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Note

On matroids without a non-Fano minor

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Abstract

We study the family of matroids that do not contain the non-Fano matroid or its dual as a minor. In particular, we prove that, for any connected matroid M, there are just finitely many minor-minimal matroids in the family that contain both an M-minor and a $U_{2,4}$-minor. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two families of matroids that are both closed under isomorphism and taking minors. It follows that both the intersection and the union of $\mathscr{M}_1$ and $\mathscr{M}_2$ are closed under isomorphism and taking minors. It is straightforward to see that, if $\mathscr{M}_1$ and $\mathscr{M}_2$ are both described by a finite set of excluded minors, then $\mathscr{M}_1 \cap \mathscr{M}_2$ is described by a finite set of excluded minors. The *intertwining problem*, posed by Brylawski [2], asks, if $\mathscr{M}_1$ and $\mathscr{M}_2$ are both described by a finite set of excluded minors, then is $\mathscr{M}_1 \cup \mathscr{M}_2$ described by a finite set of excluded minors. Vertigan [10] showed that it is often the case that $\mathscr{M}_1 \cup \mathscr{M}_2$ has infinitely many excluded minors, even when $\mathscr{M}_1$ and $\mathscr{M}_2$ are each described by one excluded minor.

Nevertheless, the intertwining problem is still of interest for certain classes of matroids. For instance, one can deduce from Seymour's decomposition of regular matroids [9], that the intertwining problem holds for the class of graphic matroids and the class of cographic matroids. That is, there are finitely many excluded minors for the union of the family of graphic matroids and the family of cographic matroids. Another interesting instance of the intertwining problem is for the families of binary and ternary

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matroids. It is not known whether there are only finitely many excluded minors for the union of the families of binary and ternary matroids, however there is strong evidence to suggest that this is the case; see Oporowski, Oxley and Whittle [6].

Let F_7^- denote the non-Fano matroid; see Oxley [7]. Let $\mathrm{EX}(M_1,\ldots,M_k)$ denote the family of matroids that do not contain a minor isomorphic to any of $M_1,\ldots,M_k$. Our main result is stated in the following theorem.

Theorem 1.1. *Let $M_1, M_2,\ldots,M_k$ be connected matroids and let $\mathscr{M} = \mathrm{EX}(M_1,\ldots,M_k)$. If $\mathscr{M}$ contains neither F_7^- nor $(F_7^-)^*$, then there are just finitely many excluded minors for the union of the family of binary matroids and $\mathscr{M}$.*

The condition that the excluded minors for $\mathscr{M}$ are connected could possibly be dropped from the theorem. However, as many interesting classes of matroids are closed under taking direct sums, this condition seems reasonable. Note that the non-Fano matroid is not representable over any field of characteristic two. Therefore, Theorem 1.1 can be applied to many families of matroids that are representable over some field of characteristic two.

Our results rely on the following crucial lemma, the proof of which is based on Gerards' simple proof of Tutte's excluded-minor characterization of regular matroids [4]. This lemma was already known for matroids representable over fields of characteristic two; see Semple and Whittle [8].

Lemma 1.2. *Let (x, y) be a coindependent pair of elements of a matroid M. If $M\backslash x$ and $M\backslash y$ are binary, and $M\backslash x, y$ is connected, then either M is binary or M contains an F_7^--minor.*

We assume that the reader is familiar with elementary notions in matroid theory, including representability, minors, duality and connectivity. For an excellent introduction to the subject read Oxley [7].

2. Twisted matroids

For convenience we work with twisted matroids, which were introduced in [3]. A twisted matroid is just a matroid viewed with respect to a particular basis. Let $\mathscr{B}$ be the set of bases of a matroid M having ground set S. For $B \in \mathscr{B}$, define $M_B = (S, \mathscr{F}_B)$, where $\mathscr{F}_B = \{B \Delta B' \colon B' \in \mathscr{B}\}$. $\mathscr{F}_B$ is the set of *feasible sets* of the *twisted matroid* M_B. If X is feasible then $|X \cap B| = |X - B|$; in particular, all feasible sets have even cardinality. M_B is also endowed with a rank function r_B where $r_B(X)$ is half the size of the largest feasible set in X. Equivalently, $r_B(X) = r(X \Delta B) - |B \backslash X|$. Duality is absorbed in the definition of a twisted matroid, since $M_B = (M^*)_{S\backslash B}$. Given $X \subseteq S$, we define $M_B[X] = (X, \mathscr{F}')$, where $\mathscr{F}' = \{F \subseteq X \colon F \in \mathscr{F}_B\}$; $M_B[X]$ is the *restriction* of M_B to X. Matroidally, this corresponds to the deletion of $(S\backslash B)\backslash X$ and contraction of $B\backslash X$ from

M. That is, $M_B[X]=N_{X\cap B}$ where $N=M\backslash((S\backslash B)\backslash X)/(B\backslash X)$. We denote by M_B-X the twisted matroid $M_B[S\backslash X]$. The following results follow from the well-known fact that, if N is a minor of M, then there exists an independent set X and a coindependent set Y of M such that $N=M\backslash Y/X$.

