1, x_2, x_3. Therefore, by the minimality of X, each $x \in X$, is contained in at least two of the paths P_{12}, P_{23}, P_{13}. However, since $\kappa_A(P_{12}) + \kappa_A(P_{13}) + \kappa_A(P_{23})$ is odd, there must exist a nonloop–vertex x that is contained in all three paths P_{12}, P_{13}, P_{23}. Then, since the paths P_{ij} are chordless, for $i = 1, 2, 3$, there is a unique (x, x_i)–path P_i in $G(A[X])$, and every edge of $G(A[X])$ is on one of these paths.

We claim that $A[X]$ reduces to B_1. We may assume that for $i = 1, 2, 3$, x_i is the only loop–vertex of $A[X]$ on path P_i, since, otherwise we replace x_i by the closest loop–vertex to x on P_i, and redefine X accordingly. Furthermore, we may assume that P_i has length 1, since otherwise we shorten P_i by pivoting on x_i, and then deleting x_i from X. Then $A[X] \times x_1 \times x - x$ is isomorphic to B_1. □

Balanceable matrices

We begin this section by proving some basic facts about circuits.

Lemma 7.8 *Let A be a loop–balanced $(0, \pm 1)$–matrix such that $G(A)$ is a circuit, and let $X \subseteq V$ such that $|X| \leq |V| - 3$. If $A[X]$ is nonsingular then $G(A * X)[V \setminus X]$ is a circuit, and $A * X[V \setminus X]$ is PU if and only if A is PU.*

Proof By Theorem 2.7 and Lemma 7.5, $A * X[V \setminus X]$ is PU if and only if A is PU. To see that $G(A * X)[V \setminus X]$ is a circuit, it suffices to check the elementary pivots, for which the result is obvious. □

Lemma 7.9 *Let A be a binary matrix such that $G(A)$ is a circuit. If A has no PU–signing then A reduces to B_1.*

Proof Suppose that A has no PU–signing. By Lemma 7.7, we may assume that A has a loop–balanced signing. By Lemma 7.8, we can reduce A to either a matrix of size 3, or a matrix of size 4 that has no loop–vertices. If $G(A)$ is a circuit of length 3, and $A \neq B_1$ then there exists a loop–vertex v of A. Thus $G(A \times v)$ is a path, so by Lemma 7.5, A has a PU–signing. If $G(A)$ is a circuit of length 4, and A has no loop–vertices then, for an edge vw of $G(A)$, $G(A \times vw)$ is a path, so A has a PU–signing. □

Lemma 7.10 *Let A be a binary matrix such that $G(A)$ is a circuit. Any two PU–signings of A are equivalent under switching.*

Proof By Lemma 7.8, it suffices to check the result for circuits of length 3 or 4; this is left to the reader. □

We call a symmetric $(0, \pm 1)$–matrix A *balanced* if A is loop–balanced and, for every induced circuit C of $G(A)$, $A[V(C)]$ is PU. A symmetric binary matrix A is called *balanceable* (otherwise *nonbalanceable*) if it has a balanced signing. The following lemma is a generalization of Theorem 6.1 for regular matroids.

Lemma 7.11 *Let A be a symmetric binary matrix, such that $G(A)$ is connected. Any two balanced signings of A are equivalent under switching. In particular, any two PU–signings of A are equivalent under switching.*

Proof Let $A_1 = (a^1_{ij})$ and $A_2 = (a^2_{ij})$ be balanced signings of A. The diagonals of A_1 and A_2 are equivalent up to reversing, so we may assume that they are the same. Define $S = \{ij : a^1_{ij} \neq a^2_{ij}\}$. By Lemma 7.10, for each chordless circuit C of G, $|E(C) \cap S|$ is even. Hence for each circuit C of G, $|E(C) \cap S|$ is even. Therefore the edge set S is a cut in $G(A)$, so A_1 and A_2 are equivalent under cut–switching. □

We define an *obstruction* to be a symmetric binary matrix, other than B_1, that does not admit a PU–signing, and that does not reduce to any smaller matrix with the same property.

Lemma 7.12 *Let A be a balanceable obstruction, and let $X \subseteq V$ such that $|X| \leq |V| - 3$ and $A[X]$ is nonsingular. Then $G(A \times X)[V \setminus X]$ is a circuit.*

Proof Let A' be a balanced signing of A. If $Y \subseteq V$ and $A'[Y]$ is not unimodular then, by Lemma 7.11, $A[Y]$ has no PU–signing. Therefore, since A is an obstruction, the only principal submatrix of A' that is not unimodular is A' itself. By Theorem 2.7, the only principal submatrix of $A' * X$ that is not unimodular is $A' * X[V \setminus X]$. If $A' * X$ is balanced then $A \times X[V \setminus X]$ has no PU–signing, contradicting that A is an obstruction. Therefore $A' * X$ is not balanced; and, since $A' * X[V \setminus X]$ is the only nonunimodular submatrix of $A' * X$, $G(A' * X)[V \setminus X]$ must be a circuit. □

The following proposition removes some trivial cases; the proof is left as an exercise. Note that if A is an obstruction, then $G(A)$ is connected, and $G(A)$ is neither a path nor a circuit. There are, up to isomorphism, just four such graphs with at most four vertices.

Proposition 7.13 *Every obstruction has size at least 5.* □

Lemma 7.14 *If A is an obstruction, then A is equivalent under binary pivoting to a nonbalanceable obstruction.*

Proof Suppose, by way of contradiction, that A is an obstruction and every matrix equivalent to A under pivoting is balanceable.
Claim *If $X \subseteq V$ such that $|X| \leq |V| - 3$, and $A[X]$ is nonsingular, then $G(A)[V \setminus X]$ and $G(A \times X)[V \setminus X]$ are both circuits.*

Since $A[X]$ and $A \times X[X]$ are nonsingular, and A and $A \times X$ are nonbalanceable, the claim follows by Lemma 7.12.

Suppose that A has a loop–vertex x. Let y be a neighbour of x in $G(A)$. We may assume that y is a not a loop–vertex, since otherwise we could make y a nonloop–vertex by pivoting on x. Both $A[\{x\}]$ and $A[\{x, y\}]$ are nonsingular. Then, by the claim, $G(A) - x$ and $G(A) - x - y$ are both circuits, which is clearly impossible. Hence A has no loop–vertices.

Since A has no loop–vertices and A does not reduce to B_1, $G(A)$ is bipartite. By the claim, for every edge vw of $G(A)$, $G(A) - v - w$ is a circuit. Let v_1, v_2, v_3, v_4 be consecutive vertices in any such circuit. We may assume that v_1v_4 is not an edge, since otherwise we can remove the edge by pivoting on v_2v_3. Since $G(A) - v_2 - v_3$ is a circuit and v_1v_4 is not an edge, v_1 has degree 3 in $G(A)$. However, v_1 is adjacent to neither v_1 nor v_2, which contradicts that $G(A) - v_1 - v_2$ is a circuit. □

Nonbalanceable matrices

The problem has now simplified to finding the nonbalanceable obstructions. This task is made easy by the following lemma.

Lemma 7.15 *Let A be a nonbalanceable obstruction. Then $G(A)$ is either a three–path configuration or a proper partial wheel.*

Proof By Lemma 7.7, A has a loop–balanced signing, say $A' = (a'_{ij})$. Let C be an induced circuit of $G(A)$, and let $H = A[V(C)]$. By Lemma 7.9, H has a PU–signing, say $H' = (h'_{ij})$. We may assume that for every loop–vertex v of H, $h'_{vv} = a'_{vv}$ (otherwise we negate H'). We now define β_C to be 0 (1) if the number of edges vw of C with $h'_{vw} = +1$ is even (odd). By Lemma 7.10, $A'[V(C)]$ is PU if and only if it is equivalent under cut–switching to H', that is, the number of edges vw of C with $a'_{vw} = +1$ is equivalent to β_C modulo 2. Hence A is balanceable if and only if $G(A)$ has a β–balanced signing. The result then follows by Theorem 7.4. □

Lemma 7.16 *Let A be a nonbalanceable obstruction, and let $X \subseteq V$ such that $|X| \leq |V| - 3$, $A[X]$ is nonsingular, and $G(A)[V \setminus X]$ is not a circuit. Then $A \times X$ is not balanceable. Furthermore, if $X = \{v\}$ then $N_{G(A)}(v)$ is not a stable set of $G(A)$.*

Proof If $A \times X$ is balanceable then, by Lemma 7.12, $G(A)[V \backslash X] = G((A \times X) \times X)[V \backslash X]$ is a circuit, a contradiction. Therefore, $A \times X$ is a nonbalanceable obstruction. Now suppose that $X = \{v\}$, and that $N_{G(A)}(v)$ is a stable set of $G(A)$. Then $N_{G(A)}(v)$ induces a clique of $G(A \times v)$. However, by Lemma 7.15, $G(A \times v)$ is a three–path configuration or a proper partial wheel, so it must be the case that $G(A \times v)$ is the complete graph on 4 vertices, contradicting Proposition 7.13. □

