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Obstructions to branch-decomposition of matroids ☆

J. Geelen [a], B. Gerards [b,c], N. Robertson [d], G. Whittle [e]

[a] *Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada*
[b] *CWI, Postbus 94079, 1090 GB Amsterdam, The Netherlands*
[c] *Department of Mathematics and Computer Science, Eindhoven University of Technology, Postbus 513, 5600 MB Eindhoven, The Netherlands*
[d] *Department of Mathematics, 231 West 18th Avenue, Ohio State University, Columbus, OH 43210, USA*
[e] *School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand*

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Abstract

A (δ,γ)-net in a matroid M is a pair $(N,\mathcal{P})$ where N is a minor of M, $\mathcal{P}$ is a set of series classes in N, $|\mathcal{P}| \geqslant \delta$, and the pairwise connectivity, in M, between any two members of $\mathcal{P}$ is at least γ. We prove that, for any finite field $\mathbb{F}$, nets provide a qualitative characterization for branch-width in the class of $\mathbb{F}$-representable matroids. That is, for an $\mathbb{F}$-representable matroid M, we prove that: (1) if M contains a (δ,γ)-net where δ and γ are both very large, then M has large branch-width, and, conversely, (2) if the branch-width of M is very large, then M or M^* contains a (δ,γ)-net where δ and γ are both large.

Keywords: Branch-width; Matroids; Connectivity

1. Introduction

For matroids representable over a given finite field, we obtain a qualitative characterization of large branch-width. For graphs, such a characterization was obtained by Robertson and Seymour [8].

Theorem 1.1 *(Robertson and Seymour). For any positive integer n there exists an integer k such that, if G is a graph with branch-width at least k, then G contains a minor isomorphic to the n by n grid.*

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Ideally we would like to prove the following conjecture of Johnson, Robertson, and Seymour [4].

Conjecture 1.2. *For any positive integer n and prime power q, there exists an integer k such that, if M is a* GF(q)*-representable matroid with branch-width at least k, then M contains a minor isomorphic to the cycle-matroid of the n by n grid.*

The cycle-matroid of the n by n grid has branch-width n. If true, the above conjecture would, given a matroid with very large branch-width (at least k), provide a succinct certificate that the branch-width is large (at least n). We provide a similar such certificate.

Let M be a matroid and let $A \subseteq E(M)$. We let $\lambda_M(A) = r_M(A) + r_M(E(M) - A) - r(M) + 1$. A partition (A, B) of $E(M)$ is called a *separation of order* $\lambda_M(A)$. For disjoint subsets A and B of $E(M)$ we let

$$\kappa_M(A, B) = \min\bigl(\lambda_M(X):\ A \subseteq X \subseteq E(M) - B\bigr).$$

A (δ, γ)*-net* of a matroid M is a pair $(N, \mathcal{P})$ where N is a minor of M, $\mathcal{P}$ is a collection of series classes of N, $|\mathcal{P}| \geqslant \delta$, and $\kappa_M(P, Q) \geqslant \gamma$ for each distinct pair of sets $P, Q \in \mathcal{P}$. The next result, proven in Section 4, shows that nets witness large branch-width.

Lemma 1.3. *Let M be a* GF(q)*-representable matroid. If M contains a (q^k, k)-net, then M has branch-width at least k.*

Our main result is that nets provide a qualitative characterization of large branch-width.

Theorem 1.4. *For all positive integers δ and γ and any finite field $\mathbb{F}$ there exists an integer k such that if M is an $\mathbb{F}$-representable matroid with branch-width at least k, then M or M^* contains a (δ, γ)-net.*

We prove a slightly stronger version of Lemma 1.3 and Theorem 1.4, namely Lemma 4.1 and Theorem 6.2, that do not require representability.

Verifying that a pair $(N, \mathcal{P})$ is a (δ, γ)-net of M can be done efficiently. Most of the work required is in verifying that $\kappa_M(P, Q) \geqslant \gamma$ for each pair (P, Q) of sets in $\mathcal{P}$. The number of such pairs is

$$\binom{\delta}{2} \leqslant \binom{|E(M)|}{2}.$$

For a given pair (P, Q) we can efficiently verify that $\kappa_M(P, Q) \geqslant \gamma$ using Tutte's Linking Theorem (Theorem 2.2). It suffices to provide a minor N' of M such that $E(N') = P \cup Q$ and $\lambda_{N'}(P) \geqslant \gamma$; this can be verified using only four rank-evaluations. For our purpose, we do not need to know how to *compute* $\kappa_M(P, Q)$ efficiently. Nevertheless, $\kappa_M(P, Q)$ can be computed efficiently via Edmonds' Matroid Intersection Algorithm; this application, due to Edmonds, is described by Bixby and Cunningham [1].

2. Preliminaries

We assume that the reader is familiar with matroid theory; we use the notation of Oxley [7].

