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MATROID MATCHING VIA MIXED SKEW-SYMMETRIC MATRICES

JAMES GEELEN, SATORU IWATA

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Tutte associates a V by V skew-symmetric matrix T, having indeterminate entries, with a graph $G=(V,E)$. This matrix, called the *Tutte matrix*, has rank exactly twice the size of a maximum cardinality matching of G. Thus, to find the size of a maximum matching it suffices to compute the rank of T. We consider the more general problem of computing the rank of $T+K$ where K is a real V by V skew-symmetric matrix. This modest generalization of the matching problem contains the linear matroid matching problem and, more generally, the linear delta-matroid parity problem. We present a tight upper bound on the rank of $T+K$ by decomposing $T+K$ into a sum of matrices whose ranks are easy to compute.

1. Introduction

Let $G=(V,E)$ be a simple graph, and let $(z_e : e \in E)$ be algebraically independent commuting indeterminates. We define a V by V skew-symmetric matrix $T=(t_{ij})$, called the *Tutte matrix* of G, such that $t_{ij}=\pm z_e$ if $ij=e\in E$, and $t_{ij}=0$ otherwise. Tutte observed that T is nonsingular (that is, its determinant is not identically zero) if and only if G admits a perfect matching. In fact, the rank of T is equal to the size of a maximum cardinality matchable set in G. (A subset X of V is called *matchable* if $G[X]$, the subgraph induced by X, admits a perfect matching.) By applying elementary linear algebra to

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the Tutte matrix, Tutte proved his famous matching theorem [17]. Similar techniques prove the following extension of Tutte's theorem.

1.1 (Tutte–Berge Formula). *If G is a graph, then*

$$2\nu(G) = \min(|V| - (\mathrm{odd}(G \setminus S) - |S|) \mid S \subseteq V).$$

Here, $\nu(G)$ is the size of a maximum cardinality matching of G, $G\setminus S$ donotes the graph obtained from G by deleting the vertices in S, and $\mathrm{odd}(G)$ denotes the number of connected components of G that have an odd number of vertices.

We call a matrix of the form $T+K$, where K is a real V by V skew-symmetric matrix, a *mixed skew-symmetric* matrix. In the present paper we consider the problem of determining the rank of a mixed skew-symmetric matrix. This is a generalization of the linear matroid matching problem; see Lovász [9–11]. In fact, this problem is equivalent to the linear delta-matroid parity problem [6]. While there is an efficient algorithm and a min-max formula for the linear delta-matroid parity problem, the min-max formula does not lend itself well to applications. We present a new min-max formula for the rank of $T+K$, and consider several applications. (While we have explicity defined K to be over the reals, the results, and their proofs, hold for any field. However, our results only provide a good characterization when there is an efficient algorithm for computing the rank of a matrix over the field.)

Suppose that $A, A_1, \ldots, A_k$ are matrices such that $A = A_1 + \cdots + A_k$. Then $\mathrm{rank}\ A \leq \mathrm{rank}\ A_1 + \cdots + \mathrm{rank}\ A_k$. This provides a convenient way to bound the rank of a matrix. We refer to $A_1, \ldots, A_k$ as a *rank-splitting decomposition* of A if $\mathrm{rank}\, A = \mathrm{rank}\, A_1 + \cdots + \mathrm{rank}\, A_k$. Every matrix admits a rank-splitting decomposition into rank-one matrices. That is, if A has rank r, then we can express A as the sum of r rank-one matrices. This is a useful fact, but it is often desirable to maintain particular matrix properties in the decomposition. For example, a skew-symmetric matrix admits a rank-splitting decomposition into skew-symmetric matrices of rank two. (Skew-symmetric matrices have even rank.)

