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Small cocircuits in matroids

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This paper is dedicated in memory of Tom Brylawski.

ABSTRACT

We prove that, for any positive integers k, n, and q, if M is a simple matroid that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor and M has sufficiently large rank, then M has a cocircuit of size at most $r(M)/k$.

1. Introduction

The main purpose of this paper is to give simpler proofs of two existing results in extremal matroid theory; we also prove the following new result:

Theorem 1.1. *For any positive integers k, n, and q, there is a positive integer R_1 such that, if M is a simple matroid of rank at least R_1 that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor, then M has a cocircuit of size at most $r(M)/k$.*

This easily implies the main result of [2], as we show immediately below.

Corollary 1.2. *For any positive integers n, k and q, there exists an integer R_2 such that, if M is a simple matroid of rank at least R_2 that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor, then M has a collection of k disjoint cocircuits.*

To prove Corollary 1.2 we use induction on k. The result is trivial for $k = 1$. For $k \geq 2$, we define $R_2(n, k, q) = \max(2R_2(n, k-1, q),\ R_1(n, 2, q))$. Let M be a matroid of rank at least $R_2(n, k, q)$ that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor. We may assume that M is simple. Then, by Theorem 1.1, M has a cocircuit C_k of size at most $r(M)/2$. Thus $r(M/C_k) \geq r(M)/2 \geq R_2(n, k-1, q)$. So, by induction, M/C_k has $k-1$ disjoint cocircuits, say $C_1, \ldots, C_{k-1}$. Thus $C_1, \ldots, C_k$ are disjoint cocircuits in M, as required.

In [3], Corollary 1.2 was used to prove the following result.

Theorem 1.3. *For any positive integers n and q, there exists an integer ρ such that, if M is a simple matroid that has neither a $U_{2,q+2}$-minor nor an $M(K_n)$-minor, then $|E(M)| \leq \rho r(M)$.*

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Note that neither $U_{2,4}$ nor $M(K_5)$ is cographic. Applying Corollary 1.2 to the class of cographic matroids gives the Erdős–Pósa theorem on edge-disjoint circuits in graphs; see [1]. Applying Theorem 1.3 to the class of graphic matroids gives Mader's theorem that, if G is a simple graph with no K_n-minor, then $|E(G)| \le \rho_n |V(G)|$; see [5].

In this paper we will use the methods of [3] to obtain a new proof of Theorem 1.3 that does not rely on Corollary 1.2. We will then use Theorem 1.3 to prove Theorem 1.1 and, hence, also Corollary 1.2. Proving the results in this order is significantly easier. We use several results from [3,2] but we include their proofs for the sake of completeness.

2. Preliminaries

For a more comprehensive introduction to extremal matroid theory, see the survey paper written by Joseph Kung [4]. We follow the notation of Oxley [6]. A rank-1 flat is referred to as a *point* and a rank-2 flat is referred to as a *line*. The number of points in M is denoted by $\epsilon(M)$. Kung [4] proved the following theorem; we include the proof since it is so nice.

Theorem 2.1. *For any integer $q \ge 2$, if M is a matroid with no $U_{2,q+2}$-minor, then $\epsilon(M) \le \frac{q^{r(M)}-1}{q-1}$.*

Proof. Let $e \in E(M)$. Inductively we may assume that $\epsilon(M/e) \le \frac{q^{r(M)-1}-1}{q-1}$. Since e is not in a $(q+2)$-point line, we have

$$\epsilon(M) \le q\epsilon(M/e) + 1 = q\left(\frac{q^{r(M)-1}-1}{q-1}\right) + 1 = \frac{q^{r(M)}-1}{q-1},$$

as required. $\square$

When q is a prime power, this bound is attained by projective geometries.

Let $\mathcal{U}(q)$ denote the set of all matroids with no $U_{2,q+2}$-minor. Our proof of Theorem 1.3 requires a bound on the number of hyperplanes in a rank-k matroid in $\mathcal{U}(q)$. Fortunately the quality of the bound is not important; we use the following crude upper bound from [3], Proposition 2.3.

Lemma 2.2. *Let $k \ge 1$ and $q \ge 2$ be integers and let $M \in \mathcal{U}(q)$ be a rank-k matroid. Then, M has at most $q^{k(k-1)}$ hyperplanes.*

Proof. Let $n = \epsilon(M)$; thus $n \le \frac{q^k-1}{q-1} \le q^k$. Each hyperplane is spanned by $k-1$ points, so the number of hyperplanes is at most $\binom{n}{k-1} \le n^{k-1} \le q^{k(k-1)}$. $\square$

The following result is from [2], Lemma 2.3.