For any positive integer q we let $\mathcal{U}(q)$ denote the class of matroids with no $U_{2,q+2}$-minor and we let $\mathcal{U}^*(q)$ denote the class of matroids with no $U_{q,q+2}$-minor. Note that, if q is a prime-power, then $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ contains all GF(q)-representable matroids. We prove the more general version of Theorem 1.4 by extending it to the class $\mathcal{U}(q) \cap \mathcal{U}^*(q)$. We use the following result of Kung [5].

Lemma 2.1. *For $q \geqslant 2$, if M is a simple rank-r matroid in $\mathcal{U}(q)$, then $|E(M)| \leqslant (q^r - 1)/(q-1)$.*

We also use the following theorem of Tutte [10].

Theorem 2.2 *(Tutte's Linking Theorem). If S and T are disjoint sets of elements in a matroid M, then there exists a minor N of M such that $E(N) = S \cup T$ and $\lambda_N(S) = \kappa_M(S,T)$.*

Let E be a finite set, and let λ be an integer-valued function defined on subsets of E. We call λ a *connectivity function* on E if:

(1) $\lambda(X) = \lambda(E - X)$ for each $X \subseteq E$, and
(2) $\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$.

The following gives some elementary properties of connectivity functions that we will use later without reference.

Lemma 2.3. *If λ is a connectivity function on E, then, for each $X, Y \subseteq E$, we have*:

- $\lambda(X) \geqslant \lambda(\emptyset)$ *and*
- $\lambda(X) + \lambda(Y) \geqslant \lambda(X - Y) + \lambda(Y - X)$.

Proof. By symmetry and submodularity we have:

$$\begin{aligned}\lambda(X) + \lambda(Y) &= \lambda(X) + \lambda(E - Y)\\ &\geqslant \lambda(X - Y) + \lambda\big(E - (Y - X)\big)\\ &= \lambda(X - Y) + \lambda(Y - X).\end{aligned}$$

Thus $\lambda(X) + \lambda(Y) \geqslant \lambda(X - Y) + \lambda(Y - X)$. When $X = Y$ this inequality reduces to $\lambda(X) \geqslant \lambda(\emptyset)$. □

A partition (A, B) of E is called a *separation of order* $\lambda(A)$. For disjoint sets $S, T \subseteq E$, we let

$$\kappa_\lambda(S,T) = \min\big(\lambda(Z)\colon\ S \subseteq Z \subseteq E - T\big).$$

Lemma 2.4. *Let λ be a connectivity function on E and let $X \subseteq A \subseteq E$. If $\kappa_\lambda(X, E - A) = \lambda(A)$, then, for each $Z \subseteq E - X$, we have $\lambda(Z - A) \leqslant \lambda(Z)$.*

Proof. Note that $X \subseteq A - Z \subseteq E - A$. Therefore $\lambda(A - Z) \geqslant \kappa_\lambda(X, E - A) = \lambda(A)$. Now

$$\lambda(A) + \lambda(Z) \geqslant \lambda(A - Z) + \lambda(Z - A).$$

Thus, $\lambda(Z) \geqslant \lambda(Z - A)$, as required. □

A tree is *cubic* if its internal vertices all have degree 3. A *partial branch-decomposition* of λ is a cubic tree T, with at least one edge, whose leaves are labelled by elements of E. That is, each element in E labels exactly one leaf of T, but leaves may be unlabelled or multiply labelled. A *branch-decomposition* is a partial branch-decomposition without multiply labelled leaves. If T' is a subgraph of T and $X \subseteq E$ is the set of labels of T', then we say that T' *displays* X. The *width* of an edge e of T, denoted $\epsilon(e, T)$, is defined to be $\lambda(X)$ where X is the set displayed by one of the components of $T - \{e\}$. The *width* of T, denoted $\epsilon(T)$, is the maximum among the widths of its edges. The *branch-width* of λ is the minimum among the widths of all branch-decompositions of λ.

The following lemma is an immediate consequence of Lemma 2.4.

Lemma 2.5. *Let λ be a connectivity function on E, let T be a partial branch-decomposition of λ, and let $X \subseteq E$ be the set labelling a vertex $v \in V(T)$. Now, let $A \subseteq E$ with $X \subseteq A$ and let T' be the branch-decomposition of λ obtained by relabelling T as follows: label v by A and label $w \in V(T) - \{v\}$ by $Y - A$ where Y is the set of labels of w in T. If $\kappa_\lambda(X, E - A) = \lambda(A)$, then $\epsilon(e, T') \leqslant \epsilon(e, T)$ for each edge e of T.*

The *branch-width* of a matroid M is the branch-width of its connectivity function λ_M. We require the following result of Oporowski [6].

Theorem 2.6. *If M is a matroid of branch-width at least $\binom{m+1}{2}$, then M contains a circuit of length at least m.*

3. Tangles

Robertson and Seymour [9] introduced branch-width for connectivity functions and showed that, for graphs, this parameter is characterized by 'tangles.' In fact, Robertson and Seymour [9, (3.5)] proved a more general duality notion for the branch-width of a connectivity function, but they did not explicitly define 'tangles' for connectivity functions. Later, Dharmatilake [2] defined tangles for matroids and proved the duality with branch-width. In this section we define tangles for connectivity functions and reprove the duality with branch-width. We remark that, when restricted to matroids, our definition, unlike that of Dharmatilake, is self-dual.