Lemma 7.17 *Let A be a nonbalanceable obstruction such that $G(A)$ is a three–path configuration. Then $G(A)$ is isomorphic to $G(B_3)$.*

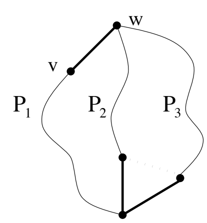

Figure 7.4: Three path configuration, Type 1 or Type 2

Proof First suppose that $G(A)$ is a three–path configuration of Type 1 or Type 2. Let w be a vertex of degree 3 in $G(A)$ such that $N_{G(A)}(w)$ is a stable set. By Lemma 7.16, w is not a loop–vertex. If all three vertices adjacent to w in $G(A)$ are loop–vertices then A is not loop–balanceable, which, by Lemma 7.7, is a contradiction. Therefore there exists a nonloop–vertex v adjacent to w in $G(A)$. This is depicted in Figure 7.4. $G(A) - v - w$ is not a circuit; so, by Lemma 7.16, $A \times vw$ is nonbalanceable. Therefore, by Lemma 7.15, $G(A \times vw)$ is a three–path configuration or a partial wheel. Note that, in $G(A \times vw)$, either v is adjacent to a vertex of degree at least 4, or w is adjacent to a vertex of degree 1. This is a contradiction, since a three–path configuration or a proper partial wheel can have neither a vertex of degree 1 nor a vertex of degree 2 that is adjacent to a vertex of degree at least 4.

Now, suppose that $G(A)$ is a three–path configuration of Type 3, and that $G(A)$ is not isomorphic to $G(B_3)$. Since $G(A)$ is not isomorphic to $G(B_3)$, one of the paths, say P_3, has length at least 2. Let v be an end vertex of P_3, and let w be the vertex of P_3 that is adjacent to v, as depicted in Figure 7.5. By Lemma 7.16, w is a nonloop–vertex. $G(A) - v$ is not a circuit, so, by Lemma 7.12, if v is a loop–vertex then $A \times v$ is nonbalanceable. However, $G(A \times v)$ is neither a three–path configuration nor a partial wheel, which is a contradiction. Therefore we may assume that v is a nonloop–vertex. Now $G(A) - v - w$ is not a circuit; so, by Lemma 7.16, $A \times vw$ is nonbalanceable. However, $G(A \times vw)$ is neither a three–path configuration nor a partial wheel, which is a contradiction. □

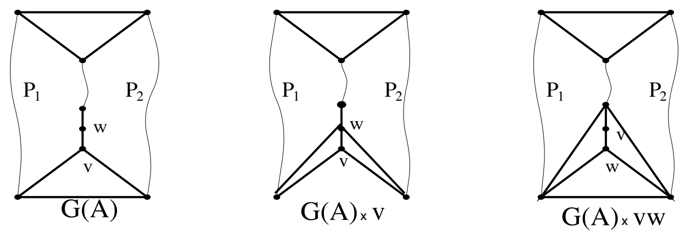

Figure 7.5: Three path configuration, Type 3

Lemma 7.18 *Let A be a nonbalanceable obstruction such that $G(A)$ is a proper partial wheel, and let C be an induced circuit of $G(A)$. Then, for every edge vw of $G(A)$ that is not an edge of C, $\left|N_{G(A)}(\{v,w\}) \cap V(C)\right| \geq 2$; in particular $G(A)$ contains no pair of adjacent vertices of degree 2.*

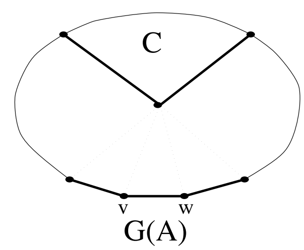

Figure 7.6: Proper partial wheel

Proof Suppose there exists an edge vw of $G(A)$ such that $\left|N_{G(A)}(\{v,w\}) \cap V(C)\right| \leq 1$. Let x be the hub of the partial wheel; C must contain the vertex x and vw must be an edge of $G(A)-x$. Suppose that v and w are adjacent vertices of degree 2. By Lemma 7.16, neither v nor w are loop–vertices. Now $G(A)-v-w$ is not a circuit, so, by Lemma 7.16, $A\times vw$ is not balanceable. However, $G(A\times vw)$ contains an edge $v'w'$ such that $G(A\times vw)-v'-w'$ is not connected, so $G(A\times vw)$ is neither a proper partial wheel nor a three–path configuration, contradicting Lemma 7.15. Thus, we may assume that at least one of v and w is adjacent to x. Then neither v nor w may be adjacent to any vertex of C other than x; this is depicted in Figure 7.6. In this case A must have size least 7.

Suppose that v is a loop–vertex. Then, by Lemma 7.16, $G(A\times v)$ is a three–path configuration or a partial wheel. However, $G(A\times v)$ has a pair of vertex disjoint circuits, so it is not a partial wheel. Therefore, $G(A\times v)$ is a three–path configuration, so, by Lemma 7.17, $G(A\times v)$ is isomorphic to $G(B_4)$, contradicting that A has size at least 7. Hence, we may assume that v (and similarly w) is not a loop–vertex.

By Lemma 7.16, $G(A\times vw)$ is a three–path configuration or a partial wheel. However, $G(A\times vw)$ has a pair of vertex disjoint circuits, so it is not a partial wheel. Therefore, $G(A\times vw)$ is a three–path configuration, so, by Lemma 7.17, $G(A\times v)$ is isomorphic to $G(B_4)$, contradicting that A has size at least 7. □

The proof is now reduced to case analysis. We hide much of it in the following lemma.

Lemma 7.19 *Let A be a nonbalanceable obstruction such that $G(A)$ is isomorphic to one of the graphs depicted in Figure 7.7. Then A is equivalent under binary pivoting to B_2, B_3 or B_4.*

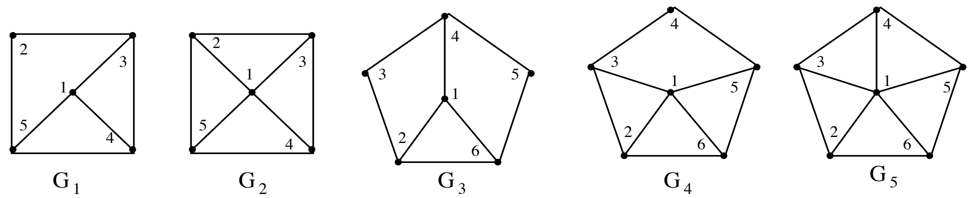

Figure 7.7: Awkward cases

Before beginning the case analysis for Lemma 7.19, we use it to prove the main result.
Proof of Theorem 7.2. Let A be an obstruction. We are required to prove that A is equivalent under pivoting to one of $B_1, \ldots, B_5$. By Lemma 7.14, we may assume that A is nonbalanceable. Then, by Lemma 7.15, $G(A)$ is either a three–path configuration, or a proper partial wheel.

Suppose that $G(A)$ is a three–path configuration. By Lemma 7.17, $G(A)$ is isomorphic to $G(B_4)$. Let x_1, x_2, x_2 be vertices that induce a triangle of $G(A)$; at least one x_i, say x_1 must be a loop–vertex (otherwise A reduces to B_1). $G(A) - x_1$ is not a circuit, so $A \times x_1$ is nonbalanceable. However, $G(A \times x_1)$ is isomorphic to G_5 of Figure 7.7, so, by Lemma 7.19, A is equivalent under binary pivoting to B_2,B_3 or B_4.

Now suppose that $G(A)$ is a proper partial wheel. By Lemmas 7.18 and 7.19 and Proposition 7.13, we may assume that A has size at least 7. Let C be a shortest circuit of $G(A)$. By Lemma 7.19, C has length 3 or 4. If $|V(G(A))| \geq |V(C)| + 4$ then there exists an edge vw of G that is not an edge of C, such that $\left|N_{G(A)}(\{v, w\})\right| \leq 1$, contradicting Lemma 7.18. Then C cannot have length 3, since otherwise A would have fewer than 7 vertices. Hence C has length 4, and A has size exactly 7. $G(B_5)$ is the unique proper partial wheel, up to isomorphism, with seven vertices and no circuit of length 3. Therefore $G(A)$ is isomorphic to $G(B_5)$. Let x be the hub of $G(A)$. By Lemma 7.16, every vertex of A other than x is a nonloop–vertex. If x is also a nonloop–vertex, then A is equivalent to B_5; otherwise if x is a loop–vertex then $(A \times x) - x$ is equivalent to B_4, a contradiction.□

Proof of Lemma 7.19. Suppose that $G(A)$ is isomorphic to G_2. Note that A must be loop–balanceable. There are, up to isomorphism, five choices for the loop–vertices of $A - 1$, and each choice uniquely determines whether or not 1 is a loop–vertex. The possibilities are depicted in Figure 7.8. $G(A_i \times 1 \times 2 \times 3)$ is a path for $i = 1, 2, 3$, so these matrices are not obstructions. $G(A_4) - 1 - 3$ is not a circuit, but $G(A_4) * 13$ is neither a proper partial wheel nor a three–path configuration, so, by Lemma 7.16, A_4 is not an obstruction. A_5 is isomorphic to B_2.

