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MATROID T-CONNECTIVITY*

JIM GEELEN†, BERT GERARDS‡, AND GEOFF WHITTLE§

Abstract. We introduce a new generalization of the maximum matching problem to matroids; this problem includes Gallai's T-path problem for graphs.

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1. Introduction.

Let $G = (V, E)$ be a simple graph and let $T \subseteq V$. A *T-path* is a path in G connecting two vertices in T. Let $\nu_G(T)$ denote the maximum number of vertex disjoint T-paths in G. This parameter was introduced by Gallai [2], who showed that determining $\nu_G(T)$ is equivalent to the maximum matching problem. (Note that $\nu_G(V)$ is the size of a maximum matching in G.) As a consequence of an exact min-max theorem for $\nu_G(T)$, Gallai [2] proved the following theorem.

THEOREM 1.1 (Gallai [2]). *Let $G = (V, E)$ be a graph and $T \subseteq V$. Then there exists a set $X \subseteq V$ that hits every T-path such that $|X| \leq 2\nu_G(T)$.*

Note that if $X \subseteq V$ hits each T-path, then $\nu_G(T) \leq |X|$. Gallai's theorem shows that this natural upper bound for $\nu_G(T)$ is within a factor of 2 of being tight. We consider a matroidal generalization of $\nu_G(T)$ and prove analogous upper bounds. This problem arose naturally in proving structural results on minor-closed classes of matroids represented over finite fields. The main result presented here is needed as a lemma in that project.

Let M be a matroid. For $X \subseteq E(M)$ we let

$$\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M).$$

For disjoint sets $S, T \subseteq E(M)$, we let

$$\kappa_M(S, T) = \min(\lambda_M(X) \, : \, S \subseteq X \subseteq E(M) - T).$$

Then, for a set $T \subseteq E(M)$, we let

$$\nu_M(T) = \max(\kappa_M(X, T - X \, : \, X \subseteq T);$$

we call $\nu_M(T)$ the *T-connectivity* of M. It is straightforward to verify that $\lambda_M(X) = \lambda_{M^*}(X)$. Therefore $\kappa_M(S, T) = \kappa_{M^*}(S, T)$ and, hence, $\nu_M(T) = \nu_{M^*}(T)$. We will consider a slightly more general parameter. Let $\mathcal{T}$ be a collection of disjoint subsets

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†Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada (jfgeelen@uwaterloo.ca).

‡CWI, Amsterdam and Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands (bert.gerards@cwi.nl).

§School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand (geoff.whittle@mcs.vuw.ac.nz).

of $E(M)$. Then we define $\nu_M(\mathcal{T})$ to be the maximum of $\kappa_M(X,Y)$, where $X = \cup\mathcal{T}_1$ and $Y = \cup\mathcal{T}_2$ for a partition $(\mathcal{T}_1,\mathcal{T}_2)$ of $\mathcal{T}$. Thus, if $\mathcal{T}$ is a partition of a set $T \subseteq E(M)$ into singletons, then $\nu_M(T) = \nu_M(\mathcal{T})$. We also call $\nu_M(\mathcal{T})$ the *$\mathcal{T}$-connectivity* of M.

Let $G = (V,E)$ be a simple graph. We can construct a matroid M on $V \cup E$ such that V is a basis of M and, for each edge $e = uv$ of G, the element e is placed freely on the line through u and v. Note that if P is a nontrivial (u,v)-path in G, then $\{u,v\} \cup E(P)$ is a circuit of M. Now it is a straightforward application of Menger's theorem to prove that for any two disjoint subsets S and T of vertices of G, $\kappa_M(S,T)$ is equal to the maximum number of vertex disjoint (S,T)-paths in G. Now it is easy to see that, for any $T \subseteq V$, we have $\nu_M(T) = \nu_G(T)$.

Let $\mathcal{T}$ be a collection of disjoint subsets of V. Let $\nu_G(\mathcal{T})$ denote the maximum, taken over all partitions $(\mathcal{T}_1,\mathcal{T}_2)$ of $\mathcal{T}$, of the connectivity between $\cup\mathcal{T}_1$ and $\cup\mathcal{T}_2$ in G. Thus $\nu_G(\mathcal{T}) = \nu_M(\mathcal{T})$. A $\mathcal{T}$-path is a path whose ends are in distinct parts of $\mathcal{T}$. Mader [5] considered the related problem of finding the maximum number, $\mu_G(\mathcal{T})$, of vertex disjoint $\mathcal{T}$-paths. It is straightforward to show that $\nu_M(\mathcal{T}) \le \mu_G(\mathcal{T}) \le 2\nu_M(\mathcal{T})$. (Indeed, the first inequality is trivial and the second comes from the fact that when taking a random partition $(\mathcal{T}_1,\mathcal{T}_2)$ of $\mathcal{T}$ we expect half of Mader's $\mathcal{T}$-paths to connect $\cup\mathcal{T}_1$ and $\cup\mathcal{T}_2$.) This bound is interesting since $\mu_G(\mathcal{T})$ can be computed efficiently (see Lovász [4] or Chudnovsky, Cunningham, and Geelen [1]), while computing $\nu_G(\mathcal{T})$ is NP-hard. Indeed, suppose that G is a graph consisting of a perfect matching, $\mathcal{T}$ is a partition of $V(G)$, and G' is obtained from G by shrinking each part of $\mathcal{T}$ to a single vertex. Then $\nu_G(\mathcal{T})$ is the size of a maximum cut in G'. Therefore computing $\nu_G(\mathcal{T})$ is NP-hard, as claimed. Moreover, this implies that computing $\nu_M(\mathcal{T})$ is NP-hard.

