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The number of points in a matroid with no n-point line as a minor ☆

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ABSTRACT

For any positive integer l we prove that if M is a simple matroid with no $(l+2)$-point line as a minor and with sufficiently large rank, then $|E(M)| \leqslant \frac{q^{r(M)}-1}{q-1}$, where q is the largest prime power less than or equal to l. Equality is attained by projective geometries over GF(q).

1. Introduction

Kung [5] proved the following theorem.

Theorem 1.1. *For any integer $l \geqslant 2$, if M is a simple matroid with no $U_{2,l+2}$-minor, then $|E(M)| \leqslant \frac{l^{r(M)}-1}{l-1}$.*

The above bound is tight in the case that l is a prime power and M is a projective geometry. In fact, among matroids of rank at least 4, projective geometries are the only matroids that attain the bound; see [5]. Therefore, the bound is not tight when l is not a prime power. We prove the following bound that was conjectured by Kung [5,4].

Theorem 1.2. *Let $l \geqslant 2$ be a positive integer and let q be the largest prime power less than or equal to l. If M is a simple matroid with no $U_{2,l+2}$-minor and with sufficiently large rank, then $|E(M)| \leqslant \frac{q^{r(M)}-1}{q-1}$.*

The case where $l = 6$ was resolved by Bonin and Kung in [1].

We will also prove that the only matroids of large rank that attain the bound in Theorem 1.2 are the projective geometries over GF(q); see Corollary 4.2.

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A matroid M is *round* if $E(M)$ cannot be partitioned into two sets of rank less than $r(M)$. We prove Theorem 1.2 by reducing it to the following result.

Theorem 1.3. *For each prime power q, there exists a positive integer n such that, if M is a round matroid with a* PG$(n-1,q)$*-minor but no U_{2,q^2+1}-minor, then $\epsilon(M) \leqslant \frac{q^{r(M)}-1}{q-1}$.*

For any integer $l \geqslant 2$, there is an integer k such that $2^{k-1} < l \leqslant 2^k$. Therefore, if q is the largest prime power less than or equal to l, then $l < 2q$. So, to prove Theorem 1.2, it would suffice to prove the weaker version of Theorem 1.3 where U_{2,q^2+1} is replaced by $U_{2,2q+1}$. With this in mind, we find the stronger version somewhat surprising.

We further reduce Theorem 1.3 to the following result.

Theorem 1.4. *For each prime power q there exists an integer n such that, if M is a round matroid that contains a $U_{2,q+2}$-restriction and a* PG$(n-1,q)$*-minor, then M contains a U_{2,q^2+1}-minor.*

The following conjecture, if true, would imply all of the results above.

Conjecture 1.5. *For each prime power q, there exists a positive integer n such that, if M is a round matroid with a* PG$(n-1,q)$*-minor but no U_{2,q^2+1}-minor, then M is* GF(q)*-representable.*

The conjecture may hold with $n = 3$ for all q. Moreover, the conjecture may also hold when "round" is replaced by "vertically 4-connected".

2. Preliminaries

We assume that the reader is familiar with matroid theory; we use the notation and terminology of Oxley [6]. A rank-1 flat in a matroid is referred to as a *point* and a rank-2 flat is a *line*. A line is *long* if it has at least 3 points. The number of points in M is denoted $\epsilon(M)$.

Let M be a matroid and let $A, B \subseteq E(M)$. We define $\sqcap_M(A,B) = r_M(A) + r_M(B) - r_M(A \cup B)$; this is the *local connectivity* between A and B. This definition is motivated by geometry. Suppose that M is a restriction of PG$(n-1,q)$ and let F_A and F_B be the flats of PG$(n-1,q)$ that are spanned by A and B respectively. Then $F_A \cap F_B$ has rank $\sqcap_M(A,B)$. We say that two sets $A, B \subseteq E(M)$ are *skew* if $\sqcap_M(A,B) = 0$.

We let $\mathcal{U}(l)$ denote the class of matroids with no $U_{2,l+2}$-minor. Our proof of Theorem 1.2 relies heavily on the following result of Geelen and Kabell [3, Theorem 2.1].

