Regular Matroid Decomposition Via Signed Graphs

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Received April 29, 2002

Published online in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20037

Abstract: The key to Seymour's Regular Matroid Decomposition Theorem is his result that each 3-connected regular matroid with no R_{10}- or R_{12}-minor is graphic or cographic. We present a proof of this in terms of signed graphs. © 2004 Wiley Periodicals, Inc. J Graph Theory 48: 74–84, 2005

This research was carried out while the second author visited the University of Waterloo.
Contract grant sponsor: Natural Sciences and Engineering Council of Canada.

Keywords: *regular; graphic and cographic matroids; decomposition; signed graphs*

1. INTRODUCTION

Seymour's Regular Matroid Decomposition Theorem [3] says that each regular matroid can be obtained from graphic matroids, their duals, and copies of R_{10} by taking 1-, 2-, and 3-sums. The key part of his proof is the following result.

(1) Theorem *Each 3-connected regular matroid that is neither graphic nor cographic contains an R_{10}- or R_{12}-minor.*

The proof of the Regular Matroid Decomposition Theorem is complemented by a result saying that the up to isomorphism, only 3-connected regular matroid with an R_{10}-minor is R_{10} and by a result saying that each regular matroid with an R_{12}-minor is a 3-sum of two proper minors of the matroid. In this paper, we present an alternative proof of Theorem (1). Seymour's proof uses "grafts": graphs and t-joins. They naturally arise in an inductive argument where one considers binary matroids with an element whose deletion is graphic. In a similar inductive approach, we instead consider binary matroids with an element whose contraction is graphic. Such matroids can be encoded by "signed graphs," which leads to the proof in this paper. Similar inductive proofs are known for Tutte's characterization of graphic matroids [6]; Seymour [4] uses grafts and Gerards [1] signed graphs.

Theorem (1) also implies a forbidden minor characterization of the matroids that are graphic or cographic. The only nonregular forbidden minors for that class of matroids are clearly U_4^2, F_7, and F_7^*; Theorem (1) says that the only 3-connected regular forbidden minors are R_{10} and R_{12}; and the non-3-connected forbidden minors—there are 13 of those—are quite easy to find. The only two proofs of Theorem (1) known to us, Seymour's original proof in [3] (see Truemper [5] for a shorter version) and the proof in this paper, are both more complicated than existing proofs of Tutte's characterization of graphic matroids [6], even though both proofs use that characterization. This may appear unexpected. However, the union of two matroid classes may have infinitely many forbidden minors even if the separate classes both have only finitely many forbidden minors (Vertigan [7]). So there is no reason to expect that Theorem (1) can be derived easily from the characterization of graphic matroids.

A *signed graph* is a pair (G, Σ) where $G = (V(G), E(G))$ is an undirected graph, possibly with loops and parallel edges, and Σ is a subset of $E(G)$. Edges in Σ are called *odd*. The other edges are called *even*. A subgraph of G, like a path or a circuit, is called *odd* (*even*) in (G, Σ), if it contains an odd (even) number of odd edges. *Resigning* (G, Σ) *on* $U \subseteq V(G)$ is replacing Σ by the symmetric difference $\Sigma \triangle \delta(U)$ of Σ and $\delta(U) := \{uv \in E(G) \mid u \in U, v \notin U\}$. A *minor* of (G, Σ) is a signed graph that comes from (G, Σ) by a series of the following operations: resigning, deletion of an edge or isolated vertex, and contraction of an even edge. In Section 2, we prove the following result on signed graphs.

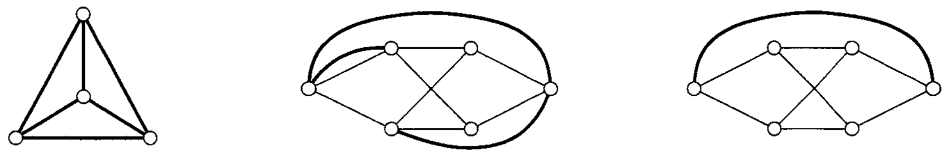

FIGURE 1. $\overline{K}_4$ (left), $\overline{R}_{12}$ (middle), and a basket (right).