Lemma 2.3. *Let $q \ge 2$ be an integer, let $M \in \mathcal{U}(q)$, and let C be a minimum-sized cocircuit of M. If C' is a cocircuit of $M \setminus C$, then $|C'| \ge |C|/q$.*

Proof. Set $F = E(M) - (C \cup C')$. Then F is a flat of M and M/F is a line with at most $q+1$ points. So there are at most $q+1$ hyperplanes of M containing F, one of which is $E(M) - C$. Let the others be $H_1, H_2, \ldots, H_{q'}$. Then $q' \le q$ and $\{H_1 - F, H_2 - F, \ldots, H_{q'} - F\}$ is a partition of C. Since C is a cocircuit of minimum size,

$$\begin{aligned} q'|C| &\le \sum_{i=1}^{q'} |E(M) - H_i| \\ &= \sum_{i=1}^{q'} (|C| + |C'| - |H_i - F|) \\ &= q'|C| + q'|C'| - |C|. \end{aligned}$$

Therefore $|C'| \ge |C|/q' \ge |C|/q$. $\square$

A *long line* is a line with at least three points. The following lemma is from [3], Lemma 3.2.

Lemma 2.4. *For integers $\alpha \geq 1$ and $q \geq 2$, if $M \in \mathcal{U}(q)$ is a matroid with $\epsilon(M) > \alpha q^2 r(M)$, then there is a minor N of M that contains more than $\alpha\epsilon(N)$ long lines.*

Proof. We may assume that M is simple. For each $v \in E(M)$, let $N_v = M/v$. Inductively, we may assume that $\epsilon(N_v) \leq \alpha q^2 r(N_v)$ for each $v \in E(M)$. Note that $r(N_v) = r(M) - 1$ and $\epsilon(M) > \alpha q^2 r(M)$, so $\epsilon(M) - \epsilon(N_v) \geq \alpha q^2 + 1$. Since $M \in \mathcal{U}(q)$, each long line in M has at most $q + 1$ points; so each parallel class in M/v has at most q elements. Thus v is on at least $\frac{\alpha q^2}{q-1}$ long lines. So the number of long lines is at least $\frac{\alpha q^2}{(q-1)(q+1)}\epsilon(M) > \alpha\epsilon(M)$. □

We use the following lemma from [2], Lemma 5.1 to recognize the cycle matroid of a clique.

Lemma 2.5. *Let M be a matroid with ground set $B \cup H$ where $B = \{b_1, \ldots, b_n\}$ is a basis of M, $H = \{e_{ij} : 1 \leq i < j \leq n\}$ is a hyperplane of M disjoint from B, and $\{b_i, e_{ij}, b_j\}$ is a triangle of M for each $i < j$. Then M is isomorphic to $M(K_{n+1})$.*

Proof. Construct a complete graph G with vertex set $V = \{v_0, \ldots, v_n\}$ and edges labelled by $B \cup H$ where $b_i \in B$ labels the edge incident with v_0 and v_i and $e_{ij} \in H$ labels the edge incident with v_i and v_j. We claim that $M = M(G)$; they clearly have the same rank. Consider a spanning tree T of G. If there exists an edge $e_{ij} \in T \cap H$ such that v_i has degree 1 in T, then $(T - \{e_{ij}\}) \cup \{b_i\}$ is a spanning tree of G and $r_M((T - \{e_{ij}\}) \cup \{b_i\}) = r_M(T)$. By repeatedly applying such changes, we see that $r_M(T) = r_M(B)$. Thus T is a basis of M. Now consider a circuit C of G, and let X be the set of edges in B that are incident with a vertex of $C - v_0$ in G. Note that $C \subseteq cl_M(X)$. If $B \cap C \neq \emptyset$ then $|X| < |C|$, so C is dependent in M. On the other hand, if $C \subseteq H$ then, since $|C| = |X|$ and $C \subseteq H \cap cl_M(X)$, we see that C is dependent in M. Hence $M = M(G)$ as required. □

3. Stratified matroids

We call a matroid M *round* if each cocircuit of M is spanning. Equivalently, M is round if and only if M does not contain a pair of disjoint cocircuits. By Corollary 1.2, every round matroid with sufficiently large rank contains either a $U_{2,q+2}$-minor or an $M(K_n)$-minor. This result is used in [3] to prove Theorem 1.3. However, the round minors produced in [3] have additional structure from which it is straightforward to extract a $U_{2,q+2}$-minor or an $M(K_n)$-minor.