2.1. *If N is a minor of M, then there exists a basis B of M such that $M_B[E(N)]=N_{B\cap E(N)}$.*

The *fundamental graph* of M_B is the graph $G_B=(S,E_B)$, where $E_B=\{ij\colon \{i,j\}\in\mathscr{F}_B\}$. Thus G_B is bipartite, with bipartition $(B,S-B)$. For $X\subseteq S$, $G_B[X]$ denotes the subgraph of G_B induced by X. The following propositions are well known.

Proposition 2.2. *M is connected if and only if G_B is connected.*

Proposition 2.3 (Brualdi [1]). *If X is a feasible set of M_B, then $G_B[X]$ has a perfect matching.*

Proposition 2.4 (Krogdahl [5]). *If $G_B[X]$ has a unique perfect matching, then X is feasible in M_B.*

Proposition 2.5. *If y is the only neighbour of x in $G_B[X]$, then X is feasible if and only if $X-\{x,y\}$ is feasible.*

2.1. Representations

Let A be a matrix, over a field $\boldsymbol{F}$, whose rows and columns are indexed by the sets B and $S\backslash B$, respectively. Then A is an $\boldsymbol{F}$*-representation* of M_B if, for each $X\subseteq B$ and $Y\subseteq(S\backslash B)$, $\text{rank}(A[X,Y])=r_B(X\cup Y)$. Equivalently, A is an $\boldsymbol{F}$-representation of M_B if and only if (I,A) is an $\boldsymbol{F}$-representation of M.

For any twisted matroid M_B, there exists a unique binary twisted matroid N_B such that M_B and N_B have the same fundamental graph. We call N the *binary approximation* of M at B. Note that M_B is binary if and only if M and N have the same bases. A set $X\subseteq S$ is said to *distinguish* M from N if X is a basis in exactly one of M and N. Similarly, X is said to *distinguish* M_B from N_B if X is feasible in exactly one of M_B and N_B. Note that, if X distinguishes M_B and N_B then, by construction, $|X|\geqslant 4$.

2.2. Pivoting

For any feasible set X of M_B, $\mathscr{F}_{B\Delta X}=\{F\Delta X\colon F\in\mathscr{F}_B\}$. For an edge xy of G_B, we refer to the shift from M_B to $M_{B\Delta\{x,y\}}$ as a *pivot* on xy. Let B' denote $B\Delta\{x,y\}$. Much of the structure of $G_{B'}$ is determined by G_B. The following observations are easy consequences of Propositions 2.3 and 2.4. (Here we denote the set of neighbours of x

in G_B by $\text{nigh}_B(x)$.)

(i) $\text{nigh}_{B'}(x) = \text{nigh}_B(y) \Delta \{x, y\}$,

(ii) If $v \notin \text{nigh}_B(x) \cup \text{nigh}_B(y)$, then $\text{nigh}_{B'}(v) = \text{nigh}_B(v)$, and

(iii) If $v \in \text{nigh}_B(x)$, $w \in \text{nigh}_B(y) \backslash \text{nigh}_B(v)$, then vw is an edge of $G_{B'}$.

Thus we can account for most edges of $G_{B'}$. The only confusion arises for elements v, w, for which $\{x, y, v, w\}$ induces a circuit in G_B. In this case, $vw \in E_{B'}$ if and only if $\{x, y, v, w\}$ is feasible in M_B; which we cannot determine from G_B alone.

3. The main lemma

In this section we prove Lemma 1.2. Suppose that M is a nonbinary matroid, and that x and y are coindependent elements of M such that $M \backslash x$ and $M \backslash y$ are both binary, and $M \backslash x, y$ is connected.

Choose $B \subseteq E(M) - x - y$, and $X \subseteq E(M)$ such that

(i) B is a basis of M,

(ii) X distinguishes M from its binary approximation N at B, and

(iii) $|X - B|$ is as small as possible with respect to (i) and (ii).