Let λ be a connectivity function on E. A *tangle* of λ of *order* k is a collection $\mathcal{T}$ of subsets of E such that:

(T1) For each $B \in \mathcal{T}$, $\lambda(B) < k$.
(T2) For each separation (A, B) of order less than k, $\mathcal{T}$ contains A or B.
(T3) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E$.
(T4) For each $e \in E$, $E - \{e\} \notin \mathcal{T}$.

Note that, by (T3), (T2) can be sharpened to say that $\mathcal{T}$ contains exactly one of A and B. The following lemma gives alterate defining conditions for a tangle that are more straightforward to verify.

Lemma 3.1. *Let λ be a connectivity function and let $k \in \mathbb{Z}$. Now let $\mathcal{T}$ be a collection of subsets of E that satisfies*:

(T1) *For each* $B \in \mathcal{T}$, $\lambda(B) < k$.
(T2) *For each separation* (A, B) *of order less than* k, $\mathcal{T}$ *contains* A *or* B.
(T3a) *If* $A \subseteq B$, $B \in \mathcal{T}$, *and* $\lambda(A) < k$, *then* $A \in \mathcal{T}$.
(T3b) *If* (A, B, C) *is a partition of* E, *then* $\mathcal{T}$ *cannot contain all three of* A, B, *and* C.
(T4) *For each* $e \in E$, $E - \{e\} \notin \mathcal{T}$.

Then $\mathcal{T}$ *is a tangle.*

Proof. If $\mathcal{T}$ is not a tangle, then there exists $A, B, C \in \mathcal{T}$ such that $A \cup B \cup C = E$. Choose such A, B, and C minimizing $|A \cap B| + |B \cap C| + |C \cap A|$. By (T3b) and symmetry, we may assume that $|A \cap B| \neq 0$. Since λ is symmetric and submodular, we have $\lambda(A - B) + \lambda(B - A) \leqslant \lambda(A) + \lambda(B)$. Then, by the symmetry between A and B, we may assume that $\lambda(A - B) < k$. Now $A - B \subseteq A$, so, by (T3a), we have $A - B \in \mathcal{T}$. Thus we have $(A - B) \cup B \cup C = E$ and $|(A - B) \cap B| + |B \cap C| + |C \cap (A - B)| < |A \cap B| + |B \cap C| + |C \cap A|$. This contradicts our choice of A, B, and C. □

The main result of this section is:

Theorem 3.2. *Let* λ *be a connectivity function on* E. *Then the maximum order of a tangle of* λ *is equal to the branch-width of* λ.

The rest of this section is devoted to the proof of Theorem 3.2. Let $\mathcal{A}$ be a collection of subsets of E. We say that $\mathcal{A}$ *extends* to a tangle $\mathcal{T}$ of order k, if $\mathcal{A} \subseteq \mathcal{T}$. We say that a partial branch-decomposition T *conforms* to $\mathcal{A}$ if, for each leaf v of T, there is a set $A \in \mathcal{A}$ that contains each of the elements labelling v. (We do not require that the set elements labelling v is contained in $\mathcal{A}$.) The following theorem is cryptomorphic to [9, (3.5)]; for completeness we will include a proof of this result later in this section.

Theorem 3.3. *Let* λ *be a connectivity function on* E, *let* $k \in \mathbb{Z}$, *and let* $\mathcal{A}$ *be a collection of subsets of* E *such that* $\lambda(A) < k$, *for each* $A \in \mathcal{A}$, *and* $\bigcup \mathcal{A} = E$. *Then either*

- $\mathcal{A}$ *extends to a tangle of order* k, *or*
- *there is a partial branch-decomposition of* λ *of width* $< k$ *that conforms to* $\mathcal{A}$.

The two possible outcomes above are in fact exclusive, as we show in the following lemma.

Lemma 3.4. *Let* λ *be a connectivity function on* E *and let* $k \in \mathbb{Z}$. *If* $\mathcal{T}$ *is a tangle of order* k *and* T *is a partial branch-decomposition of* λ *that conforms with* $\mathcal{T}$, *then* $\epsilon(T) \geqslant k$.

Proof. Suppose, by way of contradiction, that $\epsilon(T) < k$. Construct an orientation of T as follows. Consider an edge e of T; let a and b be the ends of e and let X_a and X_b be the sets displayed by the components of $T - e$ containing a and b, respectively. Thus (X_a, X_b) is a separation of order less than k. By (T2) and (T3), $\mathcal{T}$ contains exactly one of X_a and X_b. By symmetry, we may assume that $X_a \in \mathcal{T}$. Now, orient e toward b. Consider a leaf w of T. Let e be the edge of T incident with w and let $X \subseteq V$ be the set of elements labelling w. By definition, there exists $A \in \mathcal{T}$ such that $X \subseteq A$. By (T2) and (T3), we have $X \in \mathcal{T}$. Therefore e is oriented away from w.

