This document contains notes that accompanied a series of four lectures on jump systems. These lectures were presented at the Center of Parallel Computing to an audience consisting mainly of graduate students.

Lectures on jump systems

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Abstract

A jump system is a nonempty set of integral vectors that satisfy a certain exchange axiom. This notion was introduced by Bouchet and Cunningham, and popularized by recent results of Lovász. A *degree system* of a graph G is the set of degree sequences of all subgraphs of G. Degree systems are the primary example of jump systems. Other examples come from matroids and from two generalizations of matroids (polymatroids and delta–matroids). Discussion of these special cases will be kept to a minimum, and will only be used to motivate certain results.

The main result is a min–max formula of Lovász for the distance of an integral point from a jump system. This formula generalizes two of the more important min–max theorems in combinatorial optimization; namely, Tutte's f–factor–theorem, and Edmonds' matroid intersection theorem. Other points of interest are the existence of a greedy algorithm for optimizing linear functions, and a characterization of the convex hulls of jump systems. Even apart from the possibility of obtaining very general theorems, jump systems are appealing due to their simple definition and elegant structure.

1 Introduction

In this section we define jump systems, and summarize relevant information; for more detailed expositions see Bouchet and Cunningham [3] (who introduced jump systems), and Lovász [12]. We begin with the basic definitions.

Throughout, we let $V := \{1, \ldots, n\}$. For $x, y \in \mathbf{Z}^V$ define

$$[x, y] := \{x' \in \mathbf{Z}^V : \min(x_i, y_i) \leq x'_i \leq \max(x_i, y_i), \forall i\},$$

and $d(x,y) := \sum(|x_i - y_i| : i \in V)$. We call $[x,y]$ a *box*. (More generally, a *box* is a cross product of intervals in $\mathbf{Z}$, where the intervals are possibly infinite.) A point $x' \in \mathbf{Z}^V$ is a *step from x to y* (or an *(x,y)–step*) if $x' \in [x,y]$ and $d(x,x') = 1$. Let J be a nonempty subset of $\mathbf{Z}^V$, a point $x \in J$ is called a *feasible point*. J is a *jump system* if it satisfies the following axiom:

two–step axiom Given feasible points x, y and a step x' from x to y, then either x' is feasible, or there exists a feasible step x'' from x' to y.

1.1 Preliminaries

Throughout J will denote a bounded jump system, unless otherwise specified. We will assume that the reader is familiar with the following operations on jump systems:

Translation Given $a \in \mathbf{Z}^V$, we add a to each feasible point. Clearly this also gives a jump system.

Reflection Given $i \in V$, the *reflection of J in coordinate i* is the set obtained by negating x_i in each point x. This is again a jump system.

Projection Given $X \subseteq V$, the *projection of J onto X* is the set obtained by replacing each feasible point with its restriction to X. This is again a jump system.

Intersection with a box Given a box B such that $J \cap B$ is not empty, then $J \cap B$ is a jump system. This is a special case of a result of Lovász that, for any box B, the closest points in J to B form a jump system. (See Exercise 1.)

Sum Given jump systems J_1, J_2 on V, define $J_1 + J_2 = \{x_1 + x_2 : x_1 \in J_1, x_2 \in J_2\}$. We will see, in Theorem 1.1, that $J_1 + J_2$ is again a jump system.

Product Given jump systems J_1, J_2 on V_1, V_2 respectively, where V_1 and V_2 are disjoint sets. We define $J_1 \times J_2 = \{(x_1, x_2) : x_1 \in J_1, x_2 \in J_2\}$. It is straightforward to see that $J_1 \times J_2$ is a jump system on $V_1 \cup V_2$. Note that boxes are just the product of intervals.

We now introduce our main example.

Degree–sequences Let $G = (V,E)$ be a graph. For a subgraph H of G, we define the *degree–sequence of H*, to be the vector $\deg_H \in \mathbf{Z}^V$ such that, for $v \in V$, $\deg_H(v)$ is the degree of vertex v in H. Alternatively, $\deg_H = \sum(e_i + e_j : ij \text{ an edge of } H)$, where $e_i \in \mathbf{Z}^V$ is the vector having one in position i and zeros elsewhere. We define the *degree–system of* G to be $J_G := \sum(\{\mathbf{0}, e_i + e_j\} : ij \in E)$. The degree–system of G, being the sum of elementary jump systems, is also a jump system. Note that $f \in J_G$ if and only if f is the degree sequence of a subgraph of G.

Determining whether or not a vector $f \in \mathbf{Z}^V$ belongs to J_G is a nontrivial and much–studied problem, called the f–factor problem. (In the case that $b = \mathbf{1}$, the problem is to determine whether G admits a perfect matching.) The f–factor problem demonstrates that testing membership in a jump system cannot be assumed to be a straightforward operation, even when the jump system is the sum of trivial jump systems. Much of the following discussion is devoted to the problem of testing membership.

The following are important classes of jump systems.

Delta–matroids A *delta–matroid* is a collection of subsets of V whose characteristic vectors define a jump–system. For convenience, we associate a delta–matroid with the set of characteristic vectors of its sets. Thus, delta–matroids are exactly jump systems contained in $\{0,1\}^V$. Delta–matroids, which pre–date jump systems, were introduced by Bouchet [1,2]. Similar structures were defined by a number of authors, see Dunstan and Welsh [6], Dress and Havel [5], and Chandrasekaran and Kabadi [4]. Recently Kabadi [10] observed that all bounded jump systems can be constructed from delta–matroids using a "homomorphism" operation. This construction can be used to generalize many results on delta–matroids to jump systems. We favour direct proofs of such results, since we do not assume familiarity with delta–matroids.

Constant sum A jump system J is *constant sum* if there exists α sush that $\sum x_i = \alpha$ for all $x \in J$.

Constant parity A jump system J is *odd* (respectively *even*) if $\sum x_i$ is odd (resp. even). Collectively, we refer to odd and even jump systems as *constant parity* jump systems.

Matroids The set of characteristic vectors of bases of a matroid is a jump system. For convenience, we identify a matroid by the set of characteristic vectors of its bases. Thus matroids are exactly constant–parity delta–matroids (those readers who are not familiar with matroids may take this as a definition).

1.2 Key results

The following theorem is due to Bouchet and Cunningham [3]. The proof is essentially taken from the same paper, where it is attributed to Sebő.