Theorem 2.1. *There is an integer-valued function $\alpha(l,q,n)$ such that, for any positive integers l, q, n with $l \geqslant q \geqslant 2$, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) \geqslant \alpha(l,q,n)q^{r(M)}$, then M contains a* PG$(n-1,q')$*-minor for some prime-power $q' > q$.*

The following result is an important special case of Theorem 1.4.

Lemma 2.2. *If M is a round matroid that contains a $U_{2,q+2}$-restriction and a* PG$(2,q)$*-restriction, then M has a U_{2,q^2+1}-minor.*

Proof. Suppose that M is a minimum-rank counterexample. Let $L, P \subseteq E(M)$ such that $M|L = U_{2,q+2}$ and $M|P =$ PG$(2,q)$. If M has rank 3, then we may assume that $E(M) = P \cup \{e\}$. Since $M|P$ is modular, e is in at most one long line of M. Then, since $|P| = q^2+q+1$, we have $\epsilon(M/e) \geqslant q^2+1$ and, hence, M has a U_{2,q^2+1}-minor. This contradiction implies that $r(M) > 3$. Since M is round, there is an element e that is spanned by neither L nor P. Now M/e is round and contains both $M|L$ and $M|P$ as restrictions. This contradicts our choice of M. □

The base case of the following lemma is essentially proved in [3, Lemma 2.4].

Lemma 2.3. *Let $\lambda \in \mathbb{R}$. Let k and $l \geqslant q \geqslant 2$ be positive integers, and let A and B be disjoint sets of elements in a matroid $M \in \mathcal{U}(l)$ with $\sqcap_M(A,B) \leqslant k$ and $\epsilon_M(A) > \lambda q^{r_M(A)}$. Then there is a set $A' \subseteq A$ that is skew to B and satisfies $\epsilon_M(A') > \lambda l^{-k} q^{r_M(A')}$.*

Proof. By possibly contracting some elements in $B - \mathrm{cl}_M(A)$, we may assume that A spans B and thus that $r_M(B) = \sqcap_M(A,B)$. When $k=1$, this means B has rank 1. We resolve this base case first.

Let e be a non-loop element of B. We may assume that A is minimal with $\epsilon_M(A) > \lambda q^{r_M(A)}$, and that $E(M) = A \cup \{e\}$. Let W be a flat of M not containing e, such that $r_M(W) = r(M) - 2$. Let $H_0, H_1, \dots, H_m$ be the hyperplanes of M containing W, with $e \in H_0$. The sets $\{H_i - W\colon\ 1 \leqslant i \leqslant m\}$ are a disjoint cover of $E(M) - W$. Additionally, the matroid $\mathrm{si}(M/W)$ is isomorphic to the line $U_{2,m+1}$, so we know that $m \leqslant l$.

By the minimality of A, we get $\epsilon_M(H_0 \cap A) \leqslant \lambda q^{r(M)-1}$, so

$$\epsilon_M(A - H_0) > \lambda(q-1)q^{r(M)-1}.$$

Since the hyperplanes $H_1, \dots, H_m$ cover $E(M) - H_0$, a majority argument gives some $1 \leqslant i \leqslant m$ such that

$$\epsilon_M(H_i \cap A) \geqslant \frac{1}{m}\epsilon_M(A - H_0) > \frac{\lambda}{l}(q-1)q^{r(M)-1}.$$

Setting $A' = A \cap H_i$ gives a set of the required number of points that is skew to e and therefore to B, which is what we want.

Now suppose that the result holds for $k = t$ and consider the case that $k = t+1$. Let A and B be disjoint sets of elements in a matroid M with $\sqcap_M(A,B) \leqslant t+1$ and $\epsilon_M(A) > \lambda q^{r_M(A)}$. As mentioned earlier, we have $r_M(B) = \sqcap_M(A,B) \leqslant t+1$. Let e be any non-loop element of B. By the base case, there exists $A' \subseteq A$ that is skew to $\{e\}$ and satisfies $\epsilon_M(A') > \lambda l^{-1} q^{r_M(A')}$. Since $e \notin \mathrm{cl}_M(A')$ and $r_M(B) \leqslant t+1$, we have $\sqcap_M(A', B) \leqslant t$. Now the result follows routinely by the induction hypothesis. □

The following two results are used in the reduction of Theorem 1.2 to Theorem 1.3.