Therefore, there must exist an internal node v of T with all three incident edges oriented toward it. This, however, contradicts (T3). □

Before we prove Theorem 3.3, we will use it to prove Theorem 3.2.

Proof of Theorem 3.2. Let $k \in \mathbb{Z}$. By Lemma 3.4 it cannot be the case that there exists both a branch-decomposition of width $\leqslant k$ and a tangle of order k. Thus it suffices to prove that at least one of the two exist.

Case 1. There exists $e \in E$ such that $\lambda(\{e\}) \geqslant k$.

Let $\mathcal{T}$ consist of all sets $A \subseteq E - \{e\}$ with $\lambda(A) < k$. It is easy to verify that $\mathcal{T}$ is a tangle of order k.

Case 2. $\lambda(\{e\}) < k$ for each $e \in E$.

Let $\mathcal{A}$ be a partition of E into singletons. Then, by Theorem 3.3, either there exists a branch-decomposition of width $< k$ or $\mathcal{A}$ extends to a tangle of order k. □

Finally, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. We assume that:

3.4.1. *There is no partial branch-decomposition of width $< k$ that conforms with $\mathcal{A}$.*

We may also assume that:

3.4.2. *$\mathcal{A}$ is maximal subject to* 3.4.1 *and to the condition that $\lambda(A) < k$ for each $A \in \mathcal{A}$.*

From these assumptions we obtain:

3.4.3. *If $B \in \mathcal{A}$, $A \subseteq B$, and $\lambda(A) < k$, then $A \in \mathcal{A}$.*

Subproof. Since $A \subseteq B$, a partial branch-decomposition conforms with A if and only if it conforms with $\mathcal{A} \cup \{A\}$. □

Case 1. For each separation (X, Y) of λ of order $< k$, $\mathcal{A}$ contains X or Y.

In this case we will prove that $\mathcal{A}$ is, in fact, a tangle of order k. It is clear that $\mathcal{A}$ satisfies (T1) and (T2). Moreover, by 3.4.3, $\mathcal{A}$ satisfies (T3a) (of Lemma 3.1). Note that, by 3.4.1, $\mathcal{A}$ also satisfies (T3b). Finally, consider an element $e \in E$. Since $\bigcup \mathcal{A} = E$ there exists $A \in \mathcal{A}$ such that $e \in A$. If $\lambda(\{e\}) \geqslant k$, then $E - \{e\} \notin \mathcal{A}$ by (T1). If $\lambda(\{e\}) < k$, then $\{e\} \in \mathcal{A}$ by (T3a) and, hence, $E - \{e\} \notin \mathcal{A}$ by (T3b). In either case, $E - \{e\} \notin \mathcal{A}$ and, hence, $\mathcal{A}$ satisfies (T4). Then, by Lemma 3.1, $\mathcal{A}$ is a tangle.

Case 2. There exists a separation (A_1, A_2) of λ of order $< k$ such that $A_1, A_2 \notin \mathcal{A}$.

We choose such a separation (A_1, A_2) minimizing $\lambda(A_1)$. Let $i \in \{1, 2\}$. By 3.4.2, there exists a partial branch-decomposition T_i of width $< k$ that conforms with $\mathcal{A} \cup \{A_i\}$. By 3.4.1, there exists a vertex $v_i \in V(T_i)$ such the set $X_i \subseteq E$ labelling v_i is contained in A_i but is not contained in any set in $\mathcal{A}$.

3.4.4. $\kappa_\lambda(X_i, E - A_i) = \lambda(A_i)$.

Subproof. Consider a set Z such that $X_i \subseteq Z \subseteq E - A_i$. Suppose that $\lambda(Z) < \lambda(A_i)$. Then, by our choice of (X_1, X_2), we have $Z \in \mathcal{A}$ or $E - Z \in \mathcal{A}$. Since $X_i \subseteq Z$, it must be the case that $E - Z \in \mathcal{A}$. Then, by 3.4.3 and the fact that $X_2 \subseteq E - Z$, we have $X_2 \in \mathcal{A}$. This contradicts our choice of (X_1, X_2). □

Let T_i' be the branch-decomposition of λ obtained from T_i by leaving the labels in X_2 and moving the labels in X_1 to v_i. By 3.4.4 and Lemma 2.5, we have $\epsilon(T_i') \leqslant \epsilon(T_i) < k$. Now, from T_1' and T_2' we can easily construct a partial branch-decomposition of width $< k$ that conforms with $\mathcal{A}$; contrary to 3.4.1. □

4. Applications of tangles

Naturally, a *tangle* of a matroid M is a tangle of its connectivity function λ_M. The following lemma generalizes Lemma 1.3.