Let M_1 and M_2 be matroids on a common ground set E. We say that M_2 is obtained by an *elementary transformation* on M_1 if there exists a matroid N on $E \cup \{e\}$ such that either $M_1 = N \setminus e$ and $M_2 = N/e$ or $M_1 = N/e$ and $M_2 = N \setminus e$. We define $\mathrm{dist}(M_1,M_2)$ to be the minimum number of elementary transformations required to transform M_1 into M_2. The following properties are straightforward to verify; the last of these properties shows that $\mathrm{dist}(M_1,M_2)$ is well defined:

- $\mathrm{dist}(M_1,M_2) = \mathrm{dist}(M_2,M_1)$.
- $\mathrm{dist}(M_1^*,M_2^*) = \mathrm{dist}(M_1,M_2)$.
- If M' is the rank-zero matroid on E, then $\mathrm{dist}(M_1,M') = r(M_1)$.
- If M_3 is a matroid on E, then $\mathrm{dist}(M_1,M_3) \le \mathrm{dist}(M_1,M_2) + \mathrm{dist}(M_2,M_3)$.
- $\mathrm{dist}(M_1,M_2) \le |E|$.

We use the following lemma.

LEMMA 1.2. *Let M_1 and M_2 be matroids on a common ground set E and let $\mathcal{T}$ be a collection of disjoint subsets of E. Then $\nu_{M_1}(\mathcal{T}) \le \nu_{M_2}(\mathcal{T}) + dist(M_1,M_2)$.*

Proof. By a simple inductive argument we may assume that $\mathrm{dist}(M_1,M_2) = 1$. Moreover, by duality we may assume that $M_1 = N \setminus e$ and $M_2 = N/e$. Now it is easy to check that $\nu_{M_1}(\mathcal{T}) \le \nu_N(\mathcal{T}) \le \nu_{M_2}(\mathcal{T}) + 1$, as required. ∎

Note that $\nu_M(\mathcal{T}) = 0$ if and only if no component of M contains elements from two distinct parts of $\mathcal{T}$. Let $T = \cup\mathcal{T}$ and let $\delta_M(\mathcal{T}) = \max(\kappa_M(X, T - X) : X \in \mathcal{T})$. Note that $\delta_M(\mathcal{T}) \le \nu_M(\mathcal{T})$ and, when $\mathcal{T}$ contains only singletons, $\delta_M(\mathcal{T}) \le 1$. The main result of this paper is the following.

THEOREM 1.3. *Let M be a matroid and let $\mathcal{T}$ be a collection of disjoint subsets of $E(M)$. Then there exists a matroid M' on the ground set $E(M)$ such that $\nu_{M'}(\mathcal{T}) = 0$ and $dist(M,M') \le 2(\delta_M(\mathcal{T})+1)\nu_M(\mathcal{T})$.*

The next result is an easy consequence of Theorem 1.3. We say that a partition

$\mathcal{P}$ of $E(M)$ *encloses* $\mathcal{T}$ if each set in $\mathcal{T}$ is contained in some set in $\mathcal{P}$ and no set in $\mathcal{P}$ contains two or more sets in $\mathcal{T}$. The *order* of $\mathcal{P}$, denoted by $\mathrm{ord}_M(\mathcal{P})$, is defined as $\max(\lambda_M(\cup\mathcal{Q}) : \mathcal{Q} \subseteq \mathcal{P})$. Note that if $\mathcal{P}$ encloses $\mathcal{T}$, then $\mathrm{ord}_M(\mathcal{P}) \geq \nu_M(\mathcal{T})$.

COROLLARY 1.4. *Let M be a matroid and let $\mathcal{T}$ be a collection of disjoint subsets of $E(M)$. Then there exists a partition $\mathcal{P}$ of $E(M)$ enclosing $\mathcal{T}$ where $\mathrm{ord}_M(\mathcal{P}) \leq 2(\delta_M(\mathcal{T})+1)\nu_M(\mathcal{T})$.*

While Corollary 1.4 does follow from Theorem 1.3, we will not include the easy proof since Corollary 1.4 is an immediate consequence of Theorem 4.1.

We conclude the introduction by stating some open problems.