Lemma 2.4. *Let $f(k)$ be an integer-valued function such that $f(k) \geqslant 2f(k-1) - 1$ for each $k \geqslant 1$ and $f(1) \geqslant 1$. If M is a matroid with $\epsilon(M) \geqslant f(r(M))$ and $r(M) \geqslant 1$, then there is a round restriction N of M such that $\epsilon(N) \geqslant f(r(N))$ and $r(N) \geqslant 1$.*

Proof. We may assume that M is not round and, hence, there is a partition (A,B) of $E(M)$ such that $r_M(A) < r(M)$ and $r_M(B) < r(M)$. Clearly $r_M(A) \geqslant 1$ and $r_M(B) \geqslant 1$. Inductively we may assume that $\epsilon_M(A) < f(r_M(A))$ and $\epsilon_M(B) < f(r_M(B))$. Thus $\epsilon(M) \leqslant \epsilon(M|A) + \epsilon(M|B) \leqslant f(r_M(A)) + f(r_M(B)) - 2 \leqslant 2f(r(M)-1) - 2 < f(r(M))$, which is a contradiction. □

Lemma 2.5. *Let $q \geqslant 4$ and $t \geqslant 1$ be integers and let M be a matroid with $\epsilon(M) \geqslant \frac{q^{r(M)}-1}{q-1}$ and $r(M) \geqslant 3t$. If M is not round, then either M has a U_{2,q^2+2}-minor or there is a round restriction N of M such that $r(N) \geqslant t$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$.*

Proof. Let $s = r(M)$ and let $f(k) = (\frac{q}{2})^{s-k}(\frac{q^k-1}{q-1})$. For any $k \geqslant 1$,

$$\begin{aligned} f(k+1) &= \left(\frac{q}{2}\right)^{s-k-1}\left(\frac{q^{k+1}-1}{q-1}\right) \\ &> \left(\frac{q}{2}\right)^{s-k-1}\left(q\frac{q^k-1}{q-1}\right) \\ &= 2f(k). \end{aligned}$$

Moreover $f(1) \geqslant 1$ and $\epsilon(M) \geqslant f(r(M))$. Then, by Lemma 2.4, there is a round restriction N of M such that $r(N) \geqslant 1$ and $\epsilon(N) \geqslant f(r(N))$. Since M is not round, $r(N) < r(M) = s$ and, hence, $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$. We may assume that $r(N) < t$. Therefore, since $s \geqslant 3t$ and $q \geqslant 4$,

$$\begin{aligned}
\epsilon(N) &\geqslant f\big(r(N)\big) \\
&= \left(\frac{q}{2}\right)^{s-r(N)} \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&\geqslant \left(\frac{q}{2}\right)^{2t} \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&\geqslant q^t \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&> q^{r(N)} \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&\geqslant \left(\frac{q^{r(N)}+1}{q+1}\right) \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&= \left(\frac{q^{2r(N)}-1}{q^2-1}\right).
\end{aligned}$$

Therefore, by Theorem 1.1, M has a U_{2,q^2+2}-minor, as required. □

3. The main results

We start with a proof of Theorem 1.4, which we restate here.

Theorem 3.1. *There is an integer-valued function $n(q)$ such that, for each prime power q, if M is a round matroid that contains a $U_{2,q+2}$-restriction and a $\mathrm{PG}(n(q)-1,q)$-minor, then M has a U_{2,q^2+1}-minor.*

Proof. Recall that the function $\alpha(l,q,n)$ was defined in Theorem 2.1. Let q be a prime power, let $\alpha = \alpha(q^2-1, q-1, 3)$. Let n be an integer that is sufficiently large so that $(\frac{q}{q-1})^n > \alpha q^5 (q-1)^2$. We define $n(q) = n$. Suppose that the result fails for this choice of $n(q)$ and let M be a minimum-rank counterexample. Thus M is a round matroid having a line L, with at least $q+2$ points, and a minor N isomorphic to $\mathrm{PG}(n-1,q)$, but $M \in \mathcal{U}(q^2-1)$.