Theorem 1.1 *Let J_1, J_2 be jump systems on V. Then $J_1 + J_2$ is a jump system.*

Proof Let $x, y \in J_1 + J_2$, and let x' be an (x, y)–step. (We are required to prove that either $x' \in J_1 + J_2$ or there exists an (x', y)–step $x'' \in J_1 + J_2$.) Note that $y = y_1 + y_2$ for some $y_1 \in J_1$ and $y_2 \in J_2$. Now choose $z_1 \in J_1$ and $z_2 \in J_2$ minimizing $d(y_1, z_1) + d(y_2, z_2)$ subject to $d(x', z_1 + z_2) = 1$. (Note that such z_1, z_2 exist since $d(x', x) = 1$.)

Since $d(x', z_1+z_2) = 1$, then either $x' \in [z_1+z_2, y]$ or $z_1+z_2 \in [x', y]$. In the latter case, we have that z_1+z_2 is a feasible (x', y)–step, so we are done. We assume otherwise, thus $x' \in [z_1+z_2, y]$. Let $s = x' - (z_1+z_2)$. Since $z_1+z_2+s \in [z_1+z_2, y_1+y_2]$, then either $z_1+s \in [z_1, y_1]$ or $z_2+s \in [z_2, y_2]$. By symmetry, we assume that $z_1 + s \in [z_1, y_1]$. If $z_1 + s \in J_1$, then, since $x' = (z_1 + s) + z_2$, x' is feasible, so we are done. We assume otherwise, then, by the two–step axiom, there exists a (z_1+s, y_1)–step $z_1' \in J_1$. But then, $d(x', z_1'+z_2) = 1$ and $d(z_1', y_1)+d(z_2, y_2) < d(z_1, y_1)+d(z_2, y_2)$. Hence the pair z_1', z_2 is a contradiction to our choice of z_1, z_2. □

For a box B, we define $J_B := \{x \in J : d(x, B) = d(J, B)\}$. (Here, as usual, $d(J, B) := \min(d(x, y) : x \in J, y \in B)$.) The following exercise is a result of Lovász [12]. (This exercise is not elementary.)

Exercise 1 *For a box B, prove that J_B is a jump system.*

The following characterization of jump systems, due to Lovász [12], is the key to many subsequent results.

Theorem 1.2 *For $J \subseteq \mathbf{Z}^V$, the following are equivalent.*

(1) J is a jump system.

(2) Given boxes $B^1 \subseteq \ldots \subseteq B^r$, $J_{B^1} \cap \ldots \cap J_{B^r} \neq \emptyset$.

Proof Suppose J satisfies (2), and we have feasible points x, y and an (x, y)–step x'. By (2), there exists $x'' \in J_{\{x'\}} \cap J_{[x',y]}$. The box $[x', y]$ contains a feasible point (y), so $x'' \in [x', y]$. Furthermore $d(x', x) = 1$, so $d(x', x'') \leq 1$. Therefore, J satisfies the two–step axiom, as required.

We prove the converse by induction on r. The result is trivial with $r = 1$. We assume that $r > 1$ and that the result holds for lesser cases. Now, let $y \in J_{B^r}$ and choose $x \in J_{B^1} \cap \ldots \cap J_{B^{r-1}}$

minimizing $d(x,y)$. We assume that $x \notin J_{B^r}$, since otherwise we are done. Since $d(y,B^r) < d(x,B^r)$, there exists an (x,y)–step x' such that $d(x',B^r) < d(x,B^r)$. Since the boxes are nested $d(x',B^i) < d(x,B^i)$ for each i. Thus x' is not feasible, so, by the two–step axiom, there exists an (x',y)–step $x'' \in J$. Since $d(x',B^i) < d(x,B^i)$ and $d(x',x'') = 1$, we have $d(x'',B^i) \le d(x,B^i)$. However, for $i = 1,\ldots,r-1$, $x \in J_{B^i}$. Hence, $x'' \in J_{B^1} \cap \ldots \cap J_{B^{r-1}}$. Furthermore, $d(x'',y) < d(x,y)$, which contradicts our choice of x. □

1.3 Greedy Algorithm

Considering the problem of maximizing a linear weight function over a bounded jump system. More precisely, given $w \in \mathbf{R}^V$, we wish to find a feasible point x that maximizes $w^T x$. By reflection, we may assume that $w \ge 0$. We also assume that the coordinates are sorted so that $w_1 \ge \ldots \ge w_k > w_{k+1} = \ldots = w_n = 0$. Figure 1 defines a greedy algorithm for this problem. We will see that J^k is a set of optimal points; we refer to J^k as a *greedy face* of J.

Begin

$J^0 \leftarrow J$

for $i \leftarrow 1,\ldots,k$

$\alpha_i \leftarrow \max(x_i : x \in J^{i-1})$

$J^i \leftarrow \{x \in J^{i-1} : x_i = \alpha_i\}$

End.

Figure 1: Greedy Algorithm

Theorem 1.3 *Each point $x \in J^k$ maximizes $w^T x$ over J.*

We require the following notation and lemma. For $j = 1,\ldots,k$ we define $c^j = e_1 + \ldots + e_j$.

Lemma 1.4 *Each point $x \in J^k$ simultaneously maximizes $(c^j)^T x$ over J.*

Proof Since J is bounded, there exists $u \in \mathbf{Z}^V$ such that $u \ge y$ for all $y \in J$. For $j = 0,\ldots,k$ we define

$$B^j := [u_1,\infty) \times \ldots \times [u_j,\infty) \times (-\infty,\infty)^{n-j}.$$

Therefore, for each $x \in J$, we have

$$d(x,B^j) = \sum(u_i - x_i : i = 1,\ldots,j) = (c^j)^T u - (c^j)^T x.$$

Hence x maximizes $(c^j)^T x$ over J if and only if $x \in J_{B^j}$.

We now prove by induction that $J^k = J_{B^1} \cap \ldots \cap J_{B^k}$. Since $B^1 \supset \ldots \supset B^k$, $J_{B^1} \cap \ldots \cap J_{B^k}$ is not empty. Inductively we suppose that $J^{j-1} = J_{B^1} \cap \ldots \cap J_{B^{j-1}}$. Consider some $x \in J_{B^1} \cap \ldots \cap J_{B^j}$. In particular $x \in J^{j-1}$, so

$$x_j = (c^j)^T x - \sum(\alpha_i : i = 1, \ldots, j-1).$$

Since $x \in J_{B^j}$, x maximizes $(c^j)^T x$ over J, and hence x maximizes x_j over J^{j-1}. Therefore, $J^j = J_{B^1} \cap \ldots \cap J_{B^j}$. So inductively we see that $J^k = J_{B^1} \cap \ldots \cap J_{B^k}$. Hence each $x \in J^k$ simultaneously maximizes $(c^j)^T x$ over J for $j = 1, \ldots, k$, as required. □

Proof of theorem. Define $w'_k = w_k$, and, for $i = 1, \ldots, k-1$, let $w'_i = w_i - w_{i+1}$. Note that $w = \sum(w'_j c^j : j = 1, \ldots, k)$. By the lemma, each $x \in J^k$ simultaneously maximizes $(c^j)^T x$ over J for $j = 1, \ldots, k$. Then, since w is a positive linear combination of these c^j, $w^T x$ is also maximized by such x. □

The convex hull of J has many nice properties. Bouchet and Cunningham [3] proved that the convex hull is a "bisubmodular polyhedron". These bisubmodular polyhedra were studied by Fujishige [9], who presents results analogous to those of Lovász [12].