A *stratification* of a rank-r matroid M is a sequence $(X_1, \ldots, X_r)$ such that, for each $k \in \{1, \ldots, r\}$, X_k is a rank-k flat of M and, for $k < r$, $X_{k+1} - X_k$ is a spanning cocircuit of $M|X_{k+1}$. If M admits a stratification, then we say that it is *stratified*. The following lemma shows that stratified matroids are round.

Lemma 3.1. *Let C be a spanning cocircuit of a matroid M. If $M \setminus C$ is round, then M is round.*

Proof. Suppose that M is not round; thus M contains a non-spanning cocircuit C_1. Since C is spanning $C_1 \neq C$ and, since C and C_1 are both cocircuits, $C_1 - C$ is non-empty. Now $C_1 - C$ contains a cocircuit, say C_2, of $M \setminus C$. However, $M \setminus C$ is round so $r_M(C_2) = r(M \setminus C) = r(M) - 1$. Since $C_2 \subseteq C_1$ and since C_1 is non-spanning, we see that C_2 spans C_1. Therefore C_1 is contained in the hyperplane $E(M) - C$. This yields a contradiction since C_1 is a cocircuit and C is spanning. □

The main result of this section is:

Lemma 3.2. *For any integers $n \geq 1$ and $q \geq 2$, if $M \in \mathcal{U}(q)$ is stratified and $r(M) > \frac{q^{n-2}-1}{q-1}$, then M contains an $M(K_n)$-minor.*

Lemma 3.2 is an immediate consequence of the following result; we restate it in this form to facilitate induction.

Lemma 3.3. *Let $n \geq 1$ and $q \geq 2$ be integers, let $M \in \mathcal{U}(q)$, and let $(F_0, F_1, \ldots, F_k)$ be a sequence of nested flats of M such that $r_M(F_0) = n-2$ and, for each $i \in \{1, \ldots, k\}$, the set $F_i - F_{i-1}$ is a spanning cocircuit of $M|F_i$. If $\epsilon(M|F_0) + k > \frac{q^{n-2}-1}{q-1}$, then M contains an $M(K_n)$-minor.*

Proof. We prove the theorem by induction on k. By Theorem 2.1, $k \geq 1$. Let $M_1 = M|F_1$ and let $B = \{b_1, \ldots, b_{n-1}\} \subseteq F_1 - F_0$ be a basis of M_1. We may assume that M does not contain an $M(K_n)$-minor. Then, by Lemma 2.5, there are two elements in B, say b_1 and b_2, that do not span a point in F_0. It follows that $\epsilon(M_1/b_1) > \epsilon(M|F_0)$. Let $M' = M/b_1$ and, for each $i \in \{0, \ldots, k-1\}$, let $F_i' = F_{i+1} - \{b_1\}$. Note that $\epsilon(M'|F_0') + (k-1) = \epsilon(M_1/b_1) + (k-1) \geq \epsilon(M|F_0) + k$. Therefore the result follows inductively by considering M' and $(F_0', \ldots, F_{k-1}')$.

4. The density theorem

In this section we prove Theorem 1.3. We use the methods of [3] almost verbatim, except that we apply Lemma 3.2 in place of Corollary 1.2.

A flat F of a matroid M is *stratified* if $M|F$ is stratified. Let $\mathcal{F}$ be a set of stratified rank-$(k-1)$ flats in M. A rank-k flat F is called $\mathcal{F}$*-constructed* if there exist two flats $F_1, F_2 \in \mathcal{F}$ such that $F = cl_M(F_1 \cup F_2)$ and $F \neq F_1 \cup F_2$. We let $\mathcal{F}^+$ denote the set of $\mathcal{F}$-constructed flats. The following lemma shows that the flats in $\mathcal{F}^+$ are stratified.