Since $M \backslash x$ and $M \backslash y$ are binary, $M \backslash x = N \backslash x$ and $M \backslash y = N \backslash y$. Hence $x, y \in X$.

We claim $|X - B| = 2$. The proof is as follows. Suppose that $|X - B| > 2$. Then choose $a \in (X - B) - \{x, y\}$. Let $N' \in \{N, M\}$ such that X is a basis of N'. By the basis exchange axiom, there exists $b \in B - X$ such that $B - b + a$ is a basis of N'. Note that neither x nor y is in $B - b + a$, so $B - b + a$ is a basis of M. Furthermore, if Y is any set that distinguishes M and N, then $|Y \Delta (B - b + a)| \geqslant |Y \Delta B| - 2 \geqslant |X \Delta B| - 2 \geqslant 4$. Hence N is the binary approximation to M at $B - b + a$, and X distinguishes M and N. However, $|X - B| > |X - (B - a + b)|$, which contradicts our choice of X and B.

Therefore, $X - B = \{x, y\}$. Note that $|X - B| = |B - X|$. Label the elements of $B - X$ by a and b. So $\{x, y, a, b\}$ distinguishes M_B and N_B. By construction, M_B and N_B have the same fundamental graph G_B. Note that x and y are in one colour class of G_B and a and b are in the other. Furthermore, by Propositions 2.3 and 2.2 and since $\{x, y, a, b\}$ is feasible in exactly one of M_B and N_B, $G_B[x, y, a, b]$ is a circuit. Since N is binary, $\{x, y, a, b\}$ is not feasible in N_B, and hence $\{x, y, a, b\}$ is feasible in M_B.

Since $M \backslash x, y$ is connected, there exists a path from a to b in $G_B - x - y$; let $v_0 = a, v_1, \ldots, v_k, v_{k+1} = b$ be the vertices of a chordless path from a to b. Since a and b are in the same colour class, k is odd. Let M' be the minor of M associated with $M_B[\{x, y, v_0, \ldots, v_k\}]$ (that is, $M'_{\{v_0, v_3, \ldots, v_k\}} = M_B[\{x, y, v_0, \ldots, v_k\}]$), and let N' be the minor of N associated with $N_B[\{x, y, v_0, \ldots, v_k\}]$.

Consider the case that $k = 1$. Thus v_1 is adjacent to both a and b. The following matrix is a binary representation of N':

$$\begin{array}{ccccc} a & b & x & y & v_1 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since $M'\backslash x = N'\backslash x$ and $M'\backslash x = N'\backslash y$, y and x are both parallel with v_1 in M'. But then, x and y must be parallel in M', contradicting the fact that $\{x, y\}$ distinguishes M' and N'.

Now consider the case that $k \geqslant 5$. Since $v_0, \dots, v_{k+1}$ is a chordless path, v_3 is adjacent to neither $a = v_0$ nor $b = v_{k+1}$. Furthermore, v_3 is in the same colour class of G_b as x and y, so v_3 is adjacent to neither x nor y. Therefore, the only neighbour of v_3 in $G_B[\{x, y, a, b, v_2, v_3\}]$ is v_2. Hence, by Proposition 2.4, $\{x, y, a, b, v_2, v_3\}$ is feasible in M_B but not in N_B. Therefore $\{x, y, a, b\}$ distinguishes $M_{B\Delta\{v_2,v_3\}}$ from $N_{B\Delta\{v_2,v_3\}}$. Moreover, $v_0, v_1, v_4, \dots, v_{k+1}$ is a shortest path from a to b in $G_{B\Delta\{v_2,v_3\}}$. So, by replacing B with $B\Delta\{v_2, v_3\}$, we bring a and b closer together; so inductively we reduce to the case where $k = 3$.

The following matrix is a binary representation of N':

$$\begin{array}{ccccccc} a & b & v_2 & x & y & v_1 & v_3 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & \alpha & \beta & 1 & 1 \end{pmatrix},$$

where $\alpha, \beta \in \{0, 1\}$. We will show that M' is isomorphic to F_7^-.