We conclude this section by considering some weaker results. As usual, we denote by $\text{conv}(J)$ the convex hull of J.

Theorem 1.5 *The convex hull of J is described by inequalities of the form $w^T x \leq \omega$ where $w \in \{0, \pm 1\}^V$.*

Proof Let $w^T x \leq \omega$ be a valid inequality for $\text{conv}(J)$. By reflection, we suppose that $w \geq 0$. We also assume that the coordinates are sorted so that

$$w_1 \geq \ldots \geq w_k > w_{k+1} = \ldots = w_n = 0.$$

We now apply the greedy algorithm to maximize $w^T x$ over J.

By Lemma 1.4, each $x \in J^k$ simultaneously maximizes $(c^j)^T x$ over J for $j = 1, \ldots, k$. Then, since w is a positive linear combination of these c^j, the inequality $w^T x \leq \omega$ is implied by the inequalities $(c^j)^T x \leq \max((c^j)^T x : x \in J)$. □

Our last result states that the feasible points on any nonempty face of a jump system is again a jump system. It suffices to prove this for facets, and, by the previous theorem, the facet defining inequalities have the form $w^T x \leq \omega$ where $w \in \{0, \pm 1\}^V$. In the proof of Lemma 1.4, we saw that the feasible points on such a facet are the closest points to some box. Hence, by Exercise 1, these points define a jump system. Thus we have the following theorem.

Theorem 1.6 *Let the inequality* $w^T x \leq \omega$ *define a nonempty face of* $conv(J)$*. Then* $\{x \in J : w^T x = \omega\}$ *is a jump system.* □

2 The membership problem

Given $x \in \mathbf{Z}^V$, we are interested in the the problem of determining whether $x \in J$. (By translation we usually assume that $x = 0$.) Obviously the membership problem would be trivial if we were given J explicitly as a set. How then are we given a jump system? We do not have a satisfactory answer to this question. However, one might imagine that our jump system is given to us as the sum of "easy" jump systems, as is the case in the following examples.

f–factor problem Let J_G be the degree–system of graph $G = (V, E)$, and let $f \in \mathbf{Z}^V$. The membership problem for f is exactly the f–factor problem; that is, determine whether G has a spanning subgraph whose degree–sequence is f. The main result in these notes is a generalization of Tutte's f–factor theorem [14]. Recall that $J_G = \sum(\{\mathbf{0}, e_i + e_j\} : ij \in E)$. Thus, the membership problem can be quite complicated, even when the jump system is the sum of elementary jump systems.

Matroid intersection problem Let J_1, J_2 be jump systems on V. We refer to the problem of deciding whether J_1 and J_2 as the *intersection problem*. The intersection problem can be posed as a membership problem, since $J_1 \cap J_2 \neq \emptyset$ if and only if $\mathbf{0} \in J_1 - J_2$. The intersection problem is "well–solved" for matroids; see Edmonds' [7] and [8]. The main result in this section implies the matroid intersection polyhedron theorem of Edmonds [8].

Matroid parity problem The last two examples describe membership problems for which there exist efficient algorithms; now we shall see that the general situation is not so nice. Let $G = (V, E)$ be the graph where $E := \{(1, 2), (3, 4), \ldots, (n - 1, n)\}$, and let J be a matroid on the set V. The intersection problem for J and J_G is called the matroid parity problem. That is, we want a vector $x \in J$ suct that, for each $ij \in E$, $x_i = x_j$. Lovász [11] showed that there is no "efficient" algorithm to solve this problem. Of course, it depends upon how the matroid J is given to us. The result assumes that J is given as an efficiently computable function $d(J, B)$ where B is any box.

2.1 Closest points to a box

It turns out to be useful to consider a more general problem than deciding membership, namely: *Given a box B, determine $d(J,B)$.* While studying the distance from J to a box seems more general than considering a single point, the distinction is artificial since $d(\mathbf{0}, B-J) = d(J,B)$. Let $B = [a,b]$ be a box, where $a \le b$. The following sets seem fundamental in the study of this more general problem.

$$\begin{aligned} V_B^+(J) &= \{i \in V : \exists x \in J_B \text{ such that } x_i > b_i\} \\ V_B^-(J) &= \{i \in V : \exists x \in J_B \text{ such that } x_i < a_i\}. \end{aligned}$$

Usually, we denote $V_B^+(J)$ (respectively $V_B^-(J)$) by V_B^+ (resp. V_B^-).

We now define a box $\bar{B} = [\bar{a}, \bar{b}]$, where

$$\bar{a}_i = \begin{cases} a_i, & i \in V_B^- \\ -\infty, & \text{otherwise} \end{cases} \qquad \bar{b}_i = \begin{cases} b_i, & i \in V_B^+ \\ \infty, & \text{otherwise.} \end{cases}$$

Note that, for $i \in V_B^+$, $d(J,[a,b+e_i]) < d(J,B)$. Similarly, for $i \in V_B^-$, $d(J,[a-e_i,b]) < d(J,B)$. So, if B' is a box containing B such that $d(B',J) = d(B,J)$, then $B' \subseteq \bar{B}$. We prove that $d(\bar{B},J) = d(B,J)$. Therefore $\bar{B}$ is the unique maximal box containing B such that $d(J,B) = d(J,\bar{B})$; we call $\bar{B}$ the *closure* of B.

Lemma 2.1 *$d(\bar{B},J) = d(B,J)$.*

Proof Since $\bar{B} \supseteq B$, there exists $x \in J_B \cap J_{\bar{B}}$. By reflection we may assume that $x \ge a$. For $i \in V_B^+$, $\bar{b}_i = b_i$, so $d(x_i,[\bar{a}_i,\bar{b}_i]) = d(x_i,[a_i,b_i])$. For $i \in V \setminus V_b^+$, $x_i \le b_i$, so $d(x_i,[\bar{a}_i,\bar{b}_i]) = d(x_i,[a_i,b_i])$. Therefore $d(x,[\bar{a},\bar{b}]) = d(x,[a,b])$. □

From this lemma, we deduce the following lemmas.