Suppose that $N = M/C \setminus D$ where C is independent. If $e \in C - L$, then M/e is round, contains the line L, and has N as minor—contrary to our choice of M. Therefore $C \subseteq L$ and, hence, $r(M) \leqslant r(N) + 2 \leqslant n+2$.

Let $X = E(M) - L$. By our choice of n, we have $\epsilon(M|(X-D)) \geqslant \frac{q^n-1}{q-1} - (q^2+1) = q^3 \frac{q^{n-3}-1}{q-1} + q \geqslant q^{n-1} > q^4 \alpha (q-1)^{n+2} \geqslant q^4 \alpha (q-1)^{r_M(X)}$. By Lemma 2.3, there is a flat $F \subseteq X - D$ of M that is skew to L and satisfies $\epsilon(M|F) \geqslant \alpha (q-1)^{r_M(F)}$. Since F is skew to L, F is also skew to C. Therefore $M|F = N|F$ and hence $M|F$ is $\mathrm{GF}(q)$-representable. Then, by Theorem 2.1, $M|F$ has a $\mathrm{PG}(2,q)$-minor. Therefore there is a set $Y \subseteq F$ such that $(M|F)/Y$ contains a $\mathrm{PG}(2,q)$-restriction. Now M/Y is round, contains a $(q+2)$-point line, and contains a $\mathrm{PG}(2,q)$-restriction. Then, by Lemma 2.2, M has a U_{2,q^2+1}-minor. □

Now we will prove Theorem 1.3 which we reformulate here. The function $n(q)$ was defined in Theorem 3.1.

Theorem 3.2. *For each prime power q, if M is a round matroid with a $\mathrm{PG}(n(q)-1,q)$-minor but no U_{2,q^2+1}-minor, then $\epsilon(M) \leqslant \frac{q^{r(M)}-1}{q-1}$.*

Proof. Let M be a minimum-rank counterexample. By Lemma 2.2, $r(M) > n(q)$. Let $e \in E(M)$ be a non-loop element such that M/e has a $\mathrm{PG}(n-1,q)$-minor. Note that M/e is round. Then, by the minimality of M, $\epsilon(M/e) \leqslant \frac{q^{r(M)-1}-1}{q-1}$. By Theorem 3.1, each line of M containing e has at most $q+1$ points. Hence $\epsilon(M) \leqslant 1 + q\epsilon(M/e) \leqslant 1 + q(\frac{q^{r(M)-1}-1}{q-1}) = \frac{q^{r(M)}-1}{q-1}$. This contradiction completes the proof. □

We can now prove our main result, Theorem 1.2, which we restate below.

Theorem 3.3. *Let $l \geqslant 2$ be a positive integer and let q be the largest prime power less than or equal to l. If M is a matroid with no $U_{2,l+2}$-minor and with sufficiently large rank, then $\epsilon(M) \leqslant \frac{q^{r(M)}-1}{q-1}$.*