PROBLEM 1.5. *Can the bound of* $2(\delta_M(\mathcal{T})+1)\nu_M(\mathcal{T})$ *in Theorem* 1.3 *be improved to $c\nu_M(\mathcal{T})$ for some constant c?*

PROBLEM 1.6. *In the case that each element of $\mathcal{T}$ is a singleton, can the bound of* $2(\delta_M(\mathcal{T})+1)\nu_M(\mathcal{T})$ *in Theorem* 1.3 *be improved to* $2\nu_M(\mathcal{T})$*?*

We now turn to the problem of finding a tight bound on T-connectivity. If M' is a matroid on the ground set $E(M)$, then it is straightforward to prove that

$$\nu_M(T) \leq \mathrm{dist}(M, M') + \sum\left(\left\lfloor\frac{|T \cap F|}{2}\right\rfloor : F \text{ a component of } M'\right).$$

PROBLEM 1.7. *Is there always a matroid M' for which equality is attained?*

Recall that computing $\nu_M(\mathcal{T})$ is NP-hard. The final problems concern the complexity of determining $\nu_M(T)$; as usual we assume that the matroid is given by its rank oracle.

PROBLEM 1.8. *Is there a polynomial-time algorithm for computing $\nu_M(T)$?*

It is straightforward to show that $\nu_M(E(M))$ is the size of a maximum common independent set of M and M^*. So we can compute $\nu_M(E(M))$ efficiently via matroid intersection. The following special case of Problem 1.8 contains the matching problem.

PROBLEM 1.9. *Is there a polynomial-time algorithm for computing $\nu_M(B)$ where B is a basis of M?*

The above problems are all open for the class of representable matroids.

2. Submodular functions. This section contains notation, definitions, and elementary results on submodular functions.

A *set function* on a set E is an integer valued function defined on the collection of subsets of E. Let λ be a set function on E. Then

- λ is *submodular* if $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for each $X, Y \subseteq E$;
- λ is *nonnegative* if $\lambda(X) \geq 0$ for each $X \subseteq E$;
- λ is *symmetric* if $\lambda(X) = \lambda(E - X)$ for each $X \subseteq E$.

We call $K = (E, \lambda)$ a *connectivity system* if λ is a symmetric, submodular, nonnegative set function on a finite set E. For a matroid M we define $K(M) = (E(M), \lambda_M)$; $K(M)$ is readily seen to be a connectivity system. Let $K = (E, \lambda)$ be a connectivity system and let S and T be disjoint subsets of E. Now let $\kappa_K(S,T) = \min(\lambda(X) : S \subseteq X \subseteq E - T)$. Finally, for a collection $\mathcal{T}$ of disjoint subsets of E, we let $\nu_K(\mathcal{T}) = \max \kappa_K(X, Y)$ where the maximum is taken over all partitions (X, Y) of $\cup\mathcal{T}$ where X is the union of a subcollection of $\mathcal{T}$. When $\mathcal{T}$ is a partition of a set $T \subseteq E$ into singletons, then we let $\nu_M(T) = \nu_M(\mathcal{T})$. In section 4 we provide upper bounds on $\nu_K(T)$. In the remainder of this section we consider preliminary results.

A set function r on E is *nondecreasing* if $r(X) \leq r(Y)$ whenever $X \subseteq Y$.

LEMMA 2.1. *Let $K = (E, \lambda)$ be a connectivity system, let $T \subseteq E$, and let $r(S) = \kappa_K(S, T)$ for each $S \subseteq E - T$. Then r is a nondecreasing, submodular, nonnegative set function on $E - T$.*

Proof. It is clear that r is nondecreasing and nonnegative. Let $S_1, S_2 \subseteq E - T$. Then, for $i \in \{1,2\}$, there exists a set X_i such that $S_i \subseteq X_i \subseteq E - T$ and $\lambda(X_i) = \kappa_K(S_i, T) = r(S_i)$. Note that $S_1 \cap S_2 \subseteq X_1 \cap X_2 \subseteq E - T$ and $S_1 \cup S_2 \subseteq X_1 \cup X_2 \subseteq E - T$. Therefore $\lambda(X_1 \cap X_2) \geq \kappa_K(S_1 \cap S_2, T) = r(S_1 \cap S_2)$ and $\lambda(X_1 \cup X_2) \geq \kappa_K(S_1 \cup S_2, T) = r(S_1 \cup S_2)$. Hence

$$\begin{aligned} r(S_1) + r(S_2) &= \lambda(X_1) + \lambda(X_2) \\ &\geq \lambda(X_1 \cap X_2) + \lambda(X_1 \cup X_2) \\ &\geq r(S_1 \cap S_2) + r(S_1 \cup S_2). \end{aligned}$$

Therefore r is submodular, as required. ∎

The following result is well known in the context of polymatroids.

LEMMA 2.2. *Let r be a nondecreasing, submodular set function on a finite set E. If $X \subseteq Y \subseteq E$ and $r(X \cup \{e\}) = r(X)$ for each $e \in Y - X$, then $r(X) = r(Y)$.*

Proof. Suppose otherwise and choose Y' minimal such that $X \subseteq Y' \subseteq Y$ and $r(Y') > r(X)$. Clearly $|Y'| \geq |X| + 2$. Let $e \in Y' - X$. By our choice of Y', $r(Y' - \{e\}) = r(X)$ and $r(X \cup \{e\}) = r(X)$. Now, by submodularity, $r(X \cup \{e\}) + r(Y' - \{e\}) \geq r(X) + r(Y')$. But then $r(Y') \leq r(Y' - \{e\}) = r(X)$; this contradiction completes the proof. ∎