Lemma 2.2 *Let $B = [a,b]$ be a box, where $a \le b$, and let $i \in V_B^+ \setminus V_B^-$. Then, for each $x \in J_B$, $x_i \ge b_i$.*

Proof Since $J_B \subseteq J_{\bar{B}}$, we may assume that $B = \bar{B}$, so $a_i = -\infty$. Let $B' = [a, b-e_i]$. Now, suppose there exists $x \in J_B$ such that $x_i < b_i$. Then, $d(x,B') = d(x,B)$, so $d(J,B') = d(J,B)$. Furthermore, $J_{B'} = \{x \in J_B : x_i < b_i\}$. Thus, $V_{B'}^- \subseteq V_B^-$ and $V_{B'}^+ \subseteq V_B^+ \setminus \{i\}$. Therefore, $\bar{B}'$ strictly contains $\bar{B}$. This is a contradiction since $\bar{B}$ is the largest box containing B such that $d(\bar{B},J) = d(B,J)$. □

Lemma 2.3 *Let $B = [a,b]$ be a box, and let $i \in V_B^+ \cap V_B^-$. Then, $a_i = b_i$.*

Proof Suppose that $a_i < b_i$, and let $B' = [a, b - e_i]$. Since $i \in V_B^-$, there exists $x \in J_B$ such that $x_i < a_i$. Then, $d(x, B') = d(x, B)$, so $d(J, B') = d(J, B)$. Furthermore, $J_{B'} = \{x \in J_B : x_i < b_i\}$. Thus, $V_{B'}^- \subseteq V_B^-$ and $V_{B'}^+ \subseteq V_B^+ \setminus \{i\}$. Therefore, $\bar{B}'$ strictly contains $\bar{B}$, which yields a contradiction. □

2.2 A min–max theorem

We now derive a min–max theorem for the case that V_B^+ and V_B^- are disjoint. In this special case $d(J, B)$ is the distance from conv(J) to conv(B). While this says little about the interesting problem of determining what happens inside the convex hull, it does generalize Edmonds' matroid intersection theorem.

For a box B, we define $w^B \in \mathbf{Z}^V$ by

$$w_i^B = \begin{cases} -1, & i \in V_B^+ \setminus V_B^- \\ 1, & i \in V_B^- \setminus V_B^+ \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.4 *Let B be a box, and $w \in \{0, \pm 1\}^V$. Then,*

$$d(J, B) \geq \min(w^T x : x \in B) - \max(w^T y : y \in J). \tag{1}$$

Furthermore, if $V_B^+ \cap V_B^- = \emptyset$ and $w = w^B$, then equality is attained in (1).

Proof Proving (1) is straightforward, we have

$$\begin{aligned} d(J, B) &= \min(d(x, y) : x \in B, y \in J) \\ &\geq \min(w^T(x - y) : x \in B, y \in J) \\ &= \min(w^T x : x \in B) - \max(w^T y : y \in J). \end{aligned}$$

Now suppose $V_B^+ \cap V_B^- = \emptyset$. By reflection, we may assume that $V_B^- = \emptyset$, and, by translation and closure, we may assume that $B = (-\infty, 0)^{V_B^+} \times (-\infty, \infty)^{V \setminus V_B^+}$. For any $x \in J$, we define $x' \in \mathbf{Z}^V$ such that

$$x_i' = \begin{cases} x_i, & x_i < 0 \text{ and } i \in V_B^+ \\ 0, & \text{otherwise.} \end{cases}$$

Since $B + x' \subseteq B$, there exists $y \in J_B \cap J_{B+x'}$. By Lemma 2.2, $y_i \geq 0$ for all $i \in V_B^+$. Thus

$$\begin{aligned} d(J, B + x') &= d(y, B + x') \\ &= d(y, B) - \sum(x_i' : i \in V) \\ &= d(J, B) - \sum(x_i : i \in V_B^+, x_i < 0). \end{aligned} \tag{2}$$

Furthermore, $d(x, B + x') = \sum(x_i : i \in V_B^+, x_i \geq 0)$. However, $d(x, B + x') \geq d(J, B + x')$, so, by (2), $d(J, B) \leq \sum(x_i : i \in V_B^+) = -(w^B)^T x$. Therefore, since $\min((w^B)^T x : x \in B) = 0$, $d(J, B) \leq \min((w^B)^T x : x \in B) - \max((w^B)^T y : y \in J)$, which proves that (1) holds with equality. □

We get the following easy corollary.

Corollary 2.5 *Let $B = [a, b]$ be a box with $a < b$. Then $J \cap B$ is empty if and only if conv$(J) \cap B$ is empty.*

Proof If J and B are not disjoint, then B is not disjoint from conv(J). Suppose that J and B are disjoint. By Lemma 2.3, $V_B^+ \cap V_B^- = \emptyset$. Then, by Theorem 2.4, there exists $w \in \mathbf{Z}^V$ such that $\max(w^T y : y \in J) < \min(w^T x : x \in B)$. Thus there is a hyperplane separating conv(J) from B.□

Another case when the intersection problem can be solved by separation, is when J is *convex*; that is, $J = \text{conv}(J) \cap \mathbf{Z}^V$. The following result shows that constant sum jump systems are convex; this was originally conjectured by Tamir.

Corollary 2.6 *All constant sum jump systems are convex.*

Proof Let J be a constant sum jump system, and consider $x \in \text{conv}(J)$. By translation, we assume that $x = 0$. Let $B = [\mathbf{0}, \mathbf{1}]^V$. Then $B \cap \text{conv}(J)$ is not empty. Therefore, by Corollary 2.5, there exists a common element y of B and J. Clearly $y = 0$. □

While the sum of convex jump systems is not necessarily convex (consider degree–systems), the sum of constant parity jump systems is a constant parity jump system, and hence convex. This observation yields the following remarkable corollary, which for matroids is Edmonds' matroid intersection polyhedron theorem [8]. The more familiar matroid intersection theorem of Edmonds can also be obtained in this framework, see [12].

Corollary 2.7 *Let J_1, J_2 be constant sum jump systems. Then,*

$$\textit{conv}(J_1 \cap J_2) = \textit{conv}(J_1) \cap \textit{conv}(J_2).$$

Proof Since J_1 and J_2 are convex, it suffices to prove that $\text{conv}(J_1) \cap \text{conv}(J_2)$ is integral (that is, each nonempty face F of $\text{conv}(J_1) \cap \text{conv}(J_2)$ contains an integral point.) Let F be a nonempty face of $\text{conv}(J_1) \cap \text{conv}(J_2)$. Then $F = F_1 \cap F_2$, where F_i is a face of conv(J_i), $i = 1, 2$. Furthermore, by Theorem 1.6, the points of J_i that lie on F_i define a jump system. Thus we may assume that

$J_i \subseteq F_i$. All that remains to be proved is that $\text{conv}(J_1) \cap \text{conv}(J_2)$ is either empty or contains an integral point.