(2) Theorem *Let (G, Σ) be a signed graph with no blockvertex and no $\overline{K}_4$-minor, and such that G consists of a 3-connected graph with a subdivision of $K_{3,3}$ and, possibly, some extra parallel edges. Then (G, Σ) has an $\overline{R}_{12}$-minor.*

Here, a *blockvertex* is a vertex that is in each odd circuit and $\overline{K}_4$ and $\overline{R}_{12}$ are the signed graphs in Figure 1 (in all figures bold lines indicate odd paths and thin lines even paths). In Section 4, we derive Theorem (1) from Theorem (2). The link between signed graphs and binary matroids is explained in Section 3.

2. PROVING THEOREM (2)

We first give some general definitions. An *st-leg* of a graph G is an *st*-path where s and t are the only vertices of the path that have degree greater than 2 in G. An *st-link in G* of a subgraph H is an *st*-path in G that intersects H only in s and t. A *loop of H in G* is a circuit in G that intersects H in exactly one vertex. If s and t are vertices of a path P, then P_{st} is the *st*-subpath of P. If H is a subgraph of G (or just a subset of its edges or if its vertices) then $G \backslash H$ is graph obtained from G by deleting the edges of H and $G - H$ is the graph obtained by deleting the vertices of H.

A signed graph (G, Σ) is *bipartite* if $\Sigma = \delta(U)$ for some $U \subseteq V(G)$, in other words, if one can resign (G, Σ) to $(G, \emptyset)$. As resigning clearly does not affect the parity—odd or even—of any circuit in G, a signed graph is bipartite if and only if it contains no odd circuits. The following easy fact is used frequently in this proof.

(3) *Let (G, Σ) be a nonbipartite signed graph such that G is 2-connected and let H be a connected subgraph of G with at least two vertices. If H has no odd edges, then H has an odd link in G. Equivalently, if H is bipartite, then H has a link L such that $H \cup L$ is nonbipartite.*

Now we prove Theorem (2).

***Proof of* (2).** Assume that it is false and that (G, Σ) is a counterexample. We always implicitly consider a subgraph H of G as the signed graph $(H, \Sigma \cap E(H))$. A signed subdivision of $K_{3,3}$ that can be resigned such that it has exactly one odd leg is called a *basket* (see Fig. 1 (right)).

(4) *G contains a basket.*

Assume this is false. It is straightforward to check that each nonbipartite signed subdivision of $K_{3,3}$ is a basket or has a $\overline{K}_4$-minor. So, all $K_{3,3}$-subdivisions in G are bipartite. Let H be such a bipartite subdivision of $K_{3,3}$. Let U and W denote the two sets of degree-3 vertices of H that correspond to the two color classes in $K_{3,3}$ and let F be the set of edges in G with both ends in U or both ends in W. If $G\backslash F$ is bipartite, we may re-sign such that all odd edges lie in F. From that it is easy to see that $H \cup F$ contains a $\overline{K}_4$-minor or that G has a blockvertex in $U \cup W$. So we may assume that $G\backslash F$ is not bipartite.

Resign such that all edges in H are even. As $G\backslash F$ is 2-connected and nonbipartite, H has an odd st-link L in $G\backslash F$. We may assume that s and t are both in U (or both in W), as otherwise $H \cup L$ contains a subdivision of $K_{3,3}$ that uses L and thus is nonbipartite. As $G\backslash F$ contains a 3-connected spanning subgraph, $(H \cup L) - \{s,t\}$ has a link R connecting L and vertex u in H. As $L \cup R$ contains an odd link of H with u as endvertex, u lies in U as well; so $\{s,t,u\} = U$. Now $H \cup L \cup R$ contains a $K_{3,3}$-subdivision using L and R; contrary to our assumption, this $K_{3,3}$-subdivision is nonbipartite. So (4) follows.