LEMMA 2.3. *Let $K = (E, \lambda)$ be a connectivity system and let S and T be disjoint subsets of E. Then there exist sets $S' \subseteq S$ and $T' \subseteq T$ such that $\kappa_K(S', T') = \kappa_K(S, T)$ and $|S'|, |T'| \leq \kappa_K(S, T)$.*

Proof. Choose $S' \subseteq S$ maximal such that $\kappa_K(S', T) \geq |S'|$. Note that this is well defined since $\kappa_K(\emptyset, T) \geq 0$. By the definition of S' we have $\kappa_K(S' \cup \{e\}) = \kappa_K(S')$ for all $e \in S - S'$. Therefore, by Lemmas 2.1 and 2.2, $\kappa_K(S', T) = \kappa_K(S, T)$. Now choose $T' \subseteq T$ maximal such that $\kappa_K(S', T') \geq |T'|$. As above we get $\kappa_K(S', T') = \kappa_K(S', T) = \kappa_K(S, T)$, as required. ∎

LEMMA 2.4. *Let $K = (E, \lambda)$ be a connectivity system, let S and T be disjoint subsets of E with $\kappa_K(S, T) = k$, and let $\mathcal{S} = \{X : S \subseteq X \subseteq E - T \text{ and } \lambda(X) = k\}$. If $X, Y \in \mathcal{S}$, then $X \cap Y, X \cup Y \in \mathcal{S}$.*

Proof. Note that $S \subseteq X \cap Y \subseteq X \cup Y \subseteq E - T$. Then, since $\kappa_K(S, T) = k$ we have $\lambda(X \cap Y), \lambda(X \cup Y) \geq k$. Moreover, by submodularity, we have $2k = \lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y) \geq 2k$. It follows that $\lambda(X \cap Y) = k$ and $\lambda(X \cup Y) = k$. Therefore $X \cap Y, X \cup Y \in \mathcal{S}$, as required. ∎

3. Homomorphisms. Let $K = (E, \lambda)$ be a connectivity system and let $X \subseteq E$. We define a set function λ' on $(E - X) \cup \{e_X\}$ such that for each $Y \subseteq E - X$, $\lambda'(Y) = \lambda(Y)$ and $\lambda'(Y \cup \{e_X\}) = \lambda(Y \cup X)$. Now let $K \circ X = ((E - X) \cup \{e_X\}, \lambda')$. It is easy to verify that $K \circ X$ is a connectivity system; we say that $K \circ X$ is obtained from K by *identifying* X. If $\mathcal{T}$ is a collection of disjoint subsets of E, then we let $K \circ \mathcal{T}$ denote the connectivity system obtained by identifying each set in $\mathcal{T}$.

Remark. If $K = (E, \lambda)$ is a connectivity system and $\mathcal{T}$ is a collection of disjoint subsets of E, and if $T = \{e_X : X \in \mathcal{T}\}$, then $\nu_K(\mathcal{T}) = \nu_{K \circ \mathcal{T}}(T)$.

By the above remark, we can reduce the problem of computing $\nu_K(\mathcal{T})$ to the apparently easier problem of computing $\nu_K(T)$.

THEOREM 3.1. *Let $K = (E, \lambda)$ be a connectivity system and let $\mathcal{T} = \{T_1, \ldots, T_l\}$ be a partition of $T \subseteq E$. Then there exists a collection $\mathcal{T}' = \{T_1', \ldots, T_l'\}$ of disjoint sets such that $\nu_K(\mathcal{T}') = \nu_K(\mathcal{T})$ and, for each $i \in \{1, \ldots, l\}$, $T_i \subseteq T_i'$ and $\lambda(T_i') = \kappa_K(T_i, T - T_i)$.*

Note that Theorem 3.1 is an immediate corollary of the following lemma.

LEMMA 3.2. *Let $K = (E, \lambda)$ be a connectivity system, let A, B, and C be disjoint subsets of E, and let X be any set satisfying $A \subseteq X \subseteq E - (B \cup C)$ and $\lambda(X) = \kappa_K(A, B \cup C)$. Then $\nu_K(\{A, B, C\}) = \nu_K(\{X, B, C\})$.*

Proof. Note that by symmetry it suffices to prove that $\kappa_K(B, A \cup C) = \kappa_K(B, X \cup C)$. Let Y be a set satisfying $B \subseteq Y \subseteq E - (A \cup C)$ and $\lambda(Y) = \kappa_K(B, A \cup C)$. Since $A \subseteq X - Y \subseteq E - (B \cup C)$ and $B \subseteq Y - X \subseteq E - (A \cup C)$, we have $\lambda(Y) \leq \lambda(Y - X)$ and $\lambda(X) \leq \lambda(X - Y)$. However, by submodularity and symmetry, we have

$$\lambda(Y) + \lambda(X) \geq \lambda(Y - X) + \lambda(X - Y).$$

Therefore $\lambda(Y) = \lambda(Y - X)$ and $\lambda(X) = \lambda(X - Y)$. Then, since $B \subseteq Y - X \subseteq E - (X \cup C)$, we have $\kappa_K(B, X \cup C) = \kappa_K(B, A \cup C)$, as required. □

4. Connectivity systems. Let $K = (E, \lambda)$ be a connectivity system and let $\mathcal{T}$ be a collection of disjoint subsets of E. Now let $\mathcal{P}$ be a partition of E. The *order* of $\mathcal{P}$, denoted $\mathrm{ord}_K(\mathcal{P})$, is $\max(\lambda(\cup\mathcal{S}) : \mathcal{S} \subseteq \mathcal{P})$. Note that if $\mathcal{P}$ encloses $\mathcal{T}$, then $\nu_K(\mathcal{T}) \leq \mathrm{ord}_K(\mathcal{P})$. Let $T = \cup\mathcal{T}$ and let $\delta_K(\mathcal{T}) = \max(\kappa_K(X, T - X) : X \in \mathcal{T})$. One of the main results of this section is the following.