Suppose that $G(A)$ is isomorphic to G_1. By Lemma 7.16, 2 is not a loop–vertex of A. We may assume that neither 3 nor 5 are loop–vertices of A, since $G_1 * 3$ and $G_1 * 5$ are both isomorphic to G_2. Therefore one of $1, 4$ must be a loop–vertex; we assume by symmetry that 1 is a loop–vertex. However, $G(A \times 1 \times 5 \times 2)$ is a path, so A is not an obstruction.

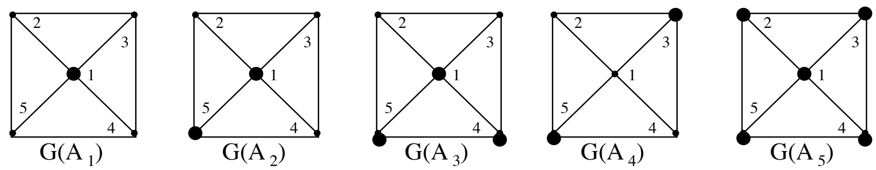

Figure 7.8: Loop–vertices for G_2

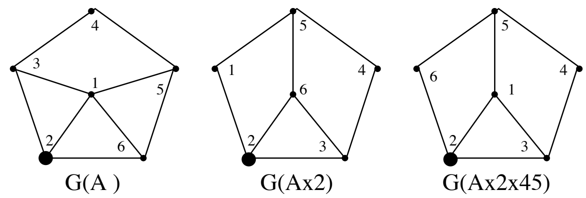

Figure 7.9: Pivoting in G_4

Suppose that $G(A)$ is isomorphic to G_4. By Lemma 7.16, 3, 4 and 5 are all not loop–vertices. However $G_4 - 1$ is an odd circuit, so either 2 or 6 must be a loop–vertex; we assume by symmetry that 2 is a loop–vertex. $G(A \times 2)$ and $G(A \times 2 \times 45)$ are depicted in Figure 7.9. By Lemma 7.16, 1 is a nonloop–vertex in $A \times 2$, and 6 is a nonloop–vertex of $A \times 2 \times 45$; hence, 1 and 6 are both loop–vertices of A. Thus, the loop–vertices of A are 1, 2 and 6, so, $A \times 1$ is isomorphic to B_4.

Suppose that $G(A)$ is isomorphic to G_3. By Lemma 7.16, 3, 4 and 5 are all not loop–vertices. However $G_4 - 1$ is an odd circuit, so either 2 or 6 must be a loop–vertex; we assume by symmetry that 2 is a loop–vertex. However $G(A \times 2)$ is isomorphic to G_4 so A reduces to B_4.

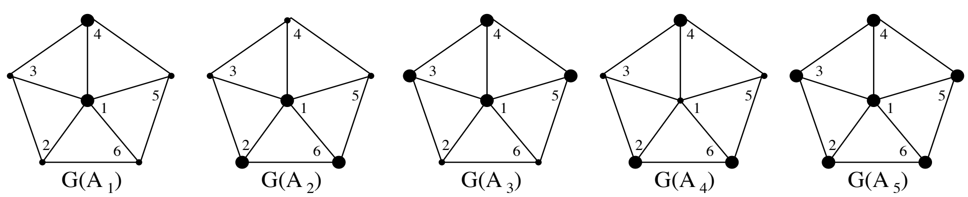

Figure 7.10: Loop–vertices for G_5

Finally, suppose that $G(A)$ is isomorphic to G_5. There are, up to isomorphism, five choices for the loop–vertices of $A - 1$ so that $A - 1$ does not reduce to B_1. Each choice uniquely determines whether or not 1 is a loop–vertex; the possibilities are depicted in Figure 7.10. $A_i \times 1 - 1$ reduces to B_1 for $i = 1, 2, 3, 5$, so these matrices are not obstructions. $A_4 \times 4$ is isomorphic to B_3. □

Chapter 8

Matching

In this chapter we consider a problem that generalizes the bipartite matching problem, and the nonbipartite matching problem in different ways. We find a min–max theorem, a totally dual integral polyhedral description, and a polynomial–time algorithm, thus generalizing standard results for the two problems. The problem arises most naturally by graphically interpreting the rank of a certain matrix of indeterminates.

Let $G = (V, E)$ be a graph and let $\{x_e : e \in E\}$ be a set of algebraically independent commuting indeterminates. Now define a V by V skew–symmetric matrix $A = (a_{ij})$ such that $a_{ij} = \pm x_{ij}$ if $ij \in E$, and $a_{ij} = 0$ otherwise. We refer to A as the *matching matrix* of G (although the construction is not unique). We recall that a subset X of V is called *matchable* if $G[X]$ has a perfect matching. Tutte observed that $\det(A[X])$ is nonzero if and only if X is a matchable set. It is a classical result in matrix theory that the rank of a skew–symmetric matrix is the size of its largest nonsingular principal submatrix. Therefore, from Tutte's result, the rank of A is the size of the largest matchable set of G. Thus Tutte developed a nice graphical interpretation for the rank of any principal submatrix of the matching matrix; we consider, in a similar fashion, the ranks of arbitrary submatrices.

More precisely, we consider the following problem: *Given subsets I, J of V, determine the rank of* $A[I, J]$. When $I = J$ the problem is just to find a maximum cardinality matchable set in $G[I]$. The other extreme is also interesting; suppose that I and J are disjoint sets. Then, since every indeterminate occurs in at most one entry of $A[I, J]$, the rank of $A[I, J]$ is the maximum number of nonzero entries in $A[I, J]$ with no two in the same row or column. Thus, the rank of $A[I, J]$ is just the size of a maximum matching in a certain bipartite graph associated with $A[I, J]$. Before proceeding further, we clear up two points.

Firstly, our problem is not well defined. The matrix A is a matrix over a ring of polynomials, whereas we use notions, like "rank", that are defined only for matrices over fields. We sweep the problem under the carpet, noting that, for the purpose of matrix manipulation, we can embed the ring of polynomials into an appropriate field.

The second point is algorithmic. An important problem in algorithmic combinatorics is to find an efficient algorithm to compute the rank of a matrix of indeterminates. It is well–known that the rank of a rational matrix can be efficiently computed using gaussian elimination. The same algorithm can be applied to calculate the rank of a matrix of indeterminates. However, while the algorithm requires only a polynomial number of elementary row operations, the entries may become rational functions of exponential size; and hence gaussian elimination cannot be performed in polynomial time. Despite this complication, one may expect there to exist an efficient combinatorial algorithm, since there exists an efficient randomized algorithm. Indeed, if M is a square nonsingular matrix of indeterminates, then the determinant of M is a nonzero polynomial in the indeterminates. It is well–known that, by substituting random numbers for the indeterminates, we are unlikely to find a zero of this polynomial. Hence, by substituting random numbers into a matrix of indeterminates, we are unlikely to decrease the rank. This idea leads to a polynomial–time algorithm for estimating the rank that is correct with high probability. Such randomized algorithms have been applied to a number of matching–related problems; see Lovász [48], Rabin and Vazirani [59] and Cheriyan [18].

The separation problem for matchable sets

Let $G = (V, E)$ be a graph, and let $M = (V, \mathcal{F})$ be the matching delta–matroid of G, that is, $\mathcal{F}$ is the set of matchable sets of G. We recall that conv($\mathcal{F}$) denotes the convex hull of incidence vectors of feasible sets of $\mathcal{F}$, and the separation problem is: *Given* $x \in \mathbf{R}^V$, *determine whether* x *is in conv*($\mathcal{F}$). Balas and Pulleyblank [2] gave a description of conv($\mathcal{F}$) using linear inequalities; their description is implied by Theorem 3.2. A combinatorial algorithm for the separation problem was given, for bipartite graphs, by Ning [55], and, for general graphs, by Cunningham and Green–Krótki [25]; however, the algorithm of Cunningham and Green–Krótki is not strongly polynomial.

Recall that the matching delta–matroid M is representable, being represented by the matching matrix A of G. Furthermore, we presented a combinatorial separation algorithm for representable delta–matroids in Chapter 4 that runs in strongly polynomial–time. The algorithm assumes the existence of a polynomial–time subroutine for determining the rank of submatrices of A. We obtain such an algorithm. Hence we have a strongly polynomial–time algorithm for the separation problem for the matchable sets polytope. However, the problem of finding an algorithm for the separation problem for the matchable set polytope that is combinatorial and runs in strongly polynomial time remains open.

A min–max formula

In this section we present a min–max formula, due essentially to Lovász (personal communication), for the rank of a submatrix of a matching matrix. The min–max formula can be viewed as a common generalization of well–known theorems of König and Tutte. We require the following classical result from linear algebra.

Proposition 8.1 *Let* $A = (a_{ij})$ *be an* I *by* J *matrix. Suppose that, for some* $i \in I$ *and* $j \in J$, $rk(A) = rk(A[I-i, J])$ *and* $rk(A) = rk(A[I, J-j])$. *Then,* $rk(A) = rk(A[I-i, J-j])$.