We recall that

$$J_1 \cap J_2 \neq \emptyset \iff \mathbf{0} \in J_1 - J_2.$$

However $J_1 - J_2$ is constant sum, and hence convex. Thus,

$$\mathbf{0} \in J_1 - J_2 \iff \mathbf{0} \in \text{conv}(J_1 - J_2) = \text{conv}(J_1) - \text{conv}(J_2).$$

Finally, we have

$$\mathbf{0} \in \text{conv}(J_1) - \text{conv}(J_2) \iff \text{conv}(J_1) \cap \text{conv}(J_2) \neq \emptyset.$$

Therefore, if $\text{conv}(J_1) \cap \text{conv}(J_2) \neq \emptyset$, then $J_1 \cap J_2$ is nonempty, and hence $\text{conv}(J_1) \cap \text{conv}(J_2)$ contains an integral point. □

3 Holes in jump systems

For $x \in \mathbf{Z}^V$, if x is not in the convex hull of J, then the membership problem becomes a matter of separating x from $\text{conv}(J)$; which is somewhat routine. The more interesting case is when $x \in \text{conv}(J)$. A *hole* of J is an integral point in $\text{conv}(J) \setminus J$. In this section we consider the structure of feasible points around holes. By translation we usually shift the hole to the origin, and for convenience we denote by V^+ and V^- the sets $V^+_{\mathbf{0}}$ and $V^-_{\mathbf{0}}$. The following result shows that the closest points to the origin exhibit reflective symmetry in the coordinates $V^+ \cap V^-$.

Theorem 3.1 (Lovász [12]) *If $j \in V^+ \cap V^-$, and $x \in J_{\mathbf{0}}$, then $x_j \in \{0, \pm 1\}$, and there exists $y \in J_{\mathbf{0}}$ such that*

$$y_i = \begin{cases} -x_j, & i = j \\ x_i, & \text{otherwise.} \end{cases}$$

Proof By reflection, we may assume that $x \geq \mathbf{0}$. Furthermore, we suppose that $x_j > 0$, since otherwise the result is immediate. Define $B^1 = [-e_j, \mathbf{0}]$ and $B^2 = [-e_j, x]$. Thus $\{\mathbf{0}\} \subset B^1 \subset B^2$, and hence there exists $y \in J_{\mathbf{0}} \cap J_{B^1} \cap J_{B^2}$. Since $j \in V^-_B$ we have $d(J, B^1) < d(J, \mathbf{0})$. Therefore $d(y, B^1) < d(y, \mathbf{0})$, and hence $y_j < 0$. Now, since $x \in B_2$, we have $y \in B_2$. So $y_j = -1$, and, for $i \neq j$, $0 \leq y_i \leq x_i$. However, $d(x, \mathbf{0}) = d(y, \mathbf{0})$, so $x_j = 1$ and, for $i \neq j$, $x_i = y_i$. □

Lemma 3.2 (Sebő [13]) *If $\mathbf{0} \in$ conv(J), then there exists $x \in J_{\mathbf{0}}$ such that $x_i = 0$ for all $i \notin V^+ \cap V^-$.*

Proof By reflection, we may assume that $V^- \subseteq V^+$. Let

$$B = (-\infty, 0]^{V^+ \setminus V^-} \times (-\infty, \infty)^{V \setminus (V^+ \setminus V^-)}.$$

Now $\mathbf{0} \in B \cap \text{conv}(J)$, and $V_B^+ \cap V_B^- = \emptyset$, then, by Theorem 2.4, $B \cap J \neq \emptyset$.

Since $\mathbf{0} \in B$, there exists $x \in J_{\mathbf{0}} \cap J_B$. By the definitions of V^+ and V^-, $x_i = 0$ for i in $V \setminus V^+$. Furthermore, by Lemma 2.2, $x_i \geq 0$ for $i \in V^+ \setminus V^-$. However, since $x \in B$, we have $x_i \leq 0$ for $i \in V^+ \setminus V^-$. Thus $x_i = 0$ for all $i \notin V_B^-$. □

If $x \in \text{conv}(J)$, then, by definition, x is the convex combination of some feasible points. The following result of Sebő [13] states that holes can be expressed as the convex combination of only two feasible points.

Theorem 3.3 *If $x \in conv(J)$ be an integral point, then there exists $y^1, y^2 \in J_{\{x\}}$ such that* $x = \frac{1}{2}(y^1 + y^2)$.

Proof By translating, we may assume that $x = \mathbf{0}$. By Lemma 3.2, there exists $y^1 \in J_{\mathbf{0}}$ such that $y_i^1 = 0$ for all $i \notin V^+ \cap V^-$. Then, by Theorem 3.1, $-y^1 \in J_{\mathbf{0}}$. Thus we choose $y^2 = -y^1$. □

As an immediate corollary we note that $x \in \text{conv}(J)$ if and only if $x \in \text{conv}(J_{\{x\}})$.

A polyhedron P is called *half–integral* if for every nonempty face F of P there exists $x \in F$ such that $2x$ is integral. The following result is due to Cunningham (unpublished). The original proof was quite involved; this easy proof is due to Sebő [13].

Corollary 3.4 *Let J_1, J_2 be jump systems on V. Then $conv(J_1) \cap conv(J_2)$ is half–integral.*

Proof Let F be a nonempty face of $\text{conv}(J_1) \cap \text{conv}(J_2)$. Then $F = F_1 \cap F_2$, where F_i is a face of $\text{conv}(J_i)$, $i = 1, 2$. Furthermore, by Theorem 1.6, the points of J_i that lie on F_i define a jump system. Thus we may assume that $J_i \subseteq F_i$. All that remains to be proved is that, if $\text{conv}(J_1) \cap \text{conv}(J_2)$ is nonempty, then $\text{conv}(J_1) \cap \text{conv}(J_2)$ contains a half–integral point.

$$\begin{aligned} \text{conv}(J_1) \cap \text{conv}(J_2) \neq \phi \quad &\rightarrow \quad \mathbf{0} \in \text{conv}(J_1) - \text{conv}(J_2) = \text{conv}(J_1 - J_2) \\ &\rightarrow \quad \exists x^1, y^1 \in J_1 \, x^2, y^2 \in J_2 \text{ such that } \mathbf{0} = \frac{1}{2}((x^1 - x^2) + (y^1 - y^2)) \end{aligned}$$

Hence $\frac{1}{2}(x^1 + y^1) = \frac{1}{2}(x^2 + y^2) \in \text{conv}(J_1) \cap \text{conv}(J_2)$. □

3.1 Intermission

The following two exercises characterize jump systems in two–dimensions.