THEOREM 4.1. *Let $K = (E, \lambda)$ be a connectivity system and let $\mathcal{T}$ be a collection of disjoint subsets of E. Then there exists a partition $\mathcal{P}$ of E enclosing $\mathcal{T}$ with $ord_K(\mathcal{P}) \leq 2(1 + \delta_K(\mathcal{T}))\nu_K(\mathcal{T})$.*

We conjecture that this bound can be sharpened from $2(1 + \delta_K(\mathcal{T}))\nu_K(\mathcal{T})$ to $2\nu_K(\mathcal{T})$.

The problem of computing $\mathrm{ord}_K(\mathcal{P})$ is easily seen to contain the max-cut problem and is therefore NP-hard. We will introduce another notion, a (T, k)-dissection, that also provides an upper bound on $\nu_K(T)$. However, the key properties of a (T, k)-dissection can be verified efficiently.

A triple $(A, B, \mathcal{P})$ is a (T, k)-*dissection* if it satisfies the following:

- $\mathcal{P} \cup \{A, B\}$ is a partition of E.
- $|A \cap T|, |B \cap T| \leq k$ and $|P \cap T| = 1$ for each $P \in \mathcal{P}$.
- $\kappa_K(A, B) = k$.
- $\lambda(A \cup P) = k$ for each $P \in \mathcal{P}$.

Note that the third property above is the only property that is nontrivial to verify. However, we can compute $\kappa_K(A, B)$ efficiently via submodular function minimization (see Iwata, Fleischer, and Fujishige [3] or Schrijver [7]). Therefore we can efficiently verify that a triple is a (T, k)-dissection.

THEOREM 4.2. *Let $K = (E, \lambda)$ be a connectivity system and let $T \subseteq E$ where $\nu_K(T) = k$. Then K admits a (T, k)-dissection.*

Proof. Let (T_1, T_2) be a partition of T such that $\kappa_K(T_1, T_2) = k$. By Lemma 2.3, there exists $A' \subseteq T_1$ and $B' \subseteq T_2$ such that $\kappa_K(A', B') = k$ and $|A'|, |B'| \leq k$. Let $\mathcal{A} = \{X : A' \subseteq X \subseteq E - B' \text{ and } \lambda(X) = k\}$. By Lemma 2.4, $\mathcal{A}$ is closed under intersections and unions.

For each set $Z \subseteq T$ with $A' \subseteq Z \subseteq T - B'$, we have $\kappa_K(Z, T - Z) = k$. Therefore there exists $X \in \mathcal{A}$ such that $X \cap T = Z$. Choose a set $A \in \mathcal{A}$ as large as possible such that $A \cap T = A'$. Now, for each element $e \in T - (A' \cup B')$, choose a set $A_e \in \mathcal{S}$ as large as possible such that $A_e \cap T = A' \cup \{e\}$. Note that $A \cup A_e \in \mathcal{A}$ and $(A \cup A_e) \cap T = A' \cup \{e\}$. Therefore, by the maximality of A_e, we have $A \subseteq A_e$. Now consider two distinct elements $e, f \in T - (A' \cup B')$. Note that $A \subseteq A_e \cap A_f \in \mathcal{A}$ and $(A_e \cap A_f) \cap T = A'$. Therefore, by the maximality of A, we have $A_e \cap A_f = A$. Now let $B = E - \cup(A_e : e \in T - (A' \cup B'))$ and let $\mathcal{P} = (A_e - A : e \in T - (A' \cup B'))$. Then $(A, B, \mathcal{P})$ is a (T, k)-dissection. □

For $T \subseteq E$ we let $\Delta_K(T) = \max(\lambda(\{e\} : e \in T))$.

THEOREM 4.3. *Let $K = (E, \lambda)$ be a connectivity system, let $T \subseteq E$, let $\mathcal{T}$ be the partition of T into singletons, and let $(A, B, \mathcal{P})$ be a (T, k)-dissection. Then there exist partitions $\mathcal{A}$ of A and $\mathcal{B}$ of B such that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{P}$ encloses $\mathcal{T}$ and $ord_K(\mathcal{A} \cup \mathcal{B} \cup \mathcal{P}) \leq 2(1 + \Delta_K(T))k$. Hence $\nu_K(T) \leq 2(1 + \Delta_K(T))k$.*

Proof. Let $\mathcal{A} = \{A - T\} \cup \{\{e\} : e \in A \cap T\}$, $\mathcal{B} = \{B - T\} \cup \{\{e\} : e \in B \cap T\}$, and $\mathcal{C} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{P}$. Note that $\mathcal{C}$ encloses $\mathcal{T}$; it remains to prove that $\text{ord}(\mathcal{C}) \leq 2(1 + \Delta_K(T))k$.