Each basket H has a unique leg, T_H, with the property that we can resign H such that T_H is odd and all other legs are even. We denote the set of vertices of H not on T_H by B_H. If T_H intersects all odd circuits in G, we call basket H *blocking*.

A *linked basket* is a triple (H, L_1, L_2) with the following properties: H is a basket; L_1 and L_2 are links or loops of H such that $(L_1 \cup L_2) - T_H$ is bipartite; and T_H contains an edge h such that $(H \cup L_1)\backslash h$ and $(H \cup L_2)\backslash h$ are nonbipartite, and such that no component of $T_H\backslash h$ intersects both L_1 and L_2. (Note that, as $(L_1 \cup L_2) - T_H$ is bipartite, this implies that one component of $T_H\backslash h$ contains at least one endvertex of L_1 and none of L_2 and the other component contains at least one endvertex of L_2 and none of L_1.)

As the only blockvertices of a basket H are the vertices on T_H, both nonblocking and linked baskets certify the nonexistence of blockvertices in G.

(5) *G contains a nonblocking basket or a linked basket.*

Assume this is false. Consider any basket H. Let s and t be the endvertices of leg T_H. As $G - s$ is nonbipartite and 2-connected, $H - s$ has a link or loop L such that $(H \cup L) - s$ is nonbipartite. As H is blocking, L has an endvertex on T_H. Let u be the endvertex of L on T_H that lies along T_H closest to s (see Fig. 2 (left); here, and in later figures, dotted lines are even paths, possibly of length zero). If L is a link, let v be the other endvertex of L; otherwise $v := u$. Assume now that H and

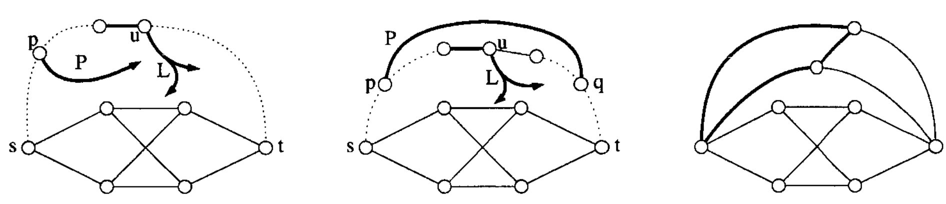

FIGURE 2.

L are chosen such that $(T_H)_{ut}$ is as short as possible and such that subject to that, if possible, $v \neq u$.

Basket H is blocking, so we may resign such that all edges not meeting T_H are even and such that the only odd edges of $H \cup L$ are the edge on $(T_H)_{su}$ incident with u and one edge on L incident with u. As u is not a blockvertex of G, $H - u$ has an odd link P. As H is blocking, this link has an endvertex on T_H. Let p be the one closest to s along T_H. Then $p \in (T_H)_{su} - u$ (by the minimality of $(T_H)_{ut}$). Let q be the other endvertex of P. As (H, P, L) is not a linked basket, $q \in (T_H)_{ut} - u$. See Figure 2 (middle).

Let H' be the basket in $H \cup P$ with $T_{H'} = (T_H)_{sp} \cup P \cup (T_H)_{qt}$. Paths L and P intersect internally, as otherwise $v \notin (T_H)_{uq} - q$ (as H' is blocking) and, hence, $L \cup (T_H)_{uq}$ is an odd loop or link of H' contradicting the minimality of $(T_H)_{ut}$. Note that P has only one odd edge and that it meets p or q. As H and L are selected with $(T_H)_{ut}$ minimal and with $v \neq u$ (if possible), this odd edge cannot meet q, so it meets p. Hence $H \cup L \cup P$ contains the signed graph in Figure 2 (right) as a minor. That signed graph has a $\overline{K}_4$-minor, so (5) follows.