The main theorem of this paper concerns rank-splitting decompositions of mixed skew-symmetric matrices. We require some definitions. For $X, Y \subseteq V$, $K[X,Y]$ is the submatrix of K induced by rows X and columns Y, and $K[X]$ denotes the principal submatrix $K[X,X]$. We say that X *supports* K if all nonzero entries of K are in the submatrix $K[X]$. We call X a *cover* of K if each term in $K[V-X]$ is zero. If X is a cover of K, then we get two upper bounds on the rank of K. First, $\mathrm{rank}\, K \leq 2|X|$, since deleting a row or column of a matrix reduces the rank by at most 1. Second, $\mathrm{rank}\, K \leq$

$|X|+\text{rank}\,K[V,V-X]=|X|+\text{rank}\,K[X,V-X]$. If $|X|+\text{rank}\,K[X,V-X]$ is odd, then this strengthens to $\text{rank}\,K\le|X|+\text{rank}\,K[X,V-X]-1$.

1.2 (Decomposition Theorem). *Let $T+K$ be a V by V mixed skew-symmetric matrix. Then there exists a rank-splitting decomposition $K_0,T_1+K_1,\ldots,T_k+K_k,T_\infty+K_\infty$ of $T+K$ and disjoint subsets $(X_1,\ldots,X_k,X_\infty)$ of V such that*

i. K_0 is a real V by V skew-symmetric matrix and $T_1+K_1,\ldots,T_k+K_k,T_\infty+K_\infty$ are V by V mixed skew-symmetric matrices,
ii. for $i=1,\ldots,k$, X_i supports T_i, X_i is a cover of K_i, $|X_i|+\text{rank}\,K_i[X_i,V-X_i]$ is odd, and $\text{rank}\,T_i+K_i=|X_i|+\text{rank}\,K_i[X_i,V-X_i]-1$.
iii. X_∞ is a cover of $T_\infty+K_\infty$, and $\text{rank}\,T_\infty+K_\infty=2|X_\infty|$.

The Decomposition Theorem provides a good characterization for the rank of a mixed skew-symmetric matrix. Indeed, by this theorem, we see that

$$\text{rank}\ T+K=\text{rank}\ K_0+2|X_\infty|+\sum_{i=1}^{k}\left(|X_i|+\text{rank}\ K_i[X_i,V-X_i]-1\right).$$

By the preceding discussion, the right hand side of this expression is a clear upper bound on $\text{rank}\,T+K$. Moreover, each of the matrices in this bound have only real entries, so the ranks are straightforward to compute. (Actually, the theorem says nothing about the size of the entries in $K_0,\ldots,K_k,K_\infty$, but these entries are all obtained via standard matrix operations on K – row and column elimination, and multiplication and inversion of submatrices – so their sizes do not grow significantly.) Hence the theorem provides a good upper bound on the rank of $T+K$. To obtain a good lower bound, we take an evaluation of T with suitably chosen real numbers in a way that does not decrease the rank of $T+K$. (In fact, there exists such an evaluation where each indeterminate is replaced with ±1.)

Our proof of Theorem 1.2 is constructive. That is, given a mixed skew-symmetric matrix we can efficiently determine the rank-splitting decomposition, provided that we have an oracle for determining the rank of a mixed skew-symmetric matrix. Fortunately, as mentioned earlier, the rank of a mixed skew-symmetric matrix can be computed efficiently; see [6].

We conclude the introduction by deriving the Tutte–Berge Formula from the Decomposition Theorem. Firstly, for any $S\subseteq V$ it is clear that

$$2\nu(G)\le|V|-(\text{odd}(G\setminus S)-|S|).$$

Hence we need only find a set S that satisfies this inequality with equality. Let T be the Tutte matrix of G, and let $K_0,T_1+K_1,\ldots,T_k+K_k,T_\infty+K_\infty$ be the rank-splitting decomposition of T and $(X_1,\ldots,X_k,X_\infty)$ be the subsets of V promised by the Decomposition Theorem. Now,

$$\begin{aligned}\text{rank } T &= \text{rank } K_0 + \sum(\text{rank } T_i + K_i \ : \ i = 1,\ldots,k,\infty)\\ &\geq \sum(\text{rank } T_i \ : \ i = 1,\ldots,k,\infty)\\ &\geq \text{rank } T.\end{aligned}$$