4.3.1. *$ord_K(\mathcal{P} \cup \{A, B\}) \leq 2k$.*

Subproof. By definition, $\lambda(A \cup P) = k$ for each $P \in \mathcal{P}$. Therefore, by Lemma 2.4, $\lambda(A \cup (\cup\mathcal{Q})) = k$ for each $\mathcal{Q} \subseteq \mathcal{P}$. By symmetry, $\lambda(B \cup (\cup\mathcal{Q})) = k$ for each $\mathcal{Q} \subseteq \mathcal{P}$. Now, by submodularity, $\lambda(\cup\mathcal{Q}) + \lambda(A \cup B \cup (\cup\mathcal{Q})) \leq \lambda(A \cup (\cup\mathcal{Q})) + \lambda(B \cup (\cup\mathcal{Q})) = 2k$ for each $\mathcal{Q} \subseteq \mathcal{P}$. Therefore $\lambda(\cup\mathcal{Q}) \leq 2k$ and $\lambda(A \cup B \cup (\cup\mathcal{Q})) \leq 2k$. Thus $\text{ord}_K(\mathcal{P} \cup \{A, B\}) \leq 2k$, as required. □

Consider a set $\mathcal{Q} \subseteq \mathcal{C}$. Let $X = \cup\mathcal{Q}$ and let $Y = E - X$. Note that either $|X \cap A| \leq k$ or $|Y \cap A| \leq k$. By symmetry we may assume that $|X \cap A| \leq k$. Similarly, either $|X \cap B| \leq k$ or $|Y \cap B| \leq k$. Consider the case that $|X \cap B| \leq k$. Then, by submodularity and statement 4.3.1, $\lambda(X) \leq 2k\Delta_K(T) + \lambda(X - (A \cup B)) \leq 2k\Delta_K(T) + 2k$. Finally, consider the case that $|Y \cap B| \leq k$. By submodularity and statement 4.3.1, $\lambda(X) \leq 2k\Delta_K(T) + \lambda((X - A) \cup B) \leq 2k\Delta_K(T) + 2k$. Therefore $\text{ord}_K(\mathcal{A} \cup \mathcal{B} \cup \mathcal{P}) \leq 2(1 + \Delta_K(T))k$, as required. □

We can now put these results together to prove Theorem 4.1. By Theorem 3.1 we may assume that $\lambda(X) \leq \delta_K(\mathcal{T})$ for each $X \in \mathcal{T}$. Then, by possibly applying a homomorphism, we may assume that each part of $\mathcal{T}$ is a singleton. Now Theorem 4.1 is an immediate consequence of Theorems 4.2 and 4.3.

5. Back to matroids.

LEMMA 5.1. *Let (S, A_1, A_2, T) be a partition of the elements of a matroid M such that $\lambda_M(S \cup A_1) + \lambda_M(S \cup A_2) = \lambda_M(S) + \lambda_M(S \cup A_1 \cup A_2)$. Then $\lambda_{M/S\backslash T}(A_1) = 0$.*

Proof. We have

$$
\begin{aligned}
0 &= \lambda_M(S \cup A_1) + \lambda_M(S \cup A_2) - \lambda_M(S) - \lambda_M(S \cup A_1 \cup A_2) \\
&= (r_M(S \cup A_1) + r_M(T \cup A_2) - r_M(E)) \\
&\quad + (r_M(S \cup A_2) + r_M(T \cup A_1) - r_M(E)) \\
&\quad - (r_M(S) + r_M(T \cup A_1 \cup A_2) - r_M(E)) \\
&\quad - (r_M(S \cup A_1 \cup A_2) + r_M(T) - r_M(E)) \\
&= (r_M(S \cup A_1) + r_M(S \cup A_2) - r_M(S) - r_M(S \cup A_1 \cup A_2)) \\
&\quad + (r_M(T \cup A_1) + r_M(T \cup A_2) - r_M(T) - r_M(T \cup A_1 \cup A_2)) \\
&= (r_{M/S}(A_1) + r_{M/S}(A_2) - r_{M/S}(A_1 \cup A_2)) \\
&\quad + (r_{M/T}(A_1) + r_{M/T}(A_2) - r_{M/T}(A_1 \cup A_2)) \\
&= \lambda_{M/S\backslash T}(A_1) + \lambda_{M\backslash S/T}(A_1).
\end{aligned}
$$

Therefore, since the last expression is the sum of two nonnegative values, we get $\lambda_{M/S\backslash T}(A_1) = 0$ and $\lambda_{M\backslash S/T}(A_1) = 0$, as required. □

LEMMA 5.2. *Let $(S, A_1, \ldots, A_l, T)$ be a partition of the elements of a matroid M such that $\kappa_M(S, T) = k$ and, for each $i \in \{1, \ldots, l\}$, $\lambda_M(S \cup A_i) = k$. Then $\lambda_{M/S\backslash T}(A_i) = 0$ for all $i \in \{1, \ldots, l\}$.*

Proof. By Lemma 2.4, $\lambda_M(S \cup (\cup_{i\in X} A_i)) = k$ for all $X \subseteq \{1,\dots,l\}$. Let $A_2' = A_2 \cup \cdots \cup A_l$. Applying Lemma 5.1 to (S, A_1, A_2', T) we see that $\lambda_{M/S\setminus T}(A_1) = 0$. Then, by symmetry, $\lambda_{M/S\setminus T}(A_i) = 0$ for all $i \in \{1,\dots,l\}$. ∎

The following result is an immediate corollary of Lemma 5.2 and Theorem 4.2.