Proof Since $\text{rk}(A) = \text{rk}(A[I-i, J])$, row i of A can be expressed as a linear combination of rows of $A[I-i, J]$. Therefore, for any subset J' of J, $\text{rk}(A[I, J']) = \text{rk}(A[I-i, J'])$. Hence, we have $\text{rk}(A[I, J]) = \text{rk}(A[I, J-j]) = \text{rk}(A[I-i, J-j])$, as required. □

Let A be the matching matrix of a graph $G = (V, E)$. We denote by *odd*(G) the number of connected components of G having an odd number of vertices. It is easy to see that the size of the largest matchable set is at most $|V| - \text{odd}(G)$.

Proposition 8.2 *Let A be the matching matrix of a graph $G = (V, E)$. Suppose, for every vertex v of G, that $rk(A[V - v, V]) = rk(A)$. Then, $rk(A) = |V| - odd(G)$.*

Proof For $v \in V$, we have $\text{rk}(A[V - v, V]) = \text{rk}(A)$. Then, since A is skew–symmetric, we also have $\text{rk}(A[V, V - v]) = \text{rk}(A)$. Therefore, by Proposition 8.1, $\text{rk}(A[V - v, V - v]) = \text{rk}(A)$. Hence, for each vertex v of G, there exists a maximum cardinality matchable set not containing v. Thus, by Gallai's Lemma (Lemma 3.9), every component of G is hypomatchable. (Recall a graph H is called *hypomatchable* if $H - x$ has a perfect matching, for every vertex x of H.) Therefore, the size of a maximum cardinality matchable set in G is $|V| - \text{odd}(G)$. □

Let I, J be subsets of V. We call I, J a *bi–stable pair* if $A[I \backslash J, J] = 0$ and $A[I, J \backslash I] = 0$. Now, let

$$D(I, J) = \{(I', J') : I' \subseteq I, J' \subseteq J, \text{ and } I', J' \text{ is a bi–stable pair }\}.$$

Theorem 8.3 (Lovász) *Let A be the matching matrix of a graph $G = (V, E)$, and let I, J be subsets of V. Then*

$$rk(A[I, J]) = \min_{(I', J') \in D(I, J)} |I' \cap J'| - odd(G[I' \cap J']) + |I \setminus I'| + |J \setminus J'|. \tag{8.1}$$

Proof The rank of a matrix decreases by at most one when we delete a row or a column; therefore, for $(I', J') \in D(I, J)$, we have $rk(A[I, J]) \leq \text{rk}(A[I', J']) + |I \setminus I'| + |J \setminus J'|$. However, since I', J' is a bi–stable pair, $\text{rk}(I', J') = \text{rk}(A[I' \cap J']) \leq |I' \cap J'| - \text{odd}(G[I' \cap J'])$. Thus

$$\text{rk}(A[I, J]) \leq |I' \cap J'| - \text{odd}(G[I' \cap J']) + |I \setminus I'| + |J \setminus J'|. \tag{8.2}$$

So now we need to prove that there exists $(I', J') \in D(I, J)$ that satisfies (8.2) with equality.

Let $I^* \subseteq I$ and $J^* \subseteq J$ be minimal such that $\text{rk}(A[I, J]) = \text{rk}(A[I^*, J^*]) + |I \setminus I^*| + |J \setminus J^*|$. Therefore, for each $i \in I^*$, $\text{rk}(A[I^*, J^*]) = \text{rk}(A[I^* - i, J^*])$, and, for each $j \in J^*$, $\text{rk}(A[I^*, J^*]) = \text{rk}(A[I^*, J^* - j])$.

Claim I^*, J^* *is a bi–stable pair.*

Suppose the claim is untrue. Then there exists an indeterminate, say x_{ij}, that occurs in exactly one entry of $A[I^*, J^*]$. By Proposition 8.1, $\text{rk}(A[I^*, J^*]) = \text{rk}(A[I^* - i, J^* - j])$. Define I', J' such that $A[I', J']$ is a largest nonsingular square submatrix of $A[I^* - i, J^* - j]$. Then, since $\text{rk}(A[I^*, J^*]) = \text{rk}(A[I^* - i, J^* - j])$, the matrix $A[I' \cup \{i\}, J' \cup \{j\}]$ must be singular. However, the coefficient of x_{ij} in the determinant of $A[I' \cup \{i\}, J' \cup \{j\}]$ is equal, up to a sign, to the determinant of $A[I', J']$, contradicting that $A[I' \cup \{i\}, J' \cup \{j\}]$ is singular. This proves the claim.

Let X denote $I^* \cap J^*$. By the claim, $\text{rk}(A[I^*, J^*]) = \text{rk}(A[X])$. However, by our choice of I^*, J^*, for any $x \in X$, $\text{rk}(A[X]) = \text{rk}(A[X - x, X])$. Then, by Proposition 8.2, $\text{rk}(A[X]) = |X| - \text{odd}(G[X])$. Thus, the bi–stable pair I^*, J^* achieves equality in (8.2), as required. □

Consider Theorem 8.3 for disjoint sets I, J. Let I^*, J^* be a bi–stable pair that attains the minimum in (8.1). Since I and J are disjoint, then so are I^* and J^*. Thus, since I^*, J^* is a bi–stable pair, $A[I^*, J^*] = 0$, and, by (8.1), we have $\text{rk}(A[I, J]) = |I \setminus I^*| + |J \setminus J^*|$. Hence, Theorem 8.3 implies König's Theorem, that is: *The maximum number of nonzero entries no two in the same line in $A[I, J]$, equals the minimum number of lines that include all the nonzero entries of $A[I, J]$.* (Here, by "line" we refer to a row or column of $A[I, J]$.)

Now, consider Theorem 8.3 for $I = J = V$. Let I^*, J^* be a bi–stable pair that attains the minimum in (8.1). Then by (8.1)

$$\begin{aligned} \text{rk}(A) &= |I^* \cap J^*| - \text{odd}(G[I^* \cap J^*]) + |V \setminus I^*| + |V \setminus J^*| \\ &= |V| - (\text{odd}(G[I^* \cap J^*]) - |V \setminus (I^* \cup J^*)|) \\ &\geq |V| - (\text{odd}(G[I^* \cup J^*]) - |V \setminus (I^* \cup J^*)|). \end{aligned}$$

However, for any subset X of V, we have

$$\text{rk}(A) \leq |V| - (\text{odd}(G[V \setminus X]) - |X|).$$

Therefore, Theorem 8.3 implies the Tutte–Berge theorem, that is: *The size of the largest matchable set in $G = (V, E)$ is*

$$\min_{X \subseteq V} |V| - (\text{odd}(G[V \setminus X]) - |X|).$$

Graphic formulation

We begin by formulating the rank problem in digraphs, and then describe the corresponding problem in G. The digraph $\vec{G} = (V, \vec{E})$ is got from G by replacing each edge ij by a pair of oppositely directed arcs ij and ji. Let I, J be subsets of V. We denote by $\vec{E}_{IJ}$ the set $\{ij \in \vec{E} : i \in I, j \in J\}$. A subset F of $\vec{E}_{IJ}$, is called an *(I, J)–factor* (*(I, J)–subfactor*) of $\vec{G}$ if i is the tail of exactly one (at most one) arc in F, for $i \in I$, and j is the head of exactly one (at most one) arc in F, for each $j \in J$. If F is an (I, J)–subfactor, then each component of (V, F) is either a directed circuit in $\vec{G}[I \cap J]$, or a directed path; furthermore, if F is an (I, J)–factor, then all of the directed paths in (V, F) start from a vertex in $I \setminus J$ and end at a vertex in $J \setminus I$. We call F *even* if every directed circuit in the digraph (V, F) has even length. Note that, if there exists an even (I, J)–subfactor in $\vec{G}$, then there exists an (I, J)–subfactor F in $\vec{G}$ such that every directed circuit in (V, F) has length two.

Lemma 8.4 *Let A be the matching matrix of $G = (V, E)$, and let I, J be subsets of V. Then the rank of $A[I, J]$ is the size of the largest even (I, J)–subfactor in $\vec{G}$.*

Proof We shall prove the equivalent result that: if $|I| = |J|$, then $A[I, J]$ is nonsingular if and only if there exists an even (I, J)–factor.