(6) *For each basket H: an odd circuit disjoint from T_H shares at most one vertex with B_H.*

Assume this is false; let H be a counterexample. Resign such that all edges of H not meeting T_H are even. Then H has an odd link L with both endvertices on B_H. Now H has a leg Q that shares an endvertex with T_H such that L has both its endvertices, b and c say, in Q (as otherwise, we have one of the two signed graphs in Fig. 3 (left and middle) as a minor; both have a $\overline{K}_4$-minor). Assume the endvertices of Q are a and t, of T_H are s and t, and of the third leg of H meeting t are d and t. See Figure 3 (right).

Choose H and L such that the length of Q_{ab} is minimal. By 3-connectivity, there exists a link P of $H \cup L$ connecting $u \in (L \cup Q_{bt}) - \{b, t\}$ with $v \in H - Q_{bt}$. Let P' be the union of P with a uc-path in $(L \cup Q_{bt}) - b$. By resigning on the set of vertices of Q_{ct} and interchanging L and Q_{bc}, and s and d, we may assume that P' is odd. Hence, by the minimality of Q_{ab}, vertex $v \notin Q_{ab} - b$, so $v \notin Q$. As argued before, this means that $v \notin B_H$. So, v lies on $T_H - t$. Contracting $(T_H)_{sv}$, the leg from t to d, Q_{ab}, and P'_{uc}, we get an $\overline{R}_{12}$-minor. So (6) follows.

(7) *If (H, L_1, L_2) is a linked basket and L_1 has a vertex in B_H, then L_2 has no endvertices in B_H and L_1 and L_2 are disjoint.*

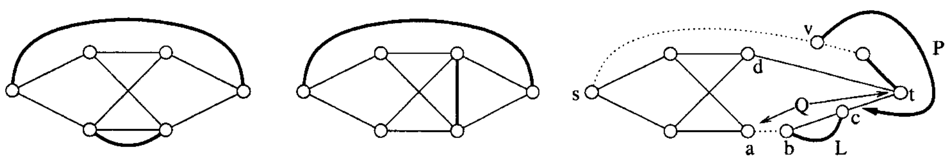

FIGURE 3.

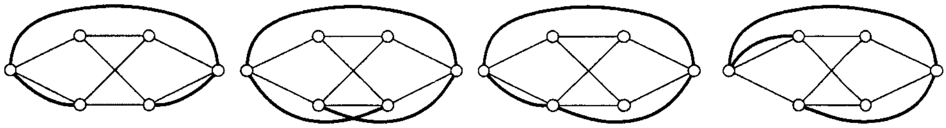

FIGURE 4.

Assume this is false; let (H, L_1, L_2) be a counterexample. If L_1 and L_2 intersect, $L_1 \cup L_2$ contains a link L_2' of H such that (H, L_1, L_2') is a linked basked and both L_1 and L_2' have an endvertex in B_H. So we may assume that L_2 has an endvertex in B_H. Then, as $(L_1 \cup L_2) - T_H$ is bipartite, G contains one of the signed graphs in Figure 4 as a minor; the first three have a $\overline{K}_4$-minor, the fourth one is $\overline{R}_{12}$. This contradiction proves (7).

(8) *If H is a basket, C is an odd circuit disjoint from T_H, and P a minimal path connecting a vertex $p \in C$ with a vertex $q \in H$, then $q \in T_H$. (In particular, C is disjoint from H.)*

Assume this is false; let H, C, and P form a counterexample with the length of P minimal. Then, by (6), C and H share no vertex, except maybe q. As $G - p$ is 2-connected, there exist two paths R_1 and R_2 connecting $C - p$ to $(H \cup P) - p$, that only meet in C, if at all. By the minimality of P and by (6), both R_1 and R_2 have an endvertex on T_H. So after resigning, if necessary, we have a configuration as in Figure 5 (left), where the parity of the dashed paths (R_1 and R_2) can be either odd or even. However, in fact, R_1 is odd and R_2 is even, as in any other case $H \cup C \cup P \cup R_1 \cup R_2$ contains a linked basket contradicting (7). So G has the signed graph in Figure 5 (right) as a minor. That signed graph has an $\overline{R}_{12}$-minor (delete e and contract f). So (8) follows.