Exercise 2 *Let $J \subseteq \mathbf{Z}^2$ be a jump system. Prove the following properties.*

i. conv(J) is defined by inequalities of the form $w^T x \leq \omega$ where $w \in \{0, \pm 1\}^2$. (This is proved in Theorem 1.5, though we rather a direct proof.)

ii. No hole of J lies on a face of the form $w^T x \leq \omega$ where $w \in \{\pm 1\}^2$. (This is implied by Theorem 1.6 and Corollary 2.6, though we rather a direct proof.)

iii. If x, y are holes of J such that $d(x, y) = 1$, then no feasible point of J is on the line spanned by x, y.

Exercise 3 *Let $J \subseteq \mathbf{Z}^2$ satisfy the properties (i), (ii), and (iii) above. Prove that J is a jump system.*

3.2 Critical jump systems

A jump system J is called *critical* if $V^+ = V^- = V$. These critical jump systems have a particularly nice structure. If J is critical, and $x \in J_{\mathbf{0}}$, then, by Lemma 3.1, $x \in \{0, \pm 1\}^V$, and every $(0, \pm 1)$–vector having the same support as x is also in $J_{\mathbf{0}}$. (By the support of a vector x, we refer to the set $\{i \in V : x_i \neq 0\}$.) That is, $J_{\mathbf{0}}$ exhibits reflective–symmetry in every coordinate hyperplane. While critical jump systems seem very special, we will see that they lie at the heart of the membership problem.

Let $M = J_{\mathbf{0}} \cap [0, 1]^V$. We call M the *local matroid* of J. (Note that M is constant sum, so it is indeed a matroid.) The following exercise shows that any loopless matroid is the local matroid of some critical jump system.

Exercise 4 *Let $M \subset \{0, 1\}^V$ be a matroid, and define $J = \{y \in \{0, \pm 1\}^V : (|y_1|, \ldots, |y_n|) \in M\}$. Prove that J is a jump system.*

Not only does the local matroid completely define the set of closest feasible points to the origin, but it also says much about the rest of the jump system. For $S \subseteq V$, we let $B^S = [0, 1]^S \times \{0\}^{V \setminus S}$.

Lemma 3.5 *For $S \subseteq V$, let $B = (-\infty, \infty)^S \times \{0\}^{V \setminus S}$. Then, $d(B^S, M) = d(B, J)$.*

Proof Since $\mathbf{0} \in B^S \subseteq B$, there exists $x \in J_{\mathbf{0}} \cap J_{B^S} \cap J_B$. Let $y = (|x_1|, \ldots, |x_n|)$. By Lemma 3.1, $y \in J$. Thus, by construction, $y \in J_{\mathbf{0}} \cap J_{B^S} \cap J_B$. Since $y \in J_{\mathbf{0}}$ and $y \geq \mathbf{0}$, $y \in [0, 1]^V$. Therefore $d(y, B) = d(y, B^S)$, and so $d(B, J) = d(B^S, J)$. Furthermore, $y \in M$, so $d(B, J) = d(B^S, M)$. □

For the following lemma we introduce some basic matroid terminology. Let $S \subseteq V$. We define $r(S) = d(\mathbf{0}, M) - d(B^S, M)$; $r(S)$ is the *rank* of S. Note that, by the last lemma, we may replace M by J in the the definition of $r(S)$. We call S a *flat* of M if $r(S') > r(S)$ for all $S' \supset S$. Equivalently, by the previous lemma, S is a flat if and only if $(-\infty, \infty)^S \times \{\mathbf{0}\}^{V \setminus S}$ is a closed box with respect to J. For us, this latter definition will be more convenient.

Lemma 3.6 *Let M be the local matroid of a critical jump system J, let S be a flat of M, and let J' be the projection of J onto $V \setminus S$. Then J' is critical.*

Proof Let $C = (-\infty, \infty)^S \times \{\mathbf{0}\}^{V \setminus S}$. Note that $J'_{\mathbf{0}}$ is the projection of J_C onto $V \setminus S$. However, since S is a flat of M, C is a closed box, and hence J' is critical. □

For those with some knowledge of matroids it is clear that the rank–one flats define a partition. We include a proof, since this partition into rank–one flats plays an important role in the f–factor problem.

Lemma 3.7 *Let M be the local matroid of a critical jump system J. Then the rank–one flats of M partition J.*

Proof Let $i \in V$. Since J is critical, $d([\mathbf{0}, e_i], J) = d(\mathbf{0}, J) - 1$. Let B be the closure of $[\mathbf{0}, e_i]$ with respect to J. By Lemma 3.5, $B = (-\infty, \infty)^S \times \{\mathbf{0}\}^{V \setminus S}$ for some $S \subseteq V$. Thus, i is in a rank–one flat of M. Furthermore, since a box has a unique closure, i is in exactly one rank–one flat. Hence the rank–one flats define a partition. □

4 Reduction to critical jump systems

Theorem 2.4 gave a weak lower bound on the distance from a jump system to a box. In this section we strengthen this to a min–max formula. Unfortunately this min–max formula is not a "good" characterization, but it does provide a reduction to the case of critical jump systems. For certain applications, like f–factors of graphs, these critical cases have a very simple structure, and can be solved by simple parity arguments. The main result is the following.

Theorem 4.1 *Let $B = [a, b]$ be a box, $w \in \{0, \pm 1\}^V$, and $S \subseteq V$ such that $w_i = 0$ for $i \in S$. Let F be a greedy face optimizing $w^T x$ over J, and let F_S, B_S be the projections of F, B onto S. Then,*

$$d(J, B) \geq d(F_S, B_S) + \min(w^T x : x \in B) - \max(w^T x : x \in J). \tag{3}$$

Furthermore, if $w = w^B$ and $S = V_B^+ \cap V_B^-$, then equality is attained in (3) and $F_S - B_S$ is critical.

It is straightforward to see that $d(J,B) = d(J-B,\mathbf{0})$, $V_B^+(J) = V_{\mathbf{0}}^+(J-B)$, and $V_B^-(J) = V_B^-(J-B)$. Thus the previous theorem can be easily obtained by applying the following lemma to the jump system $J - B$.