Lemma 4.1. *For all positive integers k and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ and M contains a (q^k, k)-net, then M has branch-width at least k.*

Proof. Let $(N, \mathcal{P})$ be a (q^k, k)-net. We define a collection of sets $\mathcal{T}$ such that $A \in \mathcal{T}$ if and only if $\lambda_M(A) < k$ and A does not contain a series class of $\mathcal{P}$.

Consider any separation (A, B) of M of order less than k. If P and Q are distinct members of $\mathcal{P}$, then, since $\kappa_M(P, Q) > \lambda_M(A)$, we cannot have $P \subseteq A$ and $Q \subseteq B$. That is, A and B cannot both contain a member of $\mathcal{P}$ and, hence, $\mathcal{T}$ satisfies (T2). Evidently, $\mathcal{T}$ also satisfies (T1), (T3a), and (T4).

Now, consider a partition (A_1, A_2, A_3) of $E(M)$ such that $\lambda_M(A_i) < k$ for each $i \in \{1, 2, 3\}$. Let $B_1 = E(M) - A_1$ and $B_2 = E(M) - A_2$. By the argument above, for each $i \in \{1, 2\}$, the number of sets $P \in \mathcal{P}$ such that either $P \cap A_1$ and $P \cap B_1$ are both non-empty or $P \cap A_2$ and $P \cap B_2$ are both non-empty is at most $2(q^{k-1} - 1) < q^k$. Therefore, there is some set in $\mathcal{P}$ that is contained in A_1, A_2, or A_3. Thus, $\mathcal{T}$ satisfies (T3b). So, by Lemma 3.1, $\mathcal{T}$ is a tangle of order k and, hence, M has branch-width at least k. □

Let X be a subset of $E(M)$. We call X an $[k, n]$-*connected set* if for each partition (X_1, X_2) of M with $|X_1|, |X_2| \geqslant n$ we have $\kappa_M(X_1, X_2) \geqslant k$.

Lemma 4.2. *Let X be a subset of $E(M)$. If X is an $[k, n]$-connected set and $|X| \geqslant 3n$, then M has branch-width at least $k + 1$.*

Proof. Let $\mathcal{T}$ be the set of all sets $A \subseteq E(M)$ such that $\lambda_M(A) \leqslant k$ and $|A \cap X| < n$. Consider a separation (A, B) of order less than k. Since X is $[k, n]$-connected, either $|A \cap X| < n$ or

$|B \cap X| < n$. That is, $\mathcal{T}$ satisfies (T2). Moreover, $\mathcal{T}$ clearly satisfies (T1), (T3), and (T4). Therefore, M has branch-width at least $k+1$. □

Let $\mathcal{T}$ be a tangle of M of order k. For $X \subseteq E(M)$, if X is a subset of a set in $\mathcal{T}$ then, we let

$$\phi_{\mathcal{T}}(X) = \min\bigl(\lambda_M(A) - 1\colon\ X \subseteq A \in \mathcal{T}\bigr),$$

otherwise we let $\phi_{\mathcal{T}}(X) = k - 1$.

Lemma 4.3. *Let M be a matroid and let $\mathcal{T}$ be a tangle of M of order k. Then $\phi_{\mathcal{T}}$ is the rank function of a matroid of rank $k-1$.*

Proof. It is straightforward to see that:

(i) $0 \leqslant \phi_{\mathcal{T}}(X) \leqslant |X|$ for any $X \subseteq E(M)$ and
(ii) $\phi_{\mathcal{T}}(X_1) \leqslant \phi_{\mathcal{T}}(X_2)$ for $X_1 \subseteq X_2 \subseteq E(M)$.

Thus it suffices to prove that $\phi_{\mathcal{T}}$ is submodular. Consider subsets Y_1 and Y_2 of $E(M)$. If $\phi_{\mathcal{T}}(Y_1) = k-1$, then $\phi_{\mathcal{T}}(Y_1 \cup Y_2) = k-1$. Moreover, $\phi_{\mathcal{T}}(Y_1 \cap Y_2) \leqslant \phi_{\mathcal{T}}(Y_2)$. Therefore, $\phi_{\mathcal{T}}(Y_1 \cup Y_2) + \phi_{\mathcal{T}}(Y_1 \cap Y_2) \leqslant \phi_{\mathcal{T}}(Y_1) + \phi_{\mathcal{T}}(Y_2)$. Now suppose that $\phi_{\mathcal{T}}(Y_1) < k-1$ and $\phi_{\mathcal{T}}(Y_2) < k-1$. Thus, for $i \in \{1,2\}$, there exists $A_i \in \mathcal{T}$ such that $Y_i \subseteq A_i$ and $\lambda_M(A_i) = \phi_{\mathcal{T}}(Y_i)$.