(9) *If (H, L_1, L_2) is a linked basket, both L_1 and L_2 have their endvertices in T_H.*

Assume this is false and that L_1 has an endvertex, v say, in B_H. Then, by (7), L_2 is disjoint from $L_1 \cup B_H$. Let u be the endvertex of L_1 on T_H and w be the endvertex of L_2 on T_H closest to u. Let s and t be the endvertices of T_H such that s, u, w, and t lie in that order along T_H; see Figure 6 (left). Then $u = s$ and v is the endvertex in B_H of one of the legs of H incident with t, as otherwise $H \cup L_1$ contains a basket containing L_1 and that basket contradicts (8).

Assume as of now that H was chosen with T_H as short as possible. Resign such that the only odd edges on $H \cup L_1$ are the edges on $T_H \cup L_1$ incident with s

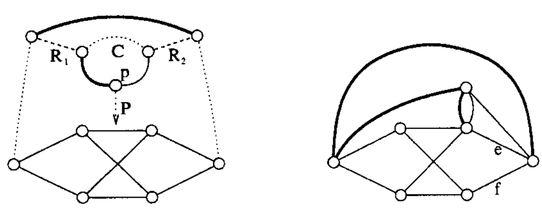

FIGURE 5.

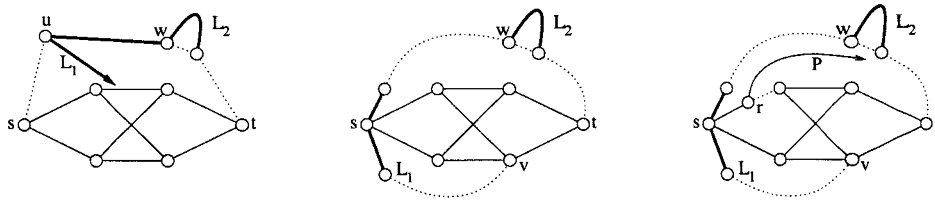

FIGURE 6.

(see Fig. 6 (middle)). Let P be a link of $H \cup L_1$ connecting a vertex p on $T_H - \{s,t\}$ with a vertex r in $(B_H \cup L_1) - \{s,t\}$. If $r \in L_1$, define $P' := P \cup (L_1)_{rv}$; otherwise $P' := P$. Then P', hence also P, is even, as otherwise (H, L_1, P) contradicts (7). Moreover, P is internally vertex disjoint with L_2 (as otherwise we could have chosen P odd). Hence, $p \in (T_H)_{wt}$, as otherwise (H, P', L_2) contradicts (9) in spite of the fact that s is not an endvertex of P' (and we argued above that this is not possible for a contradiction against (9)). The endvertex r of P lies in one of the legs of H incident with s, as otherwise $H \cup P'$ contains a basket H' with $T_{H'} = (T_H)_{sp}$ that contradicts (9) while $T_{H'}$ is shorter than T_H. See Figure 6 (right). Let H'' be the basket in $H \cup P$ containing P. Then (H'', L_1, L_2) contradicts (9) in spite of the fact that s is not an endvertex of $T_{H''}$. So (9) follows.

(10) *G contains a nonblocking basket.*

Assume this is false. Then by (5) and (9), there exists a linked basket (H, L_1, L_2) where the endvertices u and u' of L_1 and the endvertices v and v' of L_2 lie on T_H. Let s and t be the endvertices of T_H, such that s, u, u', v', v, and t lie in that order along T_H and resign such that the only odd edge on H is the one on $(T_H)_{sv'}$ incident with v'; see Figure 7 (left). Note that L_1 and L_2 need not be internally disjoint and that u may be equal to u' and v may be equal to v'. Assume (H, L_1, L_2) is chosen such that the sum of lengths of $(T_H)_{su}$ and $(T_H)_{vt}$ is minimal. Let P be a link of $H \cup L_1 \cup L_2$ connecting a vertex p in $(L_1 \cup (T_H)_{uv} \cup L_2) - \{u, v\}$ to a vertex r in $H - (T_H)_{uv}$.