Lemma 4.1. *Let F_1 and F_2 be two stratified rank-$(k-1)$ flats in a matroid M and let F be the flat spanned by $F_1 \cup F_2$. If $r_M(F) = k$ and $F - (F_1 \cup F_2)$ is non-empty, then F is stratified.*

Proof. Since F_1 is stratified, it suffices to prove that $F - F_1$ is a spanning cocircuit of $M|F$. Let $e \in F - (F_1 \cup F_2)$. By Lemma 3.1, $M|F_2$ is round. Since $r_M(F_1 \cup F_2) > r_M(F_1)$, $F_2 - F_1$ contains a cocircuit of $M|F_2$ and, since $M|F_2$ is round, $r_M(F_2 - F_1) = k-1$. Since e is not contained in the flat F_2, $r_M((F_2 - F_1) \cup \{e\}) = k$. Now $(F_2 - F_1) \cup \{e\} \subseteq F - F_1$, so $F - F_1$ is a spanning cocircuit of $M|F$, as required. □

Most of the remaining work is in the proof of the following technical lemma.

Lemma 4.2. *For all integers $k \geq 2$, $\alpha \geq 1$, and $q \geq 2$, if $M \in \mathcal{U}(q)$ is a matroid with $\epsilon(M) > \alpha q^{6\binom{k+1}{3}} r(M)$, then there exists a minor N of M and a set $\mathcal{F}$ of stratified rank-$(k-1)$ flats of N such that $|\mathcal{F}^+| > \alpha|\mathcal{F}|$.*

Proof. The proof is by induction on k. Consider the case where $k = 2$. Let $M \in \mathcal{U}(q)$ be a matroid with $\epsilon(M) > \alpha q^{6\binom{4}{3}} r(M) > \alpha q^2 r(M)$. By Lemma 2.4, there exists a minor N of M with more than $\alpha\epsilon(N)$ long lines. Now, if $\mathcal{F}$ is the set of points of N, then $\mathcal{F}^+$ is the set of long lines of N and $|\mathcal{F}^+| > \alpha|\mathcal{F}|$, as required.

Suppose that the result holds for $k = n$ and consider the case where $k = n+1$. Now let $M \in \mathcal{U}(q)$ be a matroid with $\epsilon(M) > \alpha q^{6\binom{n+2}{3}} r(M)$. We let $\alpha' = q^{n(n+1)}\alpha + q^n$. Note that

$$\begin{aligned}\alpha q^{6\binom{n+2}{3}} &= \alpha q^{6\binom{n+1}{3}} q^{6\binom{n+1}{2}} \\ &= \alpha q^{6\binom{n+1}{3}} q^{3n(n+1)} \\ &> \alpha' q^{6\binom{n+1}{3}}.\end{aligned}$$

So, by the induction hypothesis, there exists a minor N of M and a set $\mathcal{F}$ of stratified rank-$(n-1)$ flats of N such that $|\mathcal{F}^+| > \alpha'|\mathcal{F}|$. We may assume that no proper minor of N contains such a collection of flats. We may also assume that N is simple. We will prove that $|(\mathcal{F}^+)^+| \geq \alpha|\mathcal{F}^+|$.

For each $v \in E(N)$, let $N_v = N/v$ and let $\mathcal{F}_v$ denote the set of rank-$(n-1)$ flats in N_v corresponding to the set of flats in $\mathcal{F}$ in N. That is, if $F \in \mathcal{F}$ and $v \notin F$, then $cl_{N_v}(F) \in \mathcal{F}_v$. Note that a matroid that contains a stratified spanning restriction is itself stratified. Therefore the flats in $\mathcal{F}_v$ are stratified. By our choice of N, $|\mathcal{F}^+| > \alpha'|\mathcal{F}|$, and, by the minimality of N, $|\mathcal{F}_v^+| \le \alpha'|\mathcal{F}_v|$ for all $v \in E(N)$. Thus,

$$(|\mathcal{F}^+| - |(\mathcal{F}_v)^+|) > \alpha'(|\mathcal{F}| - |\mathcal{F}_v|).$$

Let

$$\Delta = \sum(|\mathcal{F}| - |\mathcal{F}_v| : v \in E(N)) \quad \text{and}$$
$$\Delta^+ = \sum(|\mathcal{F}^+| - |(\mathcal{F}_v)^+| : v \in E(N)).$$

This proves:

4.2.1. $\Delta^+ > \alpha'\Delta$.

Consider a flat $F \in \mathcal{F}^+$. By definition there exist flats $F_1, F_2 \in \mathcal{F}$ such that $F = cl_N(F_1 \cup F_2)$ and there exists an element $v \in F - (F_1 \cup F_2)$. Now $cl_{N_v}(F_1) = cl_{N_v}(F_2)$, so these two flats in $\mathcal{F}$ are reduced to a single flat in $\mathcal{F}_v$. This proves:

4.2.2. $\Delta \ge |\mathcal{F}^+|$.