Let $I = \{i_1, \ldots, i_k\}$, and $J = \{j_1, \ldots, j_k\}$. Consider the determinant expansion for $A[I, J]$. We have

$$\det(A[I, J]) = \sum_{\sigma} \text{sgn}(\sigma) \prod_{b=1}^{k} a_{i_b j_{\sigma(b)}},$$

where the sum is taken over all permutations σ of $\{1,\ldots,k\}$, and $\mathrm{sgn}(\sigma)$ denotes the "sign" of the permutation σ (see [44]). If σ is a permutation of $\{1,\ldots,k\}$, then $\{i_b j_{\sigma(b)} : b=1,\ldots,k\}$ is an (I,J)–factor of $\vec{G}$ if and only if $\prod(a_{i_b j_{\sigma(b)}} : b=1,\ldots,k) \neq 0$. For an (I,J)–factor F, we denote by $\mathrm{sgn}(F)$, the sign of the corresponding permutation. Then

$$\det(A[I,J]) = \sum_F \mathrm{sgn}(F) \prod_{ij\in F} a_{ij}, \tag{8.3}$$

where the sum is taken over all (I,J)–factors F. Let F be an (I,J)–factor, and let $C \subseteq F$ be a directed circuit in (V,F). Now define F' to be $(F\setminus C)\cup\{ji : ij \in C\}$. Now, $\mathrm{sgn}(F) = \mathrm{sgn}(F')$, and, since A is skew–symmetric

$$\prod_{ij\in F} a_{ij} = (-1)^{|C|} \prod_{ij\in F'} a_{ij}.$$

Therefore, if C has odd length, then we can cancel two terms in the determinant expansion. Furthermore, such cancellations, pair off the set of (I,J)–factors that contain C with the set of (I,J)–factors that contain $\{ji : ij\in C\}$. So, the determinant expansion (8.3) holds when the sum is taken over all *even* (I,J)–factors F. Let F be an even (I,J)–factor of $\vec{G}$. Now, the coefficient of the monomial $\prod(x_{ij} : ij \in F)$ in the determinant expansion, is $\mathrm{sgn}(F)2^r$, where r is the number of directed circuits of length at least four in (V,F). In particular, $A[I,J]$ is nonsingular if and only if there exists an even (I,J)–factor. □

Let I,J be subsets of V, and M be a subset of the edges of $G[I\cup J]$. We call M an *(I,J)–path matching* if each connected component of $(I\cup J, M)$ is a path whose ends are neither both in $I\setminus J$, nor both in $J\setminus I$, and whose internal vertices are all in $I\cap J$. An edge vw of M is called a *matching edge of M* if vw is an edge of $G[I\cap J]$, and vw is the only edge in the connected component of $(I\cup J, M)$ containing vw. Let M' denote the set of matching edges of an (I,J)–path matching M. The *value of M* is $|M\setminus M'| + 2|M'|$. (Figure 8.1 depicts an (I,J)–path matching of value 18.) Then there exists an even (I,J)–subfactor of size k, if and only if there exists an (I,J)–path matching of value k. Therefore, by Lemma 8.4, the rank of $A[I,J]$ is the largest value attained by an (I,J)–path matching.

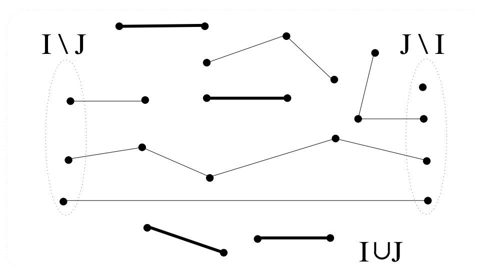

Figure 8.1: An (I,J)–path matching

Remark: In the next section, we shall see that we can efficiently find a maximum value (I,J)–path matching, by using the ellipsoid algorithm. Here we show that some closely–related problems are difficult. Consider the following problem: *Given ϵ, find an (I,J)–path matching M maximizing $v_\epsilon(M) = |M\setminus M'| + (2+\epsilon/n)|M'|$, where M' denotes the set of* *matching edges of M, and n is the number of vertices of G.* This problem is $\mathcal{NP}$–hard for all $\epsilon < 1$, *except* for $\epsilon = 0$. Indeed, suppose that $|I \setminus J| = |J \setminus I| = 1$. If $\epsilon < 0$, then for a (I, J)–path matching M, $v_\epsilon(M) \geq |I \cup J| - 1$, if and only if M is a hamilton path in $G[I, J]$ whose ends are in $I \Delta J$. Thus, the problem is $\mathcal{NP}$–hard for $\epsilon < 0$. Now suppose that $0 < \epsilon < 1$. Then it is easy to show that the problem of finding an (I, J)–path matching maximizing v_ϵ contains the following problem: *Given vertices i, j in a graph G, find the shortest (i, j)–path P such that $G[V \setminus V_P]$ has a perfect matching.* Martin Loebl, personal communication, showed that the latter problem is $\mathcal{NP}$–hard.

A *perfect (I, J)–path matching* is an (I, J)–path matching M such that each connected component of $(I \cup J, M)$ is either a matching edge, or a path with one end in $I \setminus J$ and the other end in $J \setminus I$. Then, there exists a perfect (I, J)–path matching in G, if and only if there exists an even (I, J)–factor in $\vec{G}$. Thus, for equicardinal subsets I, J of V, $A[I, J]$ is nonsingular if and only if G has a perfect (I, J)–path matching.

Polyhedra

Ideally, we wish to find an efficient combinatorial algorithm to find a maximum value (I, J)–path matching, for any given subsets I, J of V. When $I = J$, the problem is to find a maximum cardinality matching in $G[I]$, which is solved by Edmonds [31]. We have been unsuccessful in generalizing Edmonds' algorithm; the main hurdle seems to be defining an "augmenting path" for the path matching problem. In this section we generalize some polyhedral theorems concerning the matching polytope. The proofs are all generalizations of proofs of Schrijver [60] for the matching polyhedron, that do not use augmenting paths. In particular, we give a description of a polytope associated with (I, J)–path matchings, that provides an efficient algorithm for computing the rank of $A[I, J]$.

We use standard notation from polyhedral theory. For $x \in \mathbf{R}^V$ and $S \subseteq V$, we denote by $x(S)$ the sum $\sum(x_v : v \in S)$. For a subset X of V, we denote by $\gamma(X)$ (or $\gamma_G(X)$) the set of edges of G whose ends are both in X, and we denote by $\delta(X)$ (or $\delta_G(X)$) the set of edges of G that have exactly one end in X. For a directed graph $\vec{G} = (V, \vec{E})$, we define $\delta^-(X)$ to be the set of arcs leaving X, that is, $\delta^-(X) = \{vw \in \vec{E} : v \in X, w \notin X\}$. Similarly, we define $\delta^+(X)$ to be be the set of arcs of $\vec{G}$ entering X.

Let M be an (I, J)–path matching, and let M' denote the matching edges of M. We define the *path matching vector* of M, to be the vector $\psi^M \in \mathbf{R}^E$, such that, for $vw \in E$,

$$\psi^M_{vw} = \begin{cases} 2, & \text{if } vw \in M' \\ 1, & \text{if } vw \in M \setminus M' \\ 0, & \text{if } vw \notin M. \end{cases}$$

We denote by $\mathcal{M}(I, J; G)$ (or, simply, $\mathcal{M}$) the set of (I, J)–path matchings of G, and denote by $\text{conv}(\mathcal{M})$ the convex hull of path matching vectors of $\mathcal{M}$. Note that, by maximizing $x(E)$ over all $x \in \text{conv}(\mathcal{M})$, we obtain the rank of $A[I, J]$. The main result of this section is the following theorem, which generalizes Edmonds' Matching Polyhedron Theorem [29]. Given a subset K of V, and an element i of V, we let $K_i = |K \cap \{i\}|$; thus, K_i indicates whether $i \in K$.

Theorem 8.5 *Let $G = (V, E)$ be a graph, and I, J be subsets of V. Then conv($\mathcal{M}(I, J; G)$) is described by the following inequalities:*

$$\begin{aligned} x(\delta(v)) &\leq I_v + J_v && (v \in V) && (8.4)\\ x(\gamma(X)) &\leq |X \cap J| && (X : I \setminus J \subseteq X \subseteq I) && (8.5)\\ x(\gamma(X)) &\leq |X \cap I| && (X : J \setminus I \subseteq X \subseteq J) && (8.6)\\ x(\gamma(X)) &\leq |X| - 1 && (X \subseteq I \cap J, |X| \text{ odd}) && (8.7)\\ x &\geq 0. && && (8.8) \end{aligned}$$

We denote by $\mathcal{M}^*(I, J; G)$ (or, simply, $\mathcal{M}^*$) the set of perfect (I, J)–path matchings of G. We prove Theorem 8.5 as a corollary of the following theorem.

Theorem 8.6 *Let $G = (V, E)$ be a graph, and let I, J be equicardinal subsets of V. Then conv($\mathcal{M}^*(I, J; G)$) is described by the following inequalities:*

$$\begin{aligned} x(\delta(v)) &= I_v + J_v && (v \in V) && (8.9)\\ x(\delta(X)) &\geq |I \setminus J| && (X : I \setminus J \subseteq X \subseteq I) && (8.10)\\ x(\delta(X)) &\geq 2 && (X \subseteq I \cap J : 3 \leq |X|, |X| \text{ odd}) && (8.11)\\ x &\geq 0. && && (8.12) \end{aligned}$$

Finding path matchings efficiently

Let $G = (V, E)$ be a graph, and I, J be subsets of V. There exists a perfect (I, J)–path matching if and only if conv($\mathcal{M}^*$) is not empty. By Theorem 8.6, conv($\mathcal{M}^*$) is described by inequalities (8.9), (8.10), (8.11) and (8.12). Consider the separation problem for inequalities (8.9), (8.10), (8.11) and (8.12), that is: *Given $x \in \mathbf{R}^E$, either verify that x satisfies the inequalities (8.9), (8.10), (8.11) and (8.12), or find an inequality that is violated by x.* If we can solve the separation problem efficiently, then, by the ellipsoid algorithm, we can efficiently determine whether or not conv($\mathcal{M}^*$) is empty.