First consider the case that $r \in B(H)$. By symmetry, $p \in (L_1 \cup (T_H)_{uv'}) - \{u, v'\}$. If $p \in L_1$ or P is odd, then $L_1 \cup P$ contains a odd link P' of H such that the

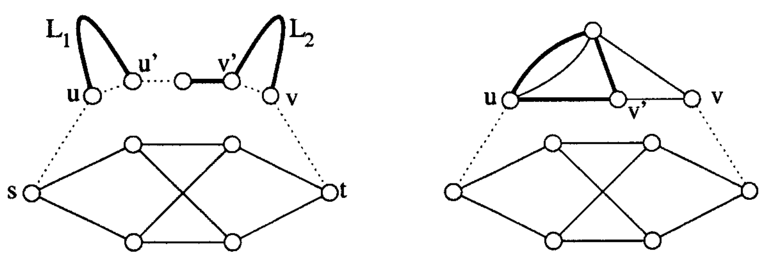

FIGURE 7.

linked basket (H, P', L_2) violates (9). If $p \in (T_H)_{u'v'} - \{u', v'\}$ and P is even then the linked basket (H, L_1, P) violates (9). So, $p \in (T_H)_{uu'}$ and P is even. Let H' be the basket in $H \cup P$ containing P. Then (H', L_1, L_2) contradicts (9).

So it remains to consider the case that $r \notin B_H$. By symmetry, we may assume that $r \in (T_H)_{su}$. By the minimality of the sum of lengths of $(T_H)_{su}$ and $(T_H)_{vt}$, it follows that $p \in ((T_H)_{v'v} \cup L_2) - v$ and that if $p \in L_2 - \{v', v\}$, then L_1 and L_2 are disjoint. As all baskets in $H \cup P \cup L_2$ are blocking, this means that $p \in (T_H)_{v'v} - v$ and that L_1 and L_2 intersect. So $v \neq v'$, and thus $H \cup L_1 \cup L_2$ contains the configuration in Figure 7 (right) as a minor. As that has a $\overline{K}_4$-minor, (10) follows.

An *extended basket* is a signed graph that can be resigned to the configuration in Figure 8 (left). If K is an extended basket, then D_K is the graph consisting of the six legs of K marked by a $*$ in Figure 8 (left), and U_K consists of the vertices of K not in D_K; the *joins* of K are the paths that are dotted in Figure 8 (left)—as indicated they may have length zero—and the endvertices of the joins that lie in D_K are the *tips* of K.

Assume as of now that G is a smallest counterexample to (2).

(11) *For each extended basket K, each leg of K that lies in D_K is a single edge.*

Assume this is false and L a leg of K in D_K with more than one edge. Let e be an edge of L that is not incident with a tip of K. Contract e to a vertex v. Any 2-vertex cutset introduced by this contraction must contain v. Thus, the extended basket K/e is contained almost entirely—except maybe for part of a leg—in one side of any such 2-vertex cutset. Hence, by replacing for each such 2-vertex cutset the side not containing U_K by a single edge of appropriate parity, we obtain a proper minor G' of G that still has an extended basket. As G' has no 2-vertex cutsets with vertices on either side, it is still a counterexample to (2). But G is a smallest counterexample, so (11) follows.

(12) *Let K be an extended basket with tips s and t and let L be a minimal path in $G \backslash \{s, t\}$ connecting $x \in D_K \backslash \{s, t\}$ to $s' \in U_K$. Then L is an even edge. Moreover, if $u \in \{s, t\}$ is the tip not adjacent in K to x, then the join of K ending in u has positive length and s' is the neighbor of u in that join.*

Link L is even and s' lies in one of the joins of K as otherwise one easily finds a basket in $K \cup L$ that violates (8) or (9). By symmetry, we may assume that $u = t$.