LEMMA 5.3. *Let $M=(E,r)$ be a matroid and let $\mathcal{T}$ be a collection of disjoint subsets of $E(M)$ with $\nu_M(\mathcal{T})=k$. Then there exist disjoint sets $A, B \subseteq E(M)$ such that*

- *each set $T \in \mathcal{T}$ is contained in A, B, or $E-(A\cup B)$;*
- *A and B each contain at most k sets from $\mathcal{T}$;*
- *$\lambda_M(A) \le k$, $\lambda_M(B) \le k$;*
- *if $\mathcal{T}'$ is the collection of sets in $\mathcal{T}$ disjoint from $A\cup B$, then $\nu_{M/A\setminus B}(\mathcal{T}') = 0$.*

We need the following lemma. (Note that the proof is not self-contained; we use Theorem 6.1 from the next section.)

LEMMA 5.4. *Let M be a matroid and let (A,B) be a partition of $E(M)$. Then there exists a matroid M' on $E(M)$ such that dist$(M,M') = \lambda_M(A)$, $\lambda_{M'}(A) = 0$, $M/B = M'/B$, and $M/A = M'/A$.*

Proof. The result is vacuous when $\lambda_M(A) = 0$, so suppose that $\lambda_M(A) > 0$. By Theorem 6.1, there exists a matroid N on ground set $E(M)\cup\{e\}$ such that $M = N\setminus e$, $e \in \mathrm{cl}_N(A)$, $e \in \mathrm{cl}_N(B)$, and e is not a loop of N. Let $M'' = N/e$. Note that e is a loop in both N/A and N/B. Therefore $M''/A = (N/e)/A = (N/A)/e = (N/A)\setminus e = M/A$ and, similarly, $M''/B = M/B$. Also note that $\lambda_{M''}(A) = \lambda_M(A) - 1$ and that $\mathrm{dist}(M, M'') = 1$. The result now follows by an easy inductive argument. ∎

We are now ready to prove our main result, which we restate here for convenience.

THEOREM 5.5. *Let $M = (E,r)$ be a matroid and let $\mathcal{T}$ be a collection of disjoint subsets of $E(M)$. Then there exists a matroid M' on ground set $E(M)$ such that $\nu_{M'}(\mathcal{T}) = 0$ and dist$(M,M') \le 2(\delta_M(\mathcal{T})+1)\nu_M(\mathcal{T})$.*

Proof. Suppose that $\mathcal{T} = \{T_1,\dots,T_l\}$ and let $k = \nu_M(\mathcal{T})$. By Theorem 3.1, there exists a collection $\mathcal{S} = \{S_1,\dots,S_l\}$ of disjoint subsets of $E(M)$ such that $\nu_M(\mathcal{S}) = k$ and, for each $i \in \{1,\dots,l\}$, $T_i \subseteq S_i$ and $\lambda_M(S_i) \le \delta_M(\mathcal{T})$. Then, by Lemma 5.3, there exist disjoint subsets A and B of $E(M)$ such that

- each set $S \in \mathcal{B}$ is contained in A, B, or $E(M) - (A\cup B)$;
- A and B each contain at most k sets from $\mathcal{S}$;
- $\lambda_M(A) \le k$, $\lambda_M(B) \le k$;
- if $\mathcal{S}'$ is the collection of sets in $\mathcal{S}$ disjoint from $A\cup B$, then $\nu_{M/A\setminus B}(\mathcal{S}') = 0$.

By Lemma 5.4 and duality, there exists a matroid M' on ground set $E(M)$ such that $\mathrm{dist}(M,M') \le 2k$, $\lambda'_M(A) = \lambda'_M(B) = 0$, and $M'/A\setminus B = M/A\setminus B$. Note that, for each $S \in \mathcal{S} - \mathcal{S}'$, we have $\lambda_{M'}(S) \le \delta_M(\mathcal{T})$. Therefore, by Lemma 5.4, there exists a matroid M'' such that $\mathrm{dist}(M', M'') \le 2k\delta_M(\mathcal{T})$, and $\nu_{M''}(\mathcal{S}) = 0$. Then, since $T_i \subseteq S_i$ for each $i \in \{1,\dots,l\}$, we have $\nu_{M''}(\mathcal{T}) = 0$, as required. ∎

6. Modular cuts.

In this section we prove the following theorem.

THEOREM 6.1. *Let M be a matroid and let (A,B) be a partition of $E(M)$. If $\lambda_M(A) > 0$, then there exists a matroid M' on ground set $E(M)\cup\{e\}$ such that $M = M'\setminus e$, $e \in \mathrm{cl}_{M'}(A)$, $e \in \mathrm{cl}_{M'}(B)$, and e is not a loop of M'.*

Note that Theorem 6.1 is trivial for representable matroids.