Given $x \in \mathbf{R}^E$, the separation problem for the inequalities (8.9) and (8.12) is trivial, so we may assume that these constraints are satisfied. However, there are exponentially many constraints of type (8.10) and (8.11), so the separation problem for these inequalities is more difficult.

Padberg and Rao [58] gave an efficient algorithm for solving the minimum odd–cut problem, that is: *Given a graph $G' = (V', E')$, an even cardinality subset V_1' of V', and nonnegative weights $w' \in \mathbf{R}^{E'}$, find a subset X' of V' such that $|X' \cap V_1'|$ is odd minimizing $w'(\delta_{G'}(X'))$.* The separation problem for inequalities (8.11) is a special case of the minimum odd–cut problem. Indeed, let $G' = G[I \cap J]$, $V_1' = I \cap J$, and w' be the restriction of x to $E_{G'}$. If X' is a minimum odd-cut for G', V_1', w', then x satisfies inequalities (8.11) if and only if $w'(\delta_{G'}(X')) \geq 2$. The separation problem for inequalities (8.10) is also a special case of the minimum odd–cut problem. (Recall that $G \circ S$ denotes the graph obtained by shrinking the vertex set S to a single vertex which we label S.) Indeed, let $G' = G[I \cup J] \circ (I \setminus J) \circ (J \setminus I)$, $V_1' = \{I \setminus J, J \setminus I\}$, and w' be the restriction of x to $E_{G'}$. If X' is a minimum odd-cut for G', V_1', w', then x satisfies inequalities (8.10) if and only if $w'(\delta_{G'}(X')) \geq |I \setminus J|$. Therefore we can efficiently solve the separation problem for inequalities (8.9), (8.10), (8.11) and (8.12).

By a standard conversion, we can also solve the separation problem for inequalities (8.4), (8.5), (8.6), (8.7) and (8.8). Thus, by Theorem 8.5 and the ellipsoid algorithm, we can optimize efficiently over conv($\mathcal{M}$). Consequently, we have an efficient algorithm for computing the rank of $A[I, J]$.

Theorem 8.7 *Let G be a graph, I, J be subsets of V, and $c \in \mathbf{R}^E$. Then there exists a polynomial–time algorithm that finds an (I, J)–path matching M maximizing $c^T\psi^M$.* □

Proof of polyhedral descriptions

We define a polyhedron $Q \subseteq \mathbf{R}^{\vec{E}}$ by

$$Q = \begin{cases} y(\delta^+(v)) & = & I_v, & (v \in V) \\ y(\delta^-(v)) & = & J_v, & (v \in V) \\ y & \geq & 0. & \end{cases}$$

By well–known results concerning total unimodularity (see [54]), the polyhedron Q is integral, that is, the extreme points of Q are all integral. Clearly, Q is the convex hull of incidence vectors of (I, J)–factors.

Remark: Y. Wang, personal communication, proved that the following problem is $\mathcal{NP}$–hard: *Given a digraph D, find a set of vertex disjoint directed even circuits that cover all nodes of D.* We can give the arcs of D weight one, and extend D to a "symmetric" directed graph D' by adding zero weight arcs to D. Then a maximum weight even (V, V)–factor in D' is a set of vertex–disjoint directed even circuits that cover all nodes of D, if one exists. Since this optimization problem is $\mathcal{NP}$–hard, it is unlikely that we can characterize the convex hull of incidence vectors of even (I, J)–factors of a graph by linear inequalities.

Let $G = (V, E)$ be a graph, and let $\vec{G} = (V, \vec{E})$ be the corresponding digraph. We define a function $\rho : \mathbf{R}^{\vec{E}} \to \mathbf{R}^E$, such that, for $y \in \mathbf{R}^{\vec{E}}$, $\rho(y)_{vw} = y_{vw} + y_{wv}$, for $vw \in E$. Let Q^ρ denote $\{\rho(y) : y \in Q\}$. Since ρ maps integral points to integral points, Q^ρ is an integral polyhedron.

Lemma 8.8 *Let $G = (V, E)$ be a graph, and $I, J \subseteq V$. Then the integral polyhedron Q^ρ is described by the inequalities (8.9), (8.10) and (8.12).*

Proof Given $y \in Q$, it is easy to show that $\rho(y)$ satisfies inequalities (8.9), (8.10) and (8.12). Conversely, suppose that $x \in \mathbf{R}^E$ satisfies inequalities (8.9), (8.10) and (8.12).

Let $\mathcal{P}$ denote the set of all paths in $G[I \cup J]$ that have one end in $I \setminus J$, the other end in $J \setminus I$, but no internal vertices in $I \Delta J$. Now, for $vw \in E$, we denote by $\mathcal{P}_{vw}$ the set of paths in $\mathcal{P}$ that use the edge vw. By the Max–flow Min–cut Theorem of Ford and Fulkerson [34], there exists a nonnegative vector $\lambda \in \mathbf{R}^{\mathcal{P}}$, such that $\lambda(\mathcal{P}) = |I \setminus J|$, and, for $vw \in E$, $\lambda(\mathcal{P}_{vw}) \leq x_{vw}$. Now, we let $f \in \mathbf{R}^{\vec{E}}$ be the $(I \setminus J, J \setminus I)$–flow in $\vec{G}$, corresponding to the path–flow λ. That is, for $vw \in \vec{E}$, $f_{vw} = \sum \lambda_P$ where the sum is over $P \in \mathcal{P}_{vw}$ such that v immediately preceeds w when travelling along P from $I \setminus J$ to $J \setminus I$. Now, define a vector

$y \in \mathbf{R}^{\vec{E}}$, such that, for $vw \in \vec{E}$,

$$y_{vw} = f_{vw} + \frac{1}{2}(x_{vw} - \rho(f)_{vw}).$$

It is easy verified that $y \in Q$, and $\rho(y) = x$. Thus, $x \in Q^\rho$, as required. □

Our interest in Lemma 8.8 is that it implies that the polyhedron described by the inequalities (8.9), (8.10) and (8.12) is integral.

The following proof is based on a proof of Edmonds' description of the perfect matching polyhedron due to Schrijver [60]; see also Green–Krótki [40].

Proof of Theorem 8.6. Let $P_1(I, J; G) \subseteq \mathbf{R}^E$ (or simply P_1) denote the polyhedron defined by the inequalities (8.9), (8.10), (8.11) and (8.12). Clearly, $\text{conv}(\mathcal{M}^*) \subseteq P_1$. For the converse, it suffices to prove that P_1 is integral. We prove this by induction on the number of vertices of G. We may assume that $V = I \cup J$.

In order to avoid using Edmonds' discription of the perfect matching polyhedra, we need to add some remarks about the case that $I = J = V$. In this case we may assume without loss of generality that V has an even number of elements. Hence, for sets X of size $|V| - 1$, the inequality $x(\delta(X)) \geq 2$ is implied by the degree constraints. Therefore, we impose the additional restriction on the inequalities 8.11 that $|X| \leq |V| - 2$. (This condition is vacuous in the case that $I \neq J$.)

Suppose that P_1 is not integral, and let $x' \in \mathbf{R}^E$ be a nonintegral extreme point of P_1. If x' does not satisfy any of the inequalities (8.11) with equality, then by Lemma 8.8, x' is integral, which is a contradiction. Choose $X \subseteq I \cap J$ such that $3 \leq |X| \leq |V| - 2$, $|X|$ is odd, and $x'(\delta(X)) = 2$.