Therefore, $T_1,\ldots,T_k,T_\infty$ is a rank-splitting decomposition of T, and, for $i=1,\ldots,k,\infty$,

$$\text{rank } T_i = \text{rank } T_i + K_i.$$

By ii, for $i=1,\ldots,k$, we see that X_i supports T_i, and

$$\begin{aligned}\text{rank } T_i &= \text{rank } T_i + K_i\\ &= |X_i| + \text{rank } K_i[X_i, V - X_i] - 1\\ &\geq |X_i| - 1.\end{aligned}$$

Therefore, either $|X_i|$ is even and $G[X_i]$ admits a perfect matching, or $|X_i|$ is odd and $G[X_i]$ has a matching covering all but one vertex. That is, $G[X_i]$ has a matching covering $|X_i|-\text{odd}(G[X_i])$ vertices. Now let $S=X_\infty$, then

$$\begin{aligned}2\nu(G) &= \text{rank } T\\ &= (\text{rank } T_1 + \cdots + \text{rank } T_k) + \text{rank } T_\infty\\ &= (|X_1 \cup \cdots \cup X_k| - \text{odd}(G[X_1 \cup \cdots \cup X_k])) + 2|S|\\ &= (|V - S| - \text{odd}(G \setminus S)) + 2|S|\\ &= |V| - (\text{odd}(G \setminus S) - |S|),\end{aligned}$$

as required.

2. Skew-symmetric matrices

The following result is elementary, and the proof is left to the reader.

2.1. *Let A be a matrix with nonzero entry $A_{i,j}=\alpha$, let u denote the row of A indexed by i and let v denote the column of A indexed by j. Then,* $\text{rank}\,(A-\frac{1}{\alpha}vu)=\text{rank}\,A-1$.

Theorem 2.1 describes a rank-splitting decomposition of A. Indeed, vu is a rank-one matrix, and, hence, $(\frac{1}{\alpha}vu, A-\frac{1}{\alpha}vu)$ is a rank-splitting decomposition of A. Repeatedly applying this theorem, we see that a rank$k$ matrix can be expressed as the sum of k rank-one matrices.

A matrix K whose row and column sets are both indexed by a finite set V is said to be *skew-symmetric* if K is equal to the transpose of $-K$, and all diagonal entries of K are zero. (For fields of characteristic different from two, the condition that K has a zero diagonal is implied by the condition that $K=-K^{\mathrm{t}}$.) The following theorem is an easy application of Theorem 2.1; again the proof is left to the reader.

2.2. *Let K be a skew-symmetric matrix with nonzero entry $K_{ij}=\alpha$, let u denote the row of K indexed by i and let v denote the column of K indexed by j. Then,* $\operatorname{rank}(K-\frac{1}{\alpha}(vu-u^{\mathrm{t}}v^{\mathrm{t}}))=\operatorname{rank}K-2$.

Note that, $vu-u^{\mathrm{t}}v^{\mathrm{t}}$ is a skew-symmetric matrix of rank 2. Thus, $(\frac{1}{\alpha}(vu-u^{\mathrm{t}}v^{\mathrm{t}}), K-\frac{1}{\alpha}(vu-u^{\mathrm{t}}v^{\mathrm{t}}))$ is a rank-splitting decomposition of K into two skew-symmetric matrices. Repeatedly applying this theorem, we see that all skew-symmetric matrices have even rank, and that a skew-symmetric matrix of rank $2k$ can be expressed as the sum of k skew-symmetric matrices of rank 2.

Let K be a V by V skew-symmetric matrix. A matrix K' is said to be *congruent* to K if there exists a nonsingular matrix Q such that $K'=Q^{\mathrm{t}}KQ$. The operation converting K to K' is called a *congruence transformation*. Note that skew-symmetry and rank are invariant under congruence transformations. Also, if K_1,K_2 is a rank-splitting decomposition of $Q^{\mathrm{t}}KQ$, then $(Q^{-1})^{\mathrm{t}}K_1Q^{-1},(Q^{-1})^{\mathrm{t}}K_2Q^{-1}$ is a rank-splitting decomposition of K.