Lemma 4.2 *Let $w \in \{0, \pm 1\}^V$, and $S \subseteq V$ such that $w_i = 0$ for $i \in S$. Let F be a greedy face optimizing $w^T x$ over J, and let F_S be the projection of F onto S. Then,*

$$d(J,\mathbf{0}) \geq d(F_S,\mathbf{0}) - \max(w^T x : x \in J). \tag{4}$$

Furthermore, if $w = w^{\mathbf{0}}$ and $S = V_{\mathbf{0}}^+ \cap V_{\mathbf{0}}^-$, then equality is attained in (3) and F_S is critical.

Proof By reflection, and by permuting coordinates, we may assume that $w = \sum(e_i : i = 1, \ldots, p)$. We now apply the greedy algorithm to maximize $w^T x$ over J. We let $J^0 = J$, and then, for $i = 1, \ldots, p$, we let $\alpha_i = \max(x_i : x \in J^{i-1})$, and $J^i = \{x \in J^{i-1} : x_i = \alpha_i\}$. Note that $F = J^p$, and each $x \in J^p$ maximizes $w^T x$ over J; thus $\max(w^T x : x \in J) = \sum(\alpha_i : i = 1, \ldots, p)$. Now let $\alpha_i^+ = \max\{\alpha_i, 0\}$, $X = V \setminus S \setminus \{1, \ldots, p\}$, and

$$A = [\alpha_1^+, \infty) \times \ldots \times [\alpha_p^+, \infty) \times \{\mathbf{0}\}^{S \cup X},$$

$$B = [\alpha_1, \infty) \times \ldots \times [\alpha_p, \infty) \times \{\mathbf{0}\}^S \times \mathbf{Z}^X,$$

$$C = [\alpha_1, \infty) \times \ldots \times [\alpha_p, \infty) \times \mathbf{Z}^{S \cup X}.$$

For each $y \in J$ we have

$$\begin{aligned} d(y,A) &\leq \sum(|y_i - \alpha +_i| : i = 1, \ldots, p) + \sum(|y_i| : i \in S \cup X) \\ &\leq \sum(|y_i| : i \in V) + \sum(\alpha_i^+ : i = 1, \ldots, p) \\ &= d(y,\mathbf{0}) + \sum(\alpha_i^+ : i = 1, \ldots, p). \end{aligned} \tag{5}$$

In particular, considering $y \in J_{\mathbf{0}}$, we have

$$d(J,\mathbf{0}) = d(y,\mathbf{0}) \geq d(y,A) - \sum(\alpha_i^+ : i = 1, \ldots, p) \geq d(J,A) - \sum(\alpha_i^+ : i = 1, \ldots, p).$$

Since $A \subseteq B \subseteq C$, there exists $x \in J$ that is simultaneously optimal with respect to A, B and C. Note that $F = J_C = J \cap C$, thus $x_i = \alpha_i$ for $i = 1, \ldots, p$. Thus, we have,

$$\begin{aligned} d(J,0) &\geq d(J,A) - \sum(\alpha_i^+ : i = 1, \ldots, p) \\ &= d(x,A) - \sum(\alpha_i^+ : i = 1, \ldots, p) \\ &= d(x,B) - \sum(\alpha_i : i = 1, \ldots, p) \\ &= d(F,B) - \max(w^T x : x \in J). \end{aligned} \tag{6}$$

However, since $x \in F \cap J_B$, we have $d(J, B) = d(F, B) = d(F_S, \mathbf{0})$. Which proves (4).

Now consider the case that $w = w^B$, and $S = V_{\mathbf{0}}^+ \cap V_{\mathbf{0}}^-$. By reflection and sorting coordinates we have assumed that $w = \sum(e_i : i = 1, \ldots, p)$. Then, by the definition of w^B, we have $V_{\mathbf{0}}^+ \subseteq V_{\mathbf{0}}^-$, and $V^- \setminus V^+ = \{1, \ldots, p\}$. Hence $\bar{\mathbf{0}} = [0, \infty)^{V^- \setminus V^+} \times \{\mathbf{0}\}^S \times \mathbf{Z}^X$ is the closure of $\{\mathbf{0}\}$. If $y \in J_{\bar{\mathbf{0}}}$, then, by Lemma 2.2, $y_i \le 0$ for $i = 1, \ldots, p$. Therefore, (5) holds with equality for each $y \in J_{\bar{\mathbf{0}}}$. Note that $A \subseteq \bar{\mathbf{0}}$. Considering some $y \in J$ that is simultaneously optimal with respect to A and $\bar{\mathbf{0}}$ we get

$$d(J, \mathbf{0}) = d(y, \bar{\mathbf{0}}) = d(y, A) - \sum(\alpha_i^+ : i = 1, \ldots, p) = d(J, A) - \sum(\alpha_i^+ : i = 1, \ldots, p).$$

Hence (6) holds with equality. However, since there exists $x \in F \cap J_B$, we have $d(J, B) = d(F, B) = d(F_S, \mathbf{0})$. Which proves that we have equality in (4).

Finally we shall prove that F_S is critical. For $i \in S$, $i \in V_{\mathbf{0}}^+$, so $d(J, \mathbf{0}) > d(J, [\mathbf{0}, e_i]) = d(J - [\mathbf{0}, e_i], \mathbf{0})$. We now apply (4) to bound $d(J - [\mathbf{0}, e_i], \mathbf{0})$, which yields

$$\begin{aligned} d(J - [\mathbf{0}, e_i], \mathbf{0}) &\ge d((F - [\mathbf{0}, e_i])_S, \mathbf{0}) - \max(w^T x : x \in J) \\ &= d(F_S - [\mathbf{0}, e_i], \mathbf{0}) - \max(w^T x : x \in J) \\ &= d(F_S, [\mathbf{0}, e_i]) - \max(w^T x : x \in J). \end{aligned}$$

Thus

$$d(F_S, [\mathbf{0}, e_i]) \le d(J, [\mathbf{0}, e_i]) + \max(w^T x : x \in J) < d(J, \mathbf{0}) + \max(w^T x : x \in J).$$

However, since (4) is satisfied with equality by J, we get $d(F_S, [\mathbf{0}, e_i]) < d(F_S, \mathbf{0})$. Hence $i \in V_{\mathbf{0}}^+(F_S)$. Similarly $i \in V_{\mathbf{0}}^-(F_S)$, so F_S is indeed critical. □

4.1 The f–factor problem

Given a graph $G = (V, E)$ and $f \in \mathbf{Z}^V$, we want to determine $d(J_G, f)$. In particular, we want to know if there is a subgraph of G whose degree sequence is f. As we have mentioned previously, this is the f–factor problem for graphs. As an application of Theorem 4.1 we will derive Tutte's f–factor theorem.