Let $X, Y \subseteq E(M)$. We call (X,Y) a *modular pair* if $r_M(X) + r_M(Y) = r_M(X\cap Y) + r_M(X\cup Y)$. A collection $\mathcal{F}$ of subsets of $E(M)$ is called a *modular cut* of M if it satisfies the following three conditions:

1. If $X \subseteq Y \subseteq E(M)$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.
2. If $X, Y \in \mathcal{F}$ and (X,Y) is a modular pair, then $X\cap Y \in \mathcal{F}$.
3. If $Y \in \mathcal{F}$ and $X \subseteq Y$ with $r_M(X) = r_M(Y)$, then $X \in \mathcal{F}$.

The following theorem is well known; see, for example, Oxley [6, Theorem 7.2.2].

THEOREM 6.2. *Let $\mathcal{F}$ be a modular cut in a matroid M. Then there exists a matroid N on ground set $E(M) \cup \{e\}$ such that $N \setminus e = M$ and, for each $X \subseteq E(M)$, $r_N(X \cup \{e\}) = r_M(X)$ if and only if $X \in \mathcal{F}$.*

LEMMA 6.3. *Let M be a matroid, let (A, B) be a partition of $E(M)$, and let $\mathcal{F}$ be the collection of all sets $X \subseteq E(M)$ such that $\lambda_{M/X}(A - X) = 0$. Then $\mathcal{F}$ is a modular cut of M.*

Proof. Note that $\mathcal{F}$ clearly satisfies the first condition.

6.3.1. *For any $X \subseteq E(M)$, $X \in \mathcal{F}$ if and only if $(A \cup X, B \cup X)$ is a modular pair in M.*

Subproof. Note that $\lambda_{M/X}(A-X) = r_{M/X}(A-X) + r_{M/X}(B-X) - r(M/X) = r_M(A \cup X) + r_M(B \cup X) - r(M) - r_M(X)$. Thus $\lambda_{M/X}(A - X) = 0$ if and only if $(A \cup X, B \cup X)$ is a modular pair. □

Now consider the third condition. Suppose that $Y \in \mathcal{F}$ and $X \subseteq Y$ with $r_M(X) = r_M(Y)$. By the claim, $(A \cup Y, B \cup Y)$ is a modular pair. Moreover, since $X \subseteq Y$ with $r_M(X) = r_M(Y)$, we have $r_M(A \cup Y) = r_M(A \cup X)$, $r_M(B \cup Y) = r_M(B \cup X)$, $r_M((A \cup Y) \cap (B \cup Y)) = r_M((A \cup X) \cap (B \cup X))$, and $r_M((A \cup Y) \cup (B \cup Y)) = r_M((A \cup X) \cup (B \cup X))$. Therefore $(A \cup X, B \cup X)$ is a modular pair and hence, by the claim, $X \in \mathcal{F}$. This verifies the third condition.

Finally consider the second condition. Let $X_1, X_2 \in \mathcal{F}$ such that (X_1, X_2) is a modular pair. By the definition of $\mathcal{F}$, $X_1 \cup X_2 \in \mathcal{F}$. Then, by statement 6.3.1, each of $(A \cup X_1, B \cup X_1)$, $(A \cup X_2, B \cup X_2)$, $(A \cup (X_1 \cup X_2), B \cup (X_1 \cup X_2))$ is a modular pair. Now

$$\begin{aligned}
&r_M(A \cup (X_1 \cap X_2)) + r_M(B \cup (X_1 \cap X_2)) \\
&\qquad = r_M((A \cup X_1) \cap (A \cup X_2)) + r_M((B \cup X_1) \cap (B \cup X_2)) \\
&\qquad \leq (r_M(A \cup X_1) + r_M(A \cup X_2) - r_M(A \cup X_1 \cup X_2)) \\
&\qquad\quad + (r_M(B \cup X_1) + r_M(B \cup X_2) - r_M(B \cup X_1 \cup X_2)) \\
&\qquad = (r_M(A \cup X_1) + r_M(B \cup X_1)) \\
&\qquad\quad + (r_M(A \cup X_2) + r_M(B \cup X_2)) \\
&\qquad\quad - (r_M(A \cup X_1 \cup X_2) + r_M(B \cup X_1 \cup X_2)) \\
&\qquad = (r_M(X_1) + r(M)) + (r_M(X_2) + r(M)) \\
&\qquad\quad - (r_M(X_1 \cup X_2) + r(M)) \\
&\qquad = (r_M(X_1) + r_M(X_2) - r_M(X_1 \cup X_2)) + r(M) \\
&\qquad = r_M(X_1 \cap X_2)) + r(M).
\end{aligned}$$

So $(A \cup (X_1 \cap X_2), B \cup (X_1 \cap X_2))$ is a modular pair. Then, by statement 6.3.1, $X_1 \cap X_2 \in \mathcal{F}$. Hence $\mathcal{F}$ is a modular cut, as required. □

Now Theorem 6.1 is an immediate consequence of Theorem 6.2 and Lemma 6.3.

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