The *support graph* of K is the graph $G(K)$ with vertex set V and edge set $\{(i,j)\mid K_{i,j}\neq 0\}$. Skew-symmetric matrices have the property that their determinants are perfect squares. The square root of the determinant is called the *Pfaffian* of K. The Pfaffian of K can be computed by taking weighted sums over all perfect matchings M of the support graph G:

2.3. $$\operatorname{Pf}K=\sum_{M}\sigma_M\prod_{(i,j)\in M}K_{i,j},$$

where σ_M takes ±1 in a suitable manner, see [7]. In particular, K is singular if G has no perfect matching (as is the case when $|V|$ is odd). Like determinants, Pfaffians can be computed using "row-expansion" [7]: if $V=\{1,\ldots,n\}$, then

2.4. $$\operatorname{Pf}K=\sum_{k=2}^{n}(-1)^kK_{1,k}\operatorname{Pf}K[V-\{1,k\}].$$

The following result is an easy consequence of 2.4.

2.5. *Let K be a real n by n skew-symmetric matrix, and let K' be the matrix obtained from K by replacing $K_{1,2}$ and $K_{2,1}$ with $K_{1,2}+a$ and $K_{2,1}-a$ respectively. Then*

$$\operatorname{Pf}K'=a\operatorname{Pf}K[V-\{1,2\}]+\operatorname{Pf}K.$$

The following result is very useful in finding applications of Theorem 1.2, and is also used in the proof.

2.6 (Murota [14]). *Let $T+K$ be a V by V mixed skew-symmetric matrix. Then, $T+K$ is nonsingular if and only if there exists a partition (X,Y) of V such that $K[X]$ and $T[Y]$ are both nonsingular.*

Note that, this result provides a convenient way to show that the matrix $T+K$ is nonsingular. Indeed, we simply provide a subset X of V such that $K[X]$ is nonsingular and $V-X$ is a matchable set in the graph represented by T. However, 2.6 is not useful for showing that $T+K$ is singular, as that would require looking at every partition of V.

We require some elementary matroid theory. Let $M(K)$ be the column-matroid of K. That is, $M(K)$ is a matroid on ground set V, and a subset X of V is independent in $M(K)$ if the columns of K indexed by X are linearly independent. The following result is elementary.

2.7. *If X is a basis of $M(K)$, then $K[X]$ is nonsingular.*

That is, if X indexes a maximal set of linearly independent columns of a skew-symmetric matrix K, then the principal submatrix $K[X]$ is nonsingular. A *coloop* of a matroid is an element whose deletion decreases the rank of the matroid. Recall that skew-symmetric matrices have even rank. Therefore, if deleting a column reduces the rank by one, then deleting the row and corresponding column must reduce the rank by two. Thus we have proved that:

2.8. *If x is a coloop of $M(K)$, then* $\operatorname{rank} K = \operatorname{rank} K[V-\{x\}]+2$.

As an immediate corollary we see that:

2.9. *If X is a maximal subset of V such that* $\operatorname{rank} K = \operatorname{rank} K[V-X]+2|X|$, *then $M(K[V-X])$ has no coloops.*

The above result can be interpreted as a result on rank-splitting decomposition. If X is a maximal subset of V such that $\operatorname{rank} K = \operatorname{rank} K[V-X]+2|X|$, then we can find a rank-splitting decomposition K_1, K_2 of K such that the nonzero entries of K_1 are in the submatrix $K_1[V-X]$, X is a cover of K_2 and $M[K_1]$ has no coloops. Therefore, in finding a rank-splitting decomposition of $T+K$ we may as well assume that $M(T+K)$ has no coloops.

The following result shows that, if K_1, K_2 is a rank-splitting decomposition of K, and $M(K)$ has no coloops, then neither $M(K_1)$ nor $M(K_2)$ has coloops.