Recall that $G \circ X$ denotes the graph obtained by shrinking X to a single vertex, which we label X. Denote by G_1 the graph $G \circ X$, and let $I_1 = (I \setminus X) \cup \{X\}$, $J_1 = (J \setminus X) \cup \{X\}$ and $x^{(1)}$ denote the restriction of x' to G_1. It is easily verified that $x^{(1)} \in P_1(I_1, J_1; G_1)$. Then, by induction, $\text{conv}(\mathcal{M}^*(I_1, J_1; G_1)) = P_1(I_1, J_1; G_1)$. Thus, there exists a nonnegative vector $\lambda^{(1)} \in \mathbf{R}^{\mathcal{M}^*(I_1,J_1;G_1)}$ such that $\lambda^{(1)}(\mathcal{M}^*(I_1, J_1; G_1)) = 1$, and

$$x^{(1)} = \sum_{M \in \mathcal{M}^*(I_1,J_1;G_1)} \lambda^{(1)}_M \psi^M.$$

Let $Y = V \setminus X$. Denote by G_2 the graph $G \circ Y$, and let $I_2 = J_2 = V_{G_2}$, and $x^{(2)}$ denote the restriction of x' to G_2. It is easily verified that $x^{(2)}$ satisfies inequalities (8.9), (8.10) and (8.12) for $P_1(I_2, J_2; G_2)$. Suppose $S \subseteq V_{G_1}$ such that $|S|$ is odd. If $Y \notin S$, then $x^{(2)}(\delta_{G_2}(S)) \geq 2$; otherwise, when $Y \in S$, $x^{(2)}(\delta_{G_2}(S)) = x^{(2)}(\delta_{G_2}(V \setminus S)) \geq 2$. Hence, $x^{(2)}$ is in $P_1(I_2, J_2; G_2)$. Then, by induction, there exists a nonnegative vector $\lambda^{(2)} \in \mathbf{R}^{\mathcal{M}^*(I_2,J_2;G_2)}$ such that $\lambda^{(2)}(\mathcal{M}^*(I_2, J_2; G_2)) = 1$, and

$$x^{(2)} = \sum_{M \in \mathcal{M}^*(I_2,J_2;G_2)} \lambda^{(2)}_M \psi^M.$$

Consider $M' \in \mathcal{M}^*(I_1, J_1; G_1)$, such that $\lambda^{(1)}_{M'} > 0$. M' is an (I_1, J_1)–path matching in G_1, so either there exist two edges $e_1, e_2 \in M'$ that are incident with X, or there exists a matching edge e_1 of M' that is incident with X. In the latter case, we take $e_2 = e_1$. Now, for $i = 1, 2$, $x^{(2)}(e_i) = x^{(1)}(e_i) > 0$, and so there exists $M_i'' \in \mathcal{M}^*(I_2, J_2; G_2)$ containing e_i such that $\lambda^{(2)}_{M_i''} > 0$. Since $I_2 = J_2$, M_i'' is a perfect matching of G_2. Let $M = M' \cup M_1'' \cup M_2''$. (For example, see Figure 8.2.) We have $\psi_e^M = \psi_e^{M'}$, for $e \in E_{G_1}$, and $\psi_e^M = (\psi_e^{M_1''} + \psi_e^{M_2''})/2$, for $e \in M_2$. Note that $M_1'' \cup M_2''$ may contain circuits of even length, so M is not necessarily a perfect (I, J)–path matching in G; however, ψ^M is the average of the path matching vectors of two perfect (I, J)–path matchings of G. By such pairings of the perfect (I_1, J_1)–path matchings of G_1 with perfect (I_2, J_2)–path matchings of G_2, we can obtain x' as a convex combination of path matching vectors of perfect (I, J)–path matchings in G. However, since x' is an extreme point, x' must be a path matching vector, contradicting that x' is fractional. □

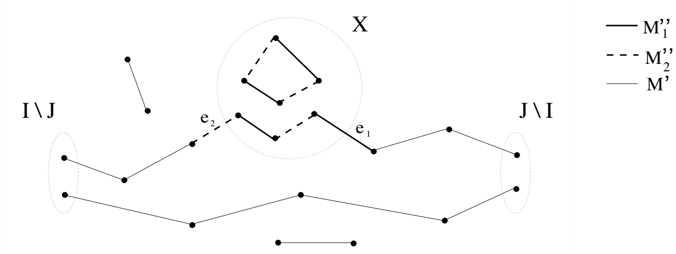

Figure 8.2: Combining solutions

As a corollary to Theorem 8.6, we get a second description of conv($\mathcal{M}^*$).

Corollary 8.9 *Let $G = (V, E)$ be a graph, and $I, J \subseteq V$. Then conv($\mathcal{M}^*(I, J; G)$) is described by the following inequalities:*

$$x(\delta(v)) = I_v + J_v \qquad (v \in V) \tag{8.13}$$

$$x(\gamma(X)) \leq |X \cap J| \qquad (X : I \setminus J \subseteq X \subseteq I) \tag{8.14}$$

$$x(\gamma(X)) \leq |X| - 1 \qquad (X \subseteq I \cap J, |X| \text{ odd}) \tag{8.15}$$

$$x \geq 0. \tag{8.16}$$

Proof Firstly, it is clear that (8.13), (8.14), (8.15) and (8.16) are valid for conv($\mathcal{M}^*$). Conversely, suppose that $x \in \mathbf{R}^E$ satisfies (8.13), (8.14), (8.15) and (8.16). Given a subset S of V, we have

$$\begin{aligned} x(\delta(S)) &= \sum_{v \in S} x(\delta(v)) - 2x(\gamma(S)) \\ &= |S \cap I| + |S \cap J| - 2x(\gamma(S)). \end{aligned}$$

Thus it is easy to check that inequalities (8.13) and (8.14) imply the inequalities (8.10). Also, inequalities (8.13) and (8.15) imply the inequalities (8.11). Trivially, x also satisfies (8.12) and (8.9). Therefore, by Theorem 8.6, $x \in$ conv($\mathcal{M}^*$), as required. □

We now prove Theorem 8.5 as a consequence of Corollary 8.9.

Proof of Theorem 8.5. It is clear that inequalities (8.4), (8.5), (8.6), (8.7) and (8.8) are valid for conv($\mathcal{M}$). Conversely, suppose that $y \in \mathbf{R}^n$ satisfies inequalities (8.4), (8.5), (8.6), (8.7) and (8.8).

Create a copy $\tilde{v}$ of each $v \in V$, and for $X \subseteq V$, denote by $\tilde{X}$ the corresponding copy of X. Similarly, for a subset S of E, we denote by $\tilde{S}$, the set $\{\tilde{v}\tilde{w} : vw \in S\}$. Now, construct a graph $G' = (V', E')$ such that $V' = V \cup \tilde{V}$, and $E' = E \cup \tilde{E} \cup \{v\tilde{v} : v \in V\}$, and let $I' = I \cup \tilde{J}$ and $J' = J \cup \tilde{I}$.

Claim *If there exists $y' \in conv(\mathcal{M}^*(I', J'; G'))$ such that y is the restriction of y' to E, then $y \in conv(\mathcal{M}(I, J; G))$.*

It suffices to prove the claim when y' is an extreme point. Thus, assume that $y' = \psi^{M'}$, for some $M' \in \mathcal{M}^*(I', J'; G')$. Let $M = M' \cap E$, and let S be the matching edges of M that are not matching edges of M'. Then, clearly, $y = \frac{1}{2}(\psi^M + \psi^{M \setminus S})$. Hence, $y \in$ conv($\mathcal{M}(I, J; G)$, which proves the claim.

Define $y' \in \mathbf{R}^{E'}$ such that, for $vw \in E$, $y'_{vw} = y_{vw}$, and $y'_{\tilde{v}\tilde{w}} = y_{vw}$, and, for $v \in V$, $y'_{v\tilde{v}} = I_v + J_v - y(\delta_G(v))$. By Theorem 8.9, conv($\mathcal{M}^*(I', J'; G)$) is defined by (8.13), (8.14), (8.15) and (8.16). Clearly, y' satisfies inequalities (8.13) and (8.16).

Let $X' \subseteq I'$ such that $I' \setminus J' \subseteq X'$. Define $X, Y \subseteq V$ such that $X' = X \cup \tilde{Y}$. Thus $I \setminus J \subseteq X \subseteq I$ and $J \setminus I \subseteq Y \subseteq J$. Then,

$$
\begin{aligned}
y'(\gamma_{G'}(X')) &= y(\gamma(X)) + y(\gamma(Y)) + |I \cap X \cap Y| + |J \cap X \cap Y| - \sum_{v \in X \cap Y} x(\delta(v)) && (8.17)\\
&= y(\gamma(X)) + y(\gamma(Y)) - 2y(\gamma(X \cap Y)) - y(\delta(X \cap Y)) \\
&\qquad + |I \cap X \cap Y| + |J \cap X \cap Y| && (8.18)\\
&\le y(\gamma(X \setminus Y)) + y(\gamma(Y \setminus X)) + |I \cap X \cap Y| + |J \cap X \cap Y| && (8.19)\\
&\le |(X \setminus Y) \cap I| + |(Y \setminus X) \cap J| + |I \cap X \cap Y| + |J \cap X \cap Y| && (8.20)\\
&= |I \cap X| + |J \cap Y| && (8.21)\\
&= |I' \cap X'|, && (8.22)
\end{aligned}
$$

where we get (8.19) from (8.18) by nonnegativity, and we get (8.20) from (8.19) by inequalities (8.5) and (8.6). Thus y' satisfies the inequalities (8.14).