Proof of Theorem 1.2. When l is a prime-power, the result follows from Theorem 1.1. Therefore we may assume that $l \geqslant 6$ and, hence, $q \geqslant 5$. Recall that $n(q)$ is defined in Theorem 3.1 and $\alpha(l, q-1, n)$ is defined in Theorem 2.1. Let $n = n(q)$ and let k be an integer that is sufficiently large so that $(\frac{q}{q-1})^k \geqslant q\alpha(l, q-1, n)$. Thus, for any $k' \geqslant k$, we get $\frac{q^{k'}-1}{q-1} \geqslant q^{k'-1} \geqslant \alpha(l, q-1, n)(q-1)^{k'}$. Let $M \in \mathcal{U}(l)$ be a matroid of rank at least $3k$ such that $\epsilon(M) > \frac{q^{r(M)}-1}{q-1}$. By Lemma 2.5, M has a round restriction N such that we have $r(N) \geqslant k$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1} \geqslant \alpha(l, q-1, n)(q-1)^{r(N)}$. By Theorem 2.1, N has a $\mathrm{PG}(n(q)-1, q')$-minor for some $q' > q-1$. If $q' > q$, then $q'+1 \geqslant l+2$, so this projective geometry has a $U_{2,l+2}$-minor, contradicting our hypothesis. We may therefore conclude that $q' = q$, so N has a $\mathrm{PG}(n(q)-1, q)$-minor. Now we get a contradiction by Theorem 3.2. □

4. Extremal matroids

In this section, we prove that the extremal matroids of large rank for Theorem 1.2 are projective geometries. We need the following result to recognize projective geometries; see Oxley [6, Theorem 6.1.1].

Lemma 4.1. *Let M be a simple matroid of rank $n \geqslant 4$ such that every line of M contains at least three points and each pair of disjoint lines of M is skew. Then M is isomorphic to $\mathrm{PG}(n-1, q)$ for some prime power q.*

We can now prove our extremal characterization.

Corollary 4.2. *Let $l \geqslant 2$ be a positive integer and let q be the largest prime power less than or equal to l. If M is a simple matroid with no $U_{2,l+2}$-minor, with $\epsilon(M) = \frac{q^{r(M)}-1}{q-1}$, and with sufficiently large rank, then M is a projective geometry over $\mathrm{GF}(q)$.*

Proof. Kung [5] proved the result for the case that l is a prime-power. Therefore we may assume that $l \geqslant 6$ and, hence, $q \geqslant 5$. By Theorem 1.2, there is an integer k_1 such that, if M is a matroid with no $U_{2,l+2}$-minor and with $r(M) \geqslant k_1$, then $\epsilon(M) \leqslant \frac{q^{r(M)}-1}{q-1}$. Recall that $n(q)$ is defined in Theorem 3.1 and $\alpha(l, q, n)$ is defined in Theorem 2.1. Let k_2 be large enough so that $(\frac{q}{q-1})^{k_2} \geqslant q\alpha(l, q-1, n(q)+2)$, and $k = \max(k_1, k_2)$.

Let $M \in \mathcal{U}(l)$ be a simple matroid of rank at least $3k$ such that $\epsilon(M) = \frac{q^{r(M)}-1}{q-1}$. If M is not round, then, by Lemma 2.5, M has a round restriction N such that $r(N) \geqslant k$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$, contrary to Theorem 1.2. Hence M is round.

From the definition of k_2, we get $\epsilon(M) \geqslant \alpha(l, q-1, n(q)+2)(q-1)^{r(M)}$, so by Theorem 2.1, M has a $\mathrm{PG}(n(q)+1, q)$-minor. Therefore, by Theorem 3.1, each line in M has at most $q+1$ points. Consider any element $e \in E(M)$. By Theorem 1.2, $\epsilon(M/e) \leqslant \frac{q^{r(M)-1}-1}{q-1}$. Then

$$\begin{aligned}\epsilon(M) &\leqslant 1 + q\epsilon(M/e)\\ &\leqslant 1 + q\left(\frac{q^{r(M)-1}-1}{q-1}\right)\\ &= \frac{q^{r(M)}-1}{q-1}\\ &= \epsilon(M).\end{aligned}$$

The inequalities above must hold with equality. Therefore each line in M has exactly $q+1$ points.

If M is not a projective geometry, then, by Lemma 4.1, there are two disjoint lines L_1 and L_2 in M such that $\sqcap_M(L_1, L_2) = 1$. Let $e \in L_1$. Then L_2 spans a line with at least $q+2$ points in M/e. Since M has a PG$(n(q)+1, q)$-minor, M/e contains a PG$(n(q)-1, q)$-minor; see [2, Lemma 5.2]. This contradicts Theorem 3.1. □

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