Suppose that $Y \subseteq \{x, y, v_0, \dots, v_{k+1}\}$ distinguishes M_B and N_B. Since $M \backslash x = N \backslash x$ and $M \backslash y = N \backslash y$, we have $x, y \in Y$. Furthermore, since $|Y - B| = |B \cap Y|$, Y must be one of $\{x, y, a, b\}$, $\{x, y, a, v_2\}$, $\{x, y, b, v_2\}$, $\{x, y, v_1, a, b, v_2\}$ or $\{x, y, v_3, a, b, v_2\}$. If $\{x, y, a, v_2\}$ distinguishes M_B and N_B, we could replace b by v_2; which puts us in the impossible case when $k = 1$. So $\{x, y, a, v_2\}$ does not distinguish M_B and N_B; by symmetry, $\{x, y, b, v_2\}$ does not distinguish M_B and N_B. If $\{x, y, v_1, a, b, v_2\}$ distinguishes M_B and N_B, then $\{x, y, a, b\}$ distinguishes $M_{B\Delta\{v_1,v_2\}}$ from $N_{B\Delta\{v_1,v_2\}}$. Moreover, v_3 is adjacent to both a and b in $G_{B\Delta\{v_2,v_3\}}$. This again reduces us to the case when $k = 1$. So $\{x, y, v_1, a, b, v_2\}$ does not distinguish M_B and N_B; by symmetry, $\{x, y, v_3, a, b, v_2\}$ does not distinguish M_B and N_B. So $\{x, y, a, b\}$ is the only subset of $\{x, y, v_0, \dots, v_{k+1}\}$ that distinguishes M_B from N_B.

Now M' is obtained from N' by relaxing the dependent set $\{x, y, v_2\}$. It follows that $\{x, y, v_2\}$ must be a circuit of N'. So we cannot have $\alpha = \beta$. By possibly interchanging x and y we may assume that $\alpha = 1$ and $\beta = 0$. Hence N' is isomorphic to the Fano matroid, and M' is isomorphic to the non-Fano matroid as claimed in Lemma 1.2.

4. Intertwining

In this section we prove Theorem 1.1.

Theorem 4.1. *Let N be a connected matroid. If M is a minor-minimal matroid in* $\mathrm{EX}(F_7^-, (F_7^-)^*)$ *that contains both a $U_{2,4}$-minor and an N-minor, then* $|E(M)| \leqslant |E(N)| + 4$.

We require the following elementary result, whose proof is left as an exercise.

4.2. *If N_1 and N_2 are connected matroids, and M is a minor-minimal matroid that contains both an N_1-minor and an N_2-minor, then either M is connected or M is isomorphic to the direct sum of N_1 and N_2.*

Proof of Theorem 4.1. If N is nonbinary, then N contains a $U_{2,4}$-minor; so any minor-minimal matroid that contains both an N-minor and a $U_{2,4}$-minor is isomorphic to N. Therefore, we may assume that N is binary. Let M be a minor-minimal matroid in $\mathrm{EX}(F_7^-,(F_7^-)^*)$ that contains both a $U_{2,4}$-minor and an N-minor; furthermore, suppose that $|E(M)| > |E(N)| + 4$. By 4.2, M is connected. Let $X = E(N)$, and let B be any basis of M such that $M_B[X] = N_{B\cap X}$.

4.3. *If $x, y \in E(M) - X$ are distinct elements in the same colour class of G_B, then $G_B - x - y$ is disconnected.*

Suppose to the contrary that $G_B - x - y$ is connected. By possibly replacing M, N and B by M^*, N^* and $E(M) - B$, we may assume that $x, y \in E - B$. Therefore, $M\backslash x, y$ is connected. Note that $M\backslash x$ and $M\backslash y$ both contain N as a minor. By our choice of M, neither $M\backslash x$ nor $M\backslash y$ contains a $U_{2,4}$-minor. Therefore, both $M\backslash x$ and $M\backslash y$ are binary. So, by Lemma 1.2, either M is binary or M contains an F_7^--minor, in either case we have a contradiction. This proves 4.3.

Let Ω_B^i denote the set of vertices in G_B that are at distance i from X; that is, a vertex v of G_B is in Ω_B^i if the number of edges in a shortest path from v to a vertex in X is i. In particular $\Omega_B^0 = X$, and Ω_B^1 is the set of vertices, not in X, that have a neighbour in X.

4.4. *For $i \geqslant 1$, $|\Omega_B^i| \leqslant 2$.*

Since M and N are connected, G_B and $G_B[X]$ are connected. Take a spanning tree of $G_B[X]$ and expand it, breadth first, to a spanning tree T of G_B. (That is, T is chosen so that, for each v§, T contains a shortest path from v to X in G_B.) If $|\Omega_B^i| > 2$ for some $i \geqslant 1$, then T will have at least 3 leaves in $E(M) - X$. Therefore, there exist two leaves, say x and y, of T in $E(M) - X$ that are in the same colour class of G_B. Then $G_B - x - y$ is connected, contradicting 4.3. This proves 4.4.