In order to state the theorem concisely, we need some notation. For subsets X, Y of V we define

$$E(X, Y) = \{ij \in E : i \in X, j \in Y\}.$$

We denote by $E(X)$ the set $E(X, X)$, and we denote by $G(X)$ the graph with vertices X and edges $E(X)$. For any vector $y \in \mathbf{Z}^V$, we define $y(X) = \sum(y_i : i \in X)$. A graph G is said to be odd with respect to f if $f(V)$ is odd. We denote by $\mathrm{odd}(G, f)$ the number of connected components of G which are odd with respect to f.

Theorem 4.3 (Tutte [14]) *Let G be a graph and $f \in \mathbf{Z}^V$. For disjoint subsets A, B of V, we let $f' = f - \sum(e_i : ij \in E, i \in V \setminus (A \cup B), j \in B)$. Then*

$$d(J_G, f) \geq f(B) - f(A) - (|E(B, V \setminus (A \cup B))| + 2|E(B)|) + odd(G(V \setminus (A \cup B)), f'). \qquad (7)$$

Furthermore, there exist A, B that attain equality.

First let us see why the inequality (7) is valid. Consider the graph $G(V \setminus A)$. The nodes in B demand a total degree of $f(B)$. The edges in $G(V \setminus A)$ can meet at most $|E(B, V \setminus (A \cup B))| + 2|E(B)|$ of the demand at B. Now consider a connected component $G(X)$ of $G(V \setminus (A \cup B))$ such that $f'(X)$ is odd. We have $f'(X) = f(X) - |E(X, B)|$. This means that if we use all of the edges in $E(X, B)$ in order to satisfy the demand at B, then the remaining demand on X is odd, and hence cannot be satisfied using only edges in $E(X)$. Thus we demand at least $f(B) - (|E(B, V \setminus (A \cup B))| + 2|E(B)|) + \mathrm{odd}(G(V \setminus (A \cup B)), f')$ edges from the cut $E(A, V \setminus A)$. However A can only meet $f(A)$ units of this demand. Inequality (7) is exactly the shortfall.

To prove that equality is attained by the appropriate choice of A, B, we require some more lemmas.

Lemma 4.4 *Let $J = J' + [\mathbf{0}, e_i]$ be a jump system. Then J is not critical.*

Proof It is straightforward to check that $V^+_{\mathbf{0}}(J) = V^+_{[-e_i, \mathbf{0}]}(J')$ and $V^-_{\mathbf{0}}(J) = V^-_{[-e_i, \mathbf{0}]}(J')$. Furthermore, by Lemma 2.3, $i \notin V^+_{[-e_i, \mathbf{0}]}(J') \cap V^-_{[-e_i, \mathbf{0}]}(J')$. Hence J is not critical. □

Lemma 4.5 *Suppose $G = (V, E)$ is a graph and $f \in \mathbf{Z}^V$ such that $J_G - f$ is critical. Then, $d(J_G, f) = odd(G, f)$.*

Proof Let $J = J_G - f$, and let $X_1, \ldots, X_k$ be the rank–one flats of the local matroid of J. Let J^1 be the projection of J onto $V \setminus X_1$, and let f' be the restriction of f to $V \setminus X_1$. By Lemma 3.6, J^1 is critical. Note that

$$J^1 = \sum(\{\mathbf{0}, e_i + e_j\} : ij \in E(V \setminus X_1)) + \sum(\{\mathbf{0}, e_i\} : ij \in E(X_1, V \setminus X_1), j \in X_1) - f'.$$

However, J^1 is critical, so, by Lemma 4.4, the cut $E(X_1, V \setminus X_1)$ is empty. Hence the graph $G(X_1)$ is the union of some components of G.

Since X_1 is rank–one, there exists $x \in J_G$ such that $|x(X_1) - f(X_1)| = 1$. Since no edges leave the set X_1, $x(X_1)$ is even, so f is odd on exactly one odd connected component of $G(X_1)$. Hence $\text{odd}(G, f) = k$. Furthermore, for each optimal $x' \in J_G$, we must have $|x'(X_1) - f(X_1)| = 1$. Hence $d(J_G, f) = k$, as required. □

We shall now apply Theorem 4.1, to determine $d(J_G, f)$. Let $S = V_f^+ \cap V_f^-$, $w = w^f$, $A = V_f^+ \setminus V_f^-$, and $B = V_f^- \setminus V_f^+$. We begin by recalling the expansion of J_G,

$$J_G = \sum(\{\mathbf{0}, e_i + e_j\} : ij \in E).$$

Now we construct a greedy face F of J_G that maximizes $w^T x$; we order the vertices such that the elements of A precede the elements of B. Note that the greedy face can be determined by considering each edge independently.

$$F = \sum(\{\mathbf{0}, e_i + e_j\} : ij \in E(V \setminus (A \cup B))) + \sum(e_i + e_j : ij \in E(B, V \setminus A)).$$

The projection of F onto S is

$$\begin{aligned} F_S &= \sum(\{\mathbf{0}, e_i + e_j\} : ij \in E(S)) + \sum(\{\mathbf{0}, e_i\} : ij \in E(S, V \setminus (A \cup B \cup S)), i \in S) + \\ &\quad \sum(e_i : ij \in E(S, B), i \in S). \end{aligned}$$

By Theorem 4.1, $F_S - f$ is critical, so, by Lemma 4.4, the cut $E(S, V \setminus (S \cup A \cup B))$ is empty. Let $h = \sum(e_i : ij \in E(S, B), i \in S)$. Therefore

$$F_S = J_{G(S)} + h.$$

Note that $d(F_S, f) = d(J_{G(S)}, f - h)$, and furthermore $J_{G(S)} - (f - h)$ is critical. Hence, by Lemma 4.5, $d(J_{G(S)}, f - h) = \text{odd}(G(S), f - h)$. Since the cut $E(S, V \setminus (S \cup A \cup B))$ is empty, we have

$$d(F_S, f) = \text{odd}(G(S), f - h) \leq \text{odd}(G(V \setminus (A \cup B)), f').$$

Furthermore, we have

$$w^T f = f(B) - f(A), \text{ and}$$

$$\max(w^T x : x \in J_G) = |E(B, V \setminus (A \cup B))| + 2|E(B)|.$$

Thus, by Theorem 4.1,

$$\begin{aligned} d(J_G, f) &= d(F_S, f) + w^T f - \max(w^T x : x \in J) \\ &\leq \text{odd}(G(V \setminus (A \cup B)), f') + f(B) - f(A) - (|E(B, V \setminus (A \cup B))| + 2|E(B)|). \end{aligned}$$

This proves that equality is attained in (7).

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