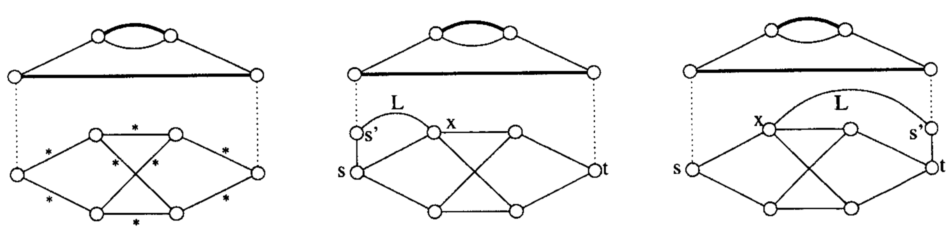

FIGURE 8.

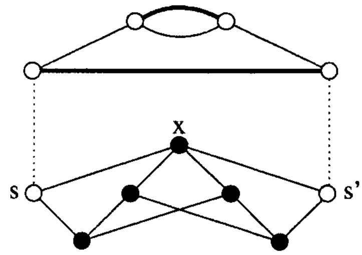

FIGURE 9.

So we are in one of the situations depicted in Figure 8 (middle and right). In both cases, $(K\backslash xs) \cup L$ is an extended basket. Applying (11) to this extended basket, it immediately follows that Figure 8 (right) applies; that L is an edge and t and s' are adjacent. Thus (12) follows.

Consider a nonblocking basket, which exists by (10). As G is 2-connected and by (8), this nonblocking basket is contained in an extended basket. By 3-connectivity, symmetry, and (12), this extended basket is contained in a configuration W as in Figure 9, where all edges with at least one black endvertex are edges. (Note that Fig. 9 is just a redrawn version of Fig. 8 (right).) Let s, s' and x be as indicated in Figure 9. By 3-connectivity, W has a pq-link L in $G\backslash\{s, s'\}$ with p black and q not. Obviously L has to violate (12) with respect to one of the two extended baskets $W\backslash sx$ and $W\backslash s'x$. This contradiction completes the proof of (2). ■

3. FROM SIGNED GRAPHS TO BINARY MATROIDS

We translate Theorem (2) to matroids. We assume the reader to be familiar with standard matroid theory (see Oxley [2] or Truemper [5]). Consider a signed graph (G, Σ). Let χ_Σ be the characteristic vector of Σ as a subset of $E(G)$ and let M_G be the vertex–edge incidence matrix of G. By $\mathcal{S}(G, \Sigma)$, we denote the binary matroid represented over $GF(2)$ by the columns of matrix

$$\begin{bmatrix} 1 & \chi_\Sigma \\ 0 & M_G \end{bmatrix} \tag{13}$$

Obviously a binary matroid M is isomorphic to a matroid of the form $\mathcal{S}(G, \Sigma)$ if and only if M/x is graphic for some element x of M. Resigning (G, Σ) on U amounts to adding the rows of M_G indexed by the vertices in U to χ_Σ, so to row operations in (13). Hence, resigning does not affect $\mathcal{S}(G, \Sigma)$. Blockvertices have the following matroidal interpretation.

(14) *If (G, Σ) has a blockvertex, $\mathcal{S}(G, \Sigma)$ is graphic.*

Indeed, let u be a blockvertex. Resign such that $\Sigma \subseteq \delta(u)$. Construct a graph H as follows: split vertex u into two new vertices u_1 and u_2; split $\delta(u)$ into $\delta(u_1)$ and $\delta(u_2)$ such that $\delta(u_1) = \Sigma$ and $\delta(u_2) = \delta(u)\backslash\Sigma$; and add an edge e_x from u_1 to u_2. It is not hard to see that $M(H) \sim \mathcal{S}(G,\Sigma)$. So (14) follows.