Define r_B to be the largest integer i for which Ω_B^i is nonempty. Since $|E(M)| - |X| > 4$, $r_B \geqslant 3$.

4.5. *There exists a choice of B such that $|\Omega_B^i| = 2$ for $i = 1, \ldots, r_B - 2$. Furthermore, for $i = 1, \ldots, r_B$, Ω_B^i is in one of the colour classes of G_B.*

Let x_0, x_1, x_2 be a path in G_B where $x_j \in \Omega_B^{i+j}$ and $i \geqslant 1$. Consider pivoting on x_0, x_1. Note that x_1 has no neighbours in $\Omega_B^0 \cup \cdots \cup \Omega_B^{i-1}$. Hence, by Proposition 2.5,

$M_B[\Omega_B^0 \cup \cdots \cup \Omega_B^{i-1} \cup \{x_0,x_1\}]$ is isomorphic to $M_{B\Delta\{x_0,x_1\}}[\Omega_B^0 \cup \cdots \cup \Omega_B^{i-1} \cup \{x_0,x_1\}]$, under swapping x_0 and x_1. It follows that $M_{B\Delta\{x_0,x_1\}}[X] = N_{B\cap X}$, and that $\Omega_B^j = \Omega_{B\Delta\{x_0,x_1\}}^j$, for $j=0,\ldots,i-1$. Furthermore x_1 and x_2 are both in $\Omega_{B\Delta\{x_0,x_1\}}^i$. By 4.4, $\Omega_{B\Delta\{x_0,x_1\}}^i = \{x_1,x_2\}$; moreover x_1 and x_2 are in the same colour class of $G_{B\Delta\{x_0,x_1\}}$. So 4.5 is now clear.

Note that deleting the vertices in $\Omega_B^{r_B}$ cannot disconnect G_B, hence, by 4.3, $|\Omega_B^{r_B}| = 1$. Suppose that $\Omega_B^{r_B} = \{z\}$ and that $\Omega_B^{r_B-2} = \{x_1,x_2\}$. By 4.3, $G_B - z - x_1$ and $G_B - z - x_2$ are both disconnected. It follows that $\Omega_B^{r_B-1}$ contains two elements, say y_1 and y_2, where y_1 is adjacent to x_1 but not to x_2, and y_2 is adjacent to x_2 but not to x_1. z is adjacent to at least one of y_1 and y_2; we may assume that z is adjacent to y_1. Consider pivoting on x_1, y_1, and let $B' = B\Delta\{x_1,y_1\}$. Since the only neighbour of y_1 in $G_B - z$ is x_1, $M_{B'} - z$ is isomorphic to $M_B - z$ under swapping x_1 and y_1. Therefore $M_{B'}[X] = N_{B'\cap X}$, and y_1 and x_2 are both in $\Omega_{B'}^{r_{B'}-2}$. However, z is also in $\Omega_{B'}^{r_{B'}-2}$, contradicting 4.4. This proves 4.5 and consequently Theorem 1.1.

Proof of Theorem 1.1. Suppose that $\mathscr{M} = \mathrm{EX}(M_1,\ldots,M_k)$, where $M_1,\ldots,M_k$ are connected, and that $\mathscr{M}$ contains neither F_7^- nor $(F_7^-)^*$ as a minor. Therefore

$$\mathscr{M} = \mathrm{EX}(F_7^-,(F_7^-)^*,M_1,\ldots,M_k) = \cap \cdots \cap \mathrm{EX}(F_7^-,(F_7^-)^*,M_k)$$
$$\mathrm{EX}(F_7^-,(F_7^-)^*,M_1).$$

Let $\mathscr{B}$ denote the family of binary matroids. Then

$$\mathscr{M} \cup \mathscr{B} = (\mathrm{EX}(F_7^-,(F_7^-)^*,M_1) \cup \mathscr{B}) \cap \cdots \cap (\mathrm{EX}(F_7^-,(F_7^-)^*,M_k) \cup \mathscr{B}).$$

By Theorem 4.1, for $i = 1,\ldots,k$, $\mathrm{EX}(F_7^-,(F_7^-)^*,M_i) \cup \mathscr{B}$ is described by a finite list of excluded minors. Therefore, $\mathscr{M}$ is described by a finite list of excluded minors. □

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