As $A_1 \in \mathcal{T}$, it follows from (T2) and (T3) that either $\lambda_M(A_1 \cap A_2) \geqslant k$ or $A_1 \cap A_2 \in \mathcal{T}$. In either case, $\phi_{\mathcal{T}}(Y_1 \cap Y_2) \leqslant \lambda_M(A_1 \cap A_2)$. Similarly, by (T2) and (T3), either $\lambda_M(A_1 \cup A_2) \geqslant k$ or $A_1 \cup A_2 \in \mathcal{T}$. In either case, $\phi_{\mathcal{T}}(Y_1 \cup Y_2) \leqslant \lambda_M(A_1 \cup A_2)$. Therefore,

$$\begin{aligned}\phi_{\mathcal{T}}(Y_1) + \phi_{\mathcal{T}}(Y_2) &= \lambda_M(A_1) + \lambda_M(A_2)\\ &\geqslant \lambda_M(A_1 \cap A_2) + \lambda_M(A_1 \cup A_2)\\ &\geqslant \phi_{\mathcal{T}}(Y_1 \cap Y_2) + \phi_{\mathcal{T}}(Y_1 \cup Y_2),\end{aligned}$$

as required. □

We obtain the following easy consequence.

Lemma 4.4. *If M is a matroid with branch-width at least $3k+1$, then there exists a $[k,k]$-connected subset X of $E(M)$ with $|X| \geqslant 3k$.*

Proof. Let $\mathcal{T}$ be a tangle of order $3k+1$, and let X be a subset of $E(M)$ such that $\phi_{\mathcal{T}}(X) = |X| = 3k$; such a set exists by Lemma 4.3. Now, consider any separation (A,B) of M of order less than k. We may assume that $A \in \mathcal{T}$. By Lemma 4.3, $|A \cap X| = \phi_{\mathcal{T}}(A \cap X) \leqslant \lambda_M(A) < k$. It follows that X is a $[k,k]$-connected set. □

Together Lemmas 4.2 and 4.4 provide a qualitative characterization of branch-width. Unfortunately, the amount of work needed to verify that a set is $[k,n]$-connected grows exponentially with respect to n and k.

5. Frames

For positive integers δ and γ, we define a (δ,γ)-*frame* in a matroid M to be a pair $(N,\mathcal{P})$ such that N is a minor of M, $\mathcal{P}$ is a set of series classes of N, $|\mathcal{P}|\geqslant\delta$, and $|P|\geqslant\gamma$ for each $P\in\mathcal{P}$. The main result of this section is the following.

Lemma 5.1. *There exists an integer-valued function $f_1(\delta,\gamma,q)$ such that for any positive integers δ, γ and $q\geqslant 2$, if M is a matroid in $\mathcal{U}(q)\cap\mathcal{U}^*(q)$ with branch-width at least $f_1(\delta,\gamma,q)$, then M or M^* contains a (δ,γ)-frame.*

We require the following preliminary results.

Lemma 5.2. *There exists an integer-valued function $f_2(\delta,\gamma,q,k)$ such that for any positive integers δ, γ, $q\geqslant 2$, and k, if M is a matroid in $\mathcal{U}^*(q)$ with branch-width at least $3(k+\delta)+1$, then either M contains a (δ,γ)-frame or there exists $Y\subseteq E(M)$ such that $M|Y$ has branch-width at least k and $|Y|\leqslant f_2(\delta,\gamma,q,k)$.*

Proof. Let $f_2(\delta,\gamma,q,k)=\binom{3(k+\delta)}{k+\delta}^2 q^{k+\delta}\gamma$. Suppose that M does not contain a (δ,γ)-frame. By Lemma 4.4, there exists a $[k+\delta,k+\delta]$-connected set Z in M with $|Z|=3(k+\delta)$.

Let S and T be disjoint subsets of Z with $|S|=|T|=k+\delta$. Then, $\kappa_M(S,T)=k+\delta$. Hence, by Tutte's Linking Theorem, there exists a partition (I,J) of $E(M)-(S\cup T)$ such that $\lambda_{M\backslash I/J}(S)=k+\delta$; we choose such a partition with J minimal. Let N denote the restriction of M to $S\cup T\cup J$. By the minimality of J, $S\cup J$ is a basis of N and N has no coloops. Since $S\cup J$ is a basis of N, we have $r(N^*)\leqslant|T|=k+\delta$. Let $\mathcal{P}$ denote the series classes of N with size at least γ. Since M does not contain a (δ,γ)-frame, we have $|\mathcal{P}|<\delta$. Let P denote the union of the sets in $\mathcal{P}$ and let $N_1=N\setminus P$. The corank of N_1 is at most $k+\delta$ and each series class of N_1 not in $\mathcal{P}$ has size at most $\gamma-1$, so, by Lemma 2.1, $|E(N_1)|\leqslant q^{k+\delta}(\gamma-1)$. Moreover, $\kappa_{N_1}(S\cap E(N_1),T\cap E(N_1))\geqslant\kappa_N(S,T)-|\mathcal{P}|\geqslant k$.

Let Y denote the set obtained by taking the union of Z and all sets of the form $E(N_1)$ taken over all possible choices of S and T. Then, Z is a $[k,k+\delta]$-connected set in $M|Y$. By Lemma 4.2, $M|Y$ has branch-width at least k. Moreover, since there are at most $\binom{3(k+\delta)}{k+\delta}^2$ different choices for S and T, we have $|Y|\leqslant f_2(\delta,\gamma,q,k)$. $\square$

For subsets X and Y of $E(M)$ we let $\sqcap_M(X,Y)$ denote $r_M(X)+r_M(Y)-r_M(X\cup Y)$.