For some $v \in E(N)$, compare $\mathcal{F}^+$ with $(\mathcal{F}_v)^+$. There are two ways to lose constructed flats; we can either contract an element in a flat or we contract two flats onto each other. Firstly, suppose $F \in \mathcal{F}^+$ and $v \in F$. Note that $F - \{v\}$ only has rank $n-1$ in N/v, so it will not determine a flat in $(\mathcal{F}_v)^+$. Now F has rank n and, by Theorem 2.1, a rank-n flat contains at most $\frac{q^n-1}{q-1} < q^n$ points; we destroy F if we contract any one of these points. Secondly, consider two flats $F_1, F_2 \in \mathcal{F}^+$ that are contracted onto each other in N_v. Let F be the flat of N spanned by $F_1 \cup F_2$ in N. Since F_1 and F_2 are contracted onto a common rank-k flat in N_v, we see that F has rank $k+1$ and $v \in F - (F_1 \cup F_2)$. Thus, $F \in (\mathcal{F}^+)^+$. Now F has rank $n+1$, so it has at most q^{n+1} points. Moreover, by Lemma 2.2, in a flat of rank $n+1$ there are at most $q^{(n+1)n}$ rank-n flats avoiding a given element. Thus, $F - \{v\}$ contains at most $q^{(n+1)n}$ flats of $\mathcal{F}$; these flats will be contracted to a single flat in $(\mathcal{F}_v)^+$. This proves:

4.2.3. $\Delta^+ \le q^n|\mathcal{F}^+| + q^{n(n+1)}|(\mathcal{F}^+)^+|$.

Combining Claims 4.2.1–4.2.3, we get

$$\begin{aligned} q^{n(n+1)}|(\mathcal{F}^+)^+| &\ge \Delta^+ - q^n|\mathcal{F}^+| \\ &> \alpha'\Delta - q^n|\mathcal{F}^+| \\ &\ge (\alpha' - q^n)|\mathcal{F}^+| \\ &= \alpha q^{n(n+1)}|\mathcal{F}^+|. \end{aligned}$$

Therefore $|(\mathcal{F}^+)^+| > \alpha|\mathcal{F}^+|$, as required. □

We are now ready to prove Theorem 1.3 which we restate here in a more convenient form.

Theorem 4.3. *For any integers $n \ge 1$ and $q \ge 2$, if $M \in \mathcal{U}(q)$ is a matroid with $\epsilon(M) > q^{q^{3n}} r(M)$, then M contains an $M(K_n)$-minor.*

Proof. Let $k = \frac{q^{n-2}-1}{q-1} + 1$. Since $k \le q^{n-2}$ we have

$$q^{6\binom{k+1}{3}} \le q^{(k+1)^3} < q^{q^{3n}}.$$

Therefore, by Lemmas 4.1 and 4.2, M contains a stratified minor N of rank k. Then, by Lemma 3.2, N contains an $M(K_n)$-minor. □

5. Small cocircuits

In this section we prove Theorem 1.1. We start with the following easy lemma.

Lemma 5.1. *For any integers $k \geq 2$ and $m \geq 2$, and real number $R \geq 1$, if M is a matroid with rank at least $(\frac{k}{k-1})^{m-2}R$ that does not contain m disjoint cocircuits, then M has a contraction-minor N with rank at least R such that each cocircuit of N has rank at least $r(N)/k$.*

Proof. The proof is by induction on m. Let M be a matroid with rank at least $(\frac{k}{k-1})^{m-2}R$. We may assume that M has a cocircuit C with rank less that $r(M)/k$. Since $r(C) < r(M)$, M has two disjoint cocircuits and, hence, we may assume that $m > 2$. Now $r(M/C) \geq (\frac{k}{k-1})^{m-3}R$. Then, by the induction hypothesis, either M/C contains $m-1$ disjoint cocircuits or M/C has a contraction-minor N with rank at least R such that each cocircuit of N has rank at least $r(N)/k$. In either case the result follows. □

The following lemma is similar to [2], Lemma 4.2.