It is straightforward to check that if the signed graph (H,Θ) is a minor of (G,Σ), then the matroid $\mathcal{S}(H,\Theta)$ is a minor of $\mathcal{S}(G,\Sigma)$. Hence, as $\mathcal{S}(\overline{K}_4) \sim F_7^*$ and $\mathcal{S}(\overline{R}_{12}) \sim R_{12}$, Theorem (2) implies the following.

(15) *Let M be a nongraphic binary matroid with no F_7^*-minor and x be an element of M such that M/x is a 3-connected graphic matroid with an $M(K_{3,3})$-minor or a parallel extension of such a matroid. Then M has an R_{12}-minor.*

4. PROVING THEOREM (1)

One of Seymour's main tools in proving the Regular Matroid Decomposition Theorem is as follows.

(16) Splitter Theorem. *Let M be a 3-connected matroid that is not a wheel or a whirl. If M has a proper 3-connected minor N, then M has an element x such that either $M\backslash x$ or M/x is 3-connected and has a minor isomorphic to N.*

One of the implications of this theorem concerns "splitters." A 3-connected matroid N is a *splitter* for a class of matroids $\mathcal{M}$ if no 3-connected member of $\mathcal{M}$ has a proper N-minor. Seymour's Splitter Theorem implies that proving a matroid N is a splitter for class $\mathcal{M}$ is just an elementary finite case check. Seymour [3] used this to prove that R_{10} is a splitter for the class of regular matroids and that $M(K_5)$ is a splitter for the class of regular matroids with no $M(K_{3,3})$-minor.

We now prove Theorem (1) from (15) following the same lines as Seymour did in [3] (but he started from (14.1) in [3] instead of (15)).

***Proof of* (1).** Assume it is false and M a minimal counterexample. Tutte's characterization of cographic matroids [6] says that M has an $M(K_{3,3})$- or $M(K_5)$-minor. As M is not graphic, and as $M(K_5)$ is a splitter for the class of regular matroids with no $M(K_{3,3})$-minor, it follows that M has a proper minor isomorphic to $M(K_{3,3})$. By the minimality of M we know the following.

(17) *Each proper 3-connected minor of M that has an $M(K_{3,3})$ minor is graphic.*

Combining this with (15) we know the following.

(18) *M has no element x such that M/x is 3-connected with an $M(K_{3,3})$-minor or a parallel extension of such a matroid.*

As M has a proper $M(K_{3,3})$-minor, it follows from the Splitter Theorem and (18) that M has an element y such that $M\backslash y$ is 3-connected and has an $M(K_{3,3})$-minor. It is straightforward to check that if $M\backslash y$ were isomorphic to $M(K_{3,3})$, then M would be graphic, isomorphic to R_{10}, or nonregular. So $M\backslash y$ has a proper $M(K_{3,3})$-minor, and hence, again by the Splitter Theorem and by (18) and (17),

$M\backslash y$ has an element z such that $M\backslash y, M\backslash z$ and $M\backslash y, z$ are cycle matroids of, uniquely determined, 3-connected graphs. From this fact, it is easy to construct a graph G with edges e_y and e_z, such that these graphs are $G\backslash e_y$, $G\backslash e_z$, and $G\backslash e_x, e_y$, respectively. As M and $M(G)$ are binary, it is straightforward to prove now that $M \sim M(G)$, a contradiction. So Theorem (1) follows. ■

REFERENCES

[1] A. M. H. Gerards, On Tutte's characterization of graphic matroids—A graphic proof, J Graph Theory 20 (1995), 351–359.
[2] J. G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[3] P. D. Seymour, Decomposition of regular matroids, J Combin Theory, Ser B 28 (1980), 305–359.
[4] P. D. Seymour, On Tutte's characterization of graphic matroids, Ann Discrete Math 8 (1980), 83–90.
[5] K. Truemper, Matroid Decomposition, Academic Press, San Diego, 1992.
[6] W. T. Tutte, Matroids and graphs, Trans Am Math Soc 90 (1959), 527–552.
[7] D. L. Vertigan, On the intertwinning conjecture for matroids, in preparation.