Lemma 5.3. *There exists an integer-valued function $f_3(\gamma,q,t)$ such that for any positive integers δ, γ, $q\geqslant 2$, and t, if M is a matroid in $\mathcal{U}^*(q)$ that does not contain a (δ,γ)-frame and $A\subseteq E(M)$ with $\lambda_M(A)\leqslant t$, then there exists $X\subseteq E(M)-A$ such that $\lambda_{M/X}(A)\leqslant\delta$ and $|X|\leqslant f_3(\gamma,q,t)$.*

Proof. Let $f_3(\gamma,q,t)=(\gamma-1)q^{t-1}$ and let M be a matroid in $\mathcal{U}^*(q)$ that does not contain a (δ,γ)-frame and let A be a subset of $E(M)$ with $\lambda_M(A)\leqslant t$.

Let J be a minimal subset of $E(M)-A$ such that $\sqcap_M(A,J)=\lambda_M(A)-1$ and let $N=(M/A)|J$. Note that N has no coloops and that, as J is independent, $r(N^*)=\lambda_M(A)-1\leqslant t-1$. Let X be the set of all elements of N that are in series classes of size at most $\gamma-1$ and let $B=J-X$. By Lemma 2.1, $|X|\leqslant(\gamma-1)q^{t-1}=f_3(\gamma,q,t)$. Since M has no (δ,γ)-frame, there are at most $\delta-1$ series classes of N that have size at least γ. Thus, $r^*(N\setminus B)\geqslant$

$r^*(N)-\delta+1=\lambda_M(A)-\delta$. It follows that $\sqcap_M(A,X)\geqslant\lambda_M(A)-\delta$ and, hence, that $\lambda_{M/X}(A)=\lambda_M(A)-\sqcap_M(A,X)\leqslant\delta$. □

We need the following result in the case that $k_1=k_2$; the more technical version facilitates induction.

Lemma 5.4. *There exists an integer-valued function $f_4(\delta,\gamma,q,k_1,k_2,n)$ such that for any positive integers δ, γ, k_1, k_2, $n\geqslant 2$ and $q\geqslant 2$, if M is a matroid in $\mathcal{U}(q)\cap\mathcal{U}^*(q)$ such that M has branch-width at least $f_4(\delta,\gamma,q,k_1,k_2,n)$ and neither M nor M^* contains a (δ,γ)-frame, then there exists a restriction N of M and a partition $(A_1,A_2,\dots,A_n)$ of $E(N)$ such that $N|A_1,\dots,N|A_{n-1}$ each have branch-width at least k_1, $N|A_n$ has branch-width at least k_2, and $\lambda_N(A_1\cup\dots\cup A_i)\leqslant\delta$ for all $i\in\{1,\dots,n-1\}$.*

Proof. Let $k_3=f_2(\delta,\gamma,q,k_1)$ and $k_4=\max(3(k_1+\delta)+1,k_2+k_3+f_3(\gamma,q,k_3+\delta))$. Now define $f_4(\delta,\gamma,q,k_1,k_2,2)=\max(3(k_1+\delta)+1,k_2+k_3+f_3(\gamma,q,k_3))$. For $n>2$, we recursively define $f_4(\delta,\gamma,q,k_1,k_2,n)=f_4(\delta,\gamma,q,k_1,k_4,n-1)$. Let M be a matroid in $\mathcal{U}(q)\cap\mathcal{U}^*(q)$ such that M has branch-width at least $f_4(\delta,\gamma,q,k_1,k_2,n)$ and neither M nor M^* contains a (δ,γ)-frame.

The proof is by induction on n; we begin with the case $n=2$. By Lemma 5.2, there exists $A_1\subseteq E(M)$ such that $M|A_1$ has branch-width at least k_1 and such that $|A_1|\leqslant k_3$. Now by dualizing Lemma 5.3, there exists $X\subseteq E(M)-A_1$ such that $\lambda_{M\setminus X}(A_1)\leqslant\delta$ and $|X|\leqslant f_3(\gamma,q,k_3)$. Let $N=M\setminus X$ and let $A_2=E(N)-A_1$. Since $|A_1\cup X|\leqslant k_3+f_3(\gamma,q,k_3)$, $N|A_2$ has branch-width at least k_2; as required.