Lemma 5.2. *There exists a integer-valued function $R_3(\rho, k, R, q)$ such that, for any positive integers $q \geq 2$, k, ρ, and R, if $M \in \mathcal{U}(q)$ is a matroid with rank at least $R_3(\rho, k, R, q)$, then either M has a cocircuit of rank at most $r(M)/k$ or M has a contraction-minor N with rank at least R such that $\epsilon(N) > \rho(r(N) - \rho)$.*

Proof. We will assume that $k \geq 2$; the result can be extended to the case $k = 1$ with $R_3(\rho, 1, R, q) = R_3(\rho, 2, R, q)$. Let $R_3(1, k, R, q) = R$, and, for $\rho > 1$, we recursively define

$$R_3(\rho, k, R, q) = \left\lceil k\left(\left(\frac{k}{k-1}\right)^{\rho kq-2} R_3(\rho - 1, k, R, q) + 1\right)\right\rceil.$$

The proof is by induction on ρ. The result is trivial when $\rho = 1$. Suppose that $\rho > 1$ and that the result holds for smaller values of ρ.

Let $M \in \mathcal{U}(q)$ be a matroid with rank at least $R_3(\rho, k, R, q)$ such that each cocircuit of M has rank greater than $r(M)/k$. Let $r_2 = R_3(\rho - 1, k, R, q)$, $m = \rho kq$, and $r_1 = (\frac{k}{k-1})^{m-2}r_2 + 1$. Let C be a minimum-size cocircuit of M. Note that $|C| \geq r_M(C) \geq r(M)/k$, so, by Lemma 2.3, each cocircuit of $M \setminus C$ has size at least $\frac{r(M)}{kq}$. We assume that M does not have a minor N with $r(N) \geq R$ and $\epsilon(N) > \rho r(N)$. It is straightforward to verify that $r(M \setminus C) = r(M) - 1 \geq R$ and, hence, $\epsilon(M \setminus C) \leq \rho r(M \setminus C) < m\frac{r(M)}{kq}$. Therefore $M \setminus C$ does not have m disjoint cocircuits. There is a contraction-minor M_1 of M such that $C \subseteq E(M_1)$ and $r(M_1) = r_{M_1}(C) = r_M(C) \geq r_1 = (\frac{k}{k-1})^{m-2}r_2+1$. Since $M_1 \setminus C$ is a contraction-minor of $M \setminus C$ and $M \setminus C$ does not have m disjoint cocircuits, $M_1 \setminus C$ does not have m disjoint cocircuits. Then, by Lemma 5.1, there is a set $X_1 \subseteq E(M_1 \setminus C)$ such that $r(M_1 \setminus C/X_1) \geq r_2 = R_3(\rho - 1, k, R, q)$ and that each cocircuit of $M_1 \setminus C/X_1$ has rank at least $r(M_1 \setminus C/X_1)/k$. Let $M_2 = M_1/X_1$. By the induction hypothesis, there is a set $X_2 \subseteq E(M_2 \setminus C)$ such that $r(M_2 \setminus C/X_2) \geq R$ and $\epsilon(M_2 \setminus C/X_2) \geq (\rho - 1)(r(M_2 \setminus C/X_2) - \rho + 1)$. Let $M_3 = M_2/X_2$. Since C is a spanning cocircuit of M_3, $\epsilon(M_3) \geq \epsilon(M_3 \setminus C) + r(M_3) \geq (\rho - 1)(r(M_3) - \rho) + r(M_3) \geq \rho(r(M_3) - \rho)$, as required. □

We are now ready to prove Theorem 1.1 which we restate here for convenience.

Theorem 5.3. *For any positive integers $q \geq 2$, k, and n, there is a positive integer R_1 such that, if $M \in \mathcal{U}(q)$ is a simple matroid of rank at least R_1 and each cocircuit of M has size greater than $r(M)/k$, then M an $M(K_n)$-minor.*

Proof. Let $\rho = q^{q^{3n}}$ and let $R_1 = \lceil R_3(\rho + 1, \rho k, (\rho + 1)^2, q)\rceil$. Now let $M \in \mathcal{U}(q)$ be a simple matroid of rank at least R_1 such that each cocircuit of M has size greater than $r(M)/k$. We assume that M has no $M(K_n)$-minor. By Theorem 4.3, for each minor N of M, $\epsilon(N) \leq \rho r(N)$. In particular, each cocircuit of M has rank greater than $r(M)/(k\rho)$. Then, by Lemma 5.2, there is a minor N of M such that $r(N) \geq (\rho + 1)^2$ and $\epsilon(N) > (\rho + 1)(r(N) - \rho - 1) = \rho r(N) + r(N) - (\rho + 1)^2 \geq \rho r(N)$. This contradiction completes the proof. □

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