Now, let $X' \subseteq I' \cap J'$ such that $|X'|$ is odd. Define $X, Y \subseteq V$ such that $X' = X \cup \tilde{Y}$. Thus $X, Y \subseteq I \cap J$, and exactly one of $|X|, |Y|$ is odd. Therefore exactly one of $|X \setminus Y|, |Y \setminus X|$ is odd. Suppose that $S \subseteq I \cap J$, then, by the inequalities (8.4), $y(\gamma(S)) \le |S|$. Then, with the inequalities (8.7),

$$
y(\gamma(X \setminus Y)) + y(\gamma(Y \setminus X)) \le |X \setminus Y| + |Y \setminus X| - 1. \qquad (8.23)
$$

Now,

$$
\begin{aligned}
y'(\gamma_{G'}(X')) &= y(\gamma(X)) + y(\gamma(Y)) + |I \cap X \cap Y| + |J \cap X \cap Y| - \sum_{v \in X \cap Y} y(\delta(v)) && (8.24)\\
&= y(\gamma(X)) + y(\gamma(Y)) - 2y(\gamma(X \cap Y)) - y(\delta(X \cap Y)) + 2|X \cap Y| && (8.25)\\
&\le y(\gamma(X \setminus Y)) + y(\gamma(Y \setminus X)) + 2|X \cap Y| && (8.26)\\
&\le |X \setminus Y| + |Y \setminus X| - 1 + 2|X \cap Y| && (8.27)\\
&= |X| + |Y| - 1 && (8.28)\\
&= |X'| - 1, && (8.29)
\end{aligned}
$$

where we get (8.26) from (8.25) by nonnegativity, and we get (8.27) from (8.26) by inequality (8.23). Therefore, y' satisfies the inequalities (8.15). So we have $y' \in$ conv($\mathcal{M}^*(I', J'; G')$); hence, by the claim, $y \in$ conv($\mathcal{M}(I, J; G)$), as required. □

Total dual integrality

By Theorem 8.5, the polyhedron defined by inequalities (8.4), (8.5), (8.6), (8.7) and (8.8) has integral vertices. Therefore, for any objective function $w \in \mathbf{R}^E$, the following linear program has an integral optimal solution

$$(P) - \begin{cases} \max w^T x \\ \text{s.t. inequalities (8.4), (8.5), (8.6), (8.7) and (8.8).} \end{cases}$$

Given subsets I, J of G, we define

$$\begin{aligned} \Omega^I &= \{X : I \setminus J \subseteq X \subseteq I\}, \\ \Omega^J &= \{X : J \setminus I \subseteq X \subseteq J\} \text{ and} \\ \Omega^{IJ} &= \{X \subseteq I \cap J : |X| \text{ is odd}\}. \end{aligned}$$

Note that Ω^I, Ω^J and Ω^{IJ} are disjoint sets, and let $\Omega = \Omega^I \cup \Omega^J \cup \Omega^{IJ}$. For a set $Y \in \Omega$, define $f(Y) \in \{0,1\}$ such that $f(Y) = 1$ exactly when $Y \in \Omega^{IJ}$. For variables $y \in \mathbf{R}^V$ and $z \in \mathbf{R}^\Omega$, it is easily checked that the dual (D) of (P) is given by

$$\min \sum_{v \in V} (I_v + J_v) y_v + \sum_{X \in \Omega} (|X \cap I \cap J| - f(X)) z_X, \tag{8.30}$$

$$y_u + y_v + \sum_{\substack{X \in \Omega \\ u,v \in X}} z_X \geq w_{uv} \quad (uv \in E) \tag{8.31}$$

$$y \geq 0, z \geq 0. \tag{8.32}$$

We will prove that, whenever w is integral, there exists an integral optimal solution to (D), in other words, the system of inequalities (8.4), (8.5), (8.6), (8.7) and (8.8), is *totally dual integral*; see Edmonds and Giles [33]. Cunningham and Marsh [24] proved that the system of inequalities in Edmonds' characterization of the matching polyhedron is totally dual integral. Our proof generalizes Schrijver's proof [60] of Cunningham and Marsh's theorem.

Let $\mathcal{S}$ be a collection of subsets of V. We call $\mathcal{S}$ a *laminar family* if, for each $X, Y \in \mathcal{S}$, either $X \subseteq Y$, $Y \subseteq X$ or $X \cap Y = \emptyset$. Let y, z be a solution of (D). We denote by $\Omega(z)$ the support of z, that is $\{X \in \Omega : z_X \neq 0\}$. We call the solution y, z of (D) a *laminar solution* if $\Omega(z)$ is a laminar family.

Theorem 8.10 *For all integral w, there exists an integral optimal solution to (D) that is laminar.*

Proof It suffices to prove the theorem for nonnegative w. Suppose that the result fails, and G, I, J, w form a counterexample with $|V| + |E| + w(E)$ as small as possible. For each edge e of G, $w_e \geq 1$, since otherwise we can delete e. Also, $V = I \cup J$, since we can delete the other vertices.

Claim 1 *For every optimal solution y, z to (D), $y = 0$.*

Let $\mathcal{F}$ denote the set of (I, J)–path matchings that attain the optimum of (P). Suppose that there exists $v \in V$ such that $\psi^M(\delta(v)) = I_v + J_v$ for each M in $\mathcal{F}$. We decrease the weight of each edge incident with v by one to get w'. Then, by our choice of w, there exists an integral optimal solution y', z' to (D), with respect to w', that is laminar. So, by increasing y'_v by one, we obtain an integral optimal solution to (D), with respect to w, that is laminar. So, for all $v \in V$, there exists $M \in \mathcal{F}$ such that $\psi^M(\delta(v)) < I_v + J_v$. Thus, by complementary slackness, $y_v = 0$, proving Claim 1.

Claim 2 *There exists an optimal solution to (D) that is laminar.*

For $z \in \mathbf{R}^\Omega$, we define $\tau(z) = \sum(z_X\,|X|\,|V \setminus X| : X \in \Omega)$. Let y, z be an optimal solution to (D) that minimizes $\tau(z)$. Suppose that $\Omega(z)$ is not laminar, and let $X, Y \in \Omega(z)$ such that $|X \setminus Y|, |Y \setminus X|, |X \cap Y| > 0$. By a simple case analysis, we find that either $X \setminus Y$ and $Y \setminus X$ are both in Ω, or $X \cap Y$ and $X \cup Y$ are both in Ω. We consider these cases separately.

Case 1: $X \setminus Y$ and $Y \setminus X$ are both in Ω. Let ϵ be the minimum of z_X and z_Y. We construct $z' \in \mathbf{R}^\Omega$ from z by decreasing z_X and z_Y by ϵ, and increasing $z_{X\setminus Y}$ and $z_{Y\setminus X}$ by ϵ. Now, construct $y' \in \mathbf{R}^V$, by increasing y_v by ϵ for all $v \in X \cap Y$. One easily checks that y', z' is an optimal solution to (D). However, $y' \neq 0$, which contradicts Claim 1.

Case 2: $X \cap Y$ and $X \cup Y$ are both in Ω. Let ϵ be the minimum of z_X and z_Y. We construct $z' \in \mathbf{R}^\Omega$ from z by decreasing z_X and z_Y by ϵ, and increasing $z_{X\cap Y}$ and $z_{X\cup Y}$ by ϵ. One easily checks that y, z' is an optimal solution to (D), and

$$\tau(z) - \tau(z') = 2\epsilon\,|X \setminus Y|\,|Y \setminus X| > 0,$$

contradicting our choice of z. This proves Claim 2.

Let y, z be an optimal solution to (D) that is laminar. By Claim 1, $y = 0$. Suppose that z is not integral, and let X be a maximum cardinality set in $\Omega(z)$ such that z_X is not integral. Now, let r be the fractional part of z_X, and $X_1, \ldots, X_k$ be the maximal proper subsets of X in $\Omega(z)$. Since $\Omega(z)$ is a laminar family, $X_1, \ldots, X_k$ are disjoint. Now define $z' \in \mathbf{R}^\Omega$ from z by decreasing z_X by r, and, for $i = 1, \ldots, k$, increasing z_{X_i} by r. For an edge $uv \in E$, the inequality (8.31) is trivially satisfied by y, z', unless $uv \in X$ and, for each $i = 1, \ldots, k$, $uv \notin \gamma(X_i)$. However, if $uv \in \gamma(X)$ and, for each $i = 1, \ldots, k$, $uv \notin \gamma(X_i)$, then, among all sets in $\Omega(z)$ that contain u, v, X is the only set for which z is fractional. Therefore, reducing z_X by r does not violate inequality (8.31), and hence y, z' are feasible for (D).

Let α and α' be the values for the dual solutions y, z and y, z' respectively. Then,

$$\begin{aligned}
\alpha' - \alpha &= r\sum_{i=1}^{k}(|X_i \cap I \cap J| - f(X_i)) - r(|X \cap I \cap J| - f(X)) \\
&= r\left(\left(f(X) - \sum_{i=1}^{k} f(X_i)\right) - \left(|X \cap I \cap J| - \sum_{i=i}^{k} |X_i \cap I \cap J|\right)\right). \qquad (8.33)
\end{aligned}$$

Note that, if $X \in \Omega^{IJ}$, then $X_1, \ldots, X_k$ are all in Ω^{IJ}; otherwise, if $X \notin \Omega^{IJ}$ then all but at most one of $X_1, \ldots, X_k$ are in Ω^{IJ}. Hence $f(X) - \sum(f(X_i) : i = 1, \ldots, k) \leq 1 - k$. Then, from (8.33), we easily check that $\alpha' - \alpha < 0$, contradicting that y, z is optimal. □

By considering the weight function $w = (1, \ldots, 1)$ in Theorem 8.10, one easily obtains an alternative proof of Theorem 8.3.

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