Now consider the case that $n>2$. By induction, there exists a restriction N_1 of M and a partition $(A_1,\dots,A_{n-2},B)$ of $E(N_1)$ such that, for each $i\in\{1,\dots,n-2\}$, $N_1|A_i$ has branch-width at least k_1, $N_1|B$ has branch-width at least k_4, and, for each $i\in\{1,\dots,n-2\}$, $\lambda_{N_1}(A_1\cup\dots\cup A_i)\leqslant\delta$. By Lemma 5.2, there exists $A_{n-1}\subseteq B$ such that $M|A_{n-1}$ has branch-width exactly k_1 and such that $|A_{n-1}|\leqslant k_3$. Note that $\lambda_{N_1}(A_1\cup\dots\cup A_{n-1})\leqslant\lambda_{N_1}(A_1\cup\dots\cup A_{n-2})+|A_{n-1}|\leqslant\delta+k_3$. Thus, by dualizing Lemma 5.3, there exists $X\subseteq E(N_1)-(A_1\cup\dots\cup A_{n-1})$ such that $\lambda_{N_1\setminus X}(A_1\cup\dots\cup A_{n-1})\leqslant\delta$ and $|X|\leqslant f_3(\gamma,q,\delta+k_3)$. Let $N=N_1\setminus X$ and let $A_n=E(N)-(A_1\cup\dots\cup A_{n-1})$. Since $|A_{n-1}\cup X|\leqslant k_3+f_3(\gamma,q,k_3+\delta)$ and $N|A_n=(N_1|B)\setminus(A_{n-1}\cup X)$, $N|A_n$ has branch-width at least k_2; as required. □

Proof of Lemma 5.1. Let $m=\gamma q^{2\delta}$, $k=\binom{m+1}{2}$, and $f_1(\delta,\gamma,q)=f_4(\delta,\gamma,q,k,k,\delta)$. Now let M be a matroid in $\mathcal{U}(q)\cap\mathcal{U}^*(q)$ such that M has branch-width at least $f_1(\delta,\gamma,q)$ and neither M nor M^* contains a (δ,γ)-frame. By Lemma 5.4, there exists a minor N_1 of M and a partition $(A_1,A_2,\dots,A_\delta)$ of $E(N_1)$ such that $N|A_1,\dots,N|A_\delta$ each have branch-width at least k, and $\lambda_{N_1}(A_1\cup\dots\cup A_i)\leqslant\delta$ for all $i\in\{1,\dots,\delta-1\}$. Now, by Theorem 2.6, for each $i\in\{1,\dots,\delta\}$ there exists a circuit $C_i\subseteq A_i$ of N_1 of length at least m. Let $N=N|(C_1\cup\dots\cup C_\delta)$. For each $i\in\{1,\dots,\delta\}$, we have

$$\begin{aligned}\lambda_N(C_i)&\leqslant\lambda_N(C_1\cup\dots\cup C_{i-1})+\lambda_N(C_{i+1}\cup\dots\cup C_\delta)\\&=\lambda_N(C_1\cup\dots\cup C_{i-1})+\lambda_N(C_1\cup\dots\cup C_i)\\&\leqslant\lambda_{N_1}(A_1\cup\dots\cup A_{i-1})+\lambda_{N_1}(A_1\cup\dots\cup A_i)\\&\leqslant 2\delta.\end{aligned}$$

It follows easily by definitions that

$$r_N^*(C_i) = \lambda_N(C_i) - 1 + r^*(N|C_i) \leqslant 2\delta.$$

Thus, there exists a series class $S_i \subseteq C_i$ with $|S_i| \geqslant \gamma$. So $(N, \{S_1, \ldots, S_\delta\})$ is a (δ, γ)-frame. □

6. Nets

Let f be an integer valued function defined on the set of positive integers. A matroid M is called (m, f)-*connected* if whenever (A, B) is a separation of order $\ell < m$, then either $|A| \leqslant f(\ell)$ or $|B| \leqslant f(\ell)$. The following result was proved in [3].

Lemma 6.1. *Let $g(\ell) = (6^{\ell-1} - 1)/5$ for all positive integers ℓ. If M is a minor-minimal matroid with branch-width k, then M is $(k+1, g)$-connected.*

We are finally ready to prove the main result.

Theorem 6.2. *For all positive integers δ, γ and $q \geqslant 2$, there exists an integer k such that if M is a matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ with branch-width at least k, then M or M^* contains a (δ, γ)-net.*

Proof. Let $\gamma' = g(\gamma - 1) + 1$ and let $k = f_1(\delta, \gamma', q)$. Now let M be a matroid in $\mathcal{U}(q) \cap \mathcal{U}^*(q)$ with branch-width at least k. Evidently, we may assume that M is minor-minimal with branch-width k. Thus, by Lemma 6.1, M is $(k+1, g)$-connected. By Lemma 5.1 and duality, we may assume that M contains a (δ, γ')-frame $(N, \mathcal{P})$. Consider a pair of distinct sets $P_1, P_2 \in \mathcal{P}$. Let (X_1, X_2) be a partition of $E(M)$ with $P_1 \subseteq X_1$ and $P_2 \subseteq X_2$. Now, $|X_1|, |X_2| \geqslant g(\gamma - 1) + 1$. Thus, $\lambda_M(X_1) \geqslant \gamma$. It follows that $\kappa_M(P_1, P_2) \geqslant \gamma$. That is, $(N, \mathcal{P})$ is a (δ, γ)-net in M. □

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