2.10. *If K_1, K_2 is a rank-splitting decomposition of K and x is a coloop of $M(K_1)$, then x is a coloop of $M(K)$.*

Proof. If x is a coloop of $M(K_1)$, then

$$\begin{aligned}\text{rank } K[V, V-\{x\}] &\le \text{rank } K_1[V, V-\{x\}] + \text{rank } K_2[V, V-\{x\}] \\ &\le \text{rank } K_1 + \text{rank } K_2 - 1 \\ &= \text{rank } K - 1.\end{aligned}$$

Hence x is a coloop of $M(K)$. ∎

A pair of elements x, y of $M(K)$ are said to be in *series* if neither x nor y are coloops but $V-\{x,y\}$ has rank less than that of V. It is wellknown that series pairs are transitive.

2.11. *Let M be a matroid on the set V. If $x, y, z \in V$ such that x, y are in series in M and y, z are in series in M, then x, z are in series in M.*

The transitivity of series pairs allows us to partition the elements of a matroid without coloops into sets, such that two elements are in series if and only if they are in the same part of the partition; the sets in this partition are called *series-classes*. Note that, if X is the series-class of $M(K)$ that contains an element x, then $X-\{x\}$ is the set of coloops of $M(K[V-\{x\}])$, since $M(K[V-\{x\}]) = M(K[V, V-\{x\}])$. The following result is an easy consequence of this observation.

2.12. *If $x, y \in V$ are not coloops of $M(K)$, then* $\text{rank } K[V-\{x,y\}] = \text{rank } K - 2$ *if x, y are in series in $M(K)$, and* $\text{rank } K[V-\{x,y\}] = \text{rank } K$ *otherwise.*

From this result we easily obtain the following corollary. (In the case that $K = 0$, this result is tantamout to Gallai's Lemma; see Lovász and Plummer [13, Theorem 3.1.13].)

2.13. *If $M(T+K)$ has no coloops, and x and y are in different series-classes of $M(T+K)$, then $T_{x,y} = 0$.*

That is, if T is the Tutte matrix of a graph G, then each edge (x, y) of G has both of its ends in the same series class of $M(T+K)$. The next result is a little technical, but is also an easy consequence of 2.12.

2.14. *Let T be the Tutte matrix of a graph G, and let K be a real V by V skew-symmetric matrix such that $M(T+K)$ has no coloops. Let x and y be distinct nonadjacent vertices of G that are in the same series-class of $M(T+K)$, let G' be the graph obtained by adding the edge (x, y) to G and let T' be the Tutte matrix of G'. Then* $\text{rank} T' + K = \text{rank} T + K$.

Let T, G, T', and K be as in the previous result, and let $e=(x,y)$ be the edge added to G. It is easy to obtain a rank-splitting decomposition of $T+K$ from a rank-splitting decomposition of $T'+K$. Indeed, suppose that $T_1'+K_1, T_2'+K_2$ is a rank-splitting decomposition of $T'+K$, where $T_1'+K_1$ and $T_2'+K_2$ are mixed skew-symmetric matrices. Now, let T_1 and T_2 be the matrices obtained from T_1' and T_2', respectively, by substituting $z_e=0$. Thus, $T+K=(T_1+K_1)+(T_2+K_2)$. Moreover,

$$\begin{aligned}\text{rank } T'+K &= \text{rank } T+K\\ &\leq \text{rank } T_1+K_1+\text{rank } T_2+K_2\\ &\leq \text{rank } T_1'+K_1+\text{rank } T_2'+K_2\\ &= \text{rank } T'+K.\end{aligned}$$

Hence, T_1+K_1, T_2+K_2 is a rank-splitting decomposition of $T+K$. It is also straightforward to see that, if $M(T+K)$ has no coloops, then $M(T'+K)$ has no coloops. Therefore, to find a rank-splitting decomposition of $T+K$ we may as well assume that, for each series-class X of $T+K$, $G[X]$ is complete.

The following result shows that, if K_1, K_2 is a rank-splitting decomposition of K, then any series class of $M(K)$ is the union of series classes of $M(K_1)$.

2.15. *Let K_1, K_2 be a rank-splitting decomposition of K, and let x, y be in series in $M(K_1)$ such that neither x nor y are coloops in $M(K)$, then x, y are in series in $M(K)$.*

Proof. By 2.10, x is not a coloop of either $M(K_1)$ nor $M(K_2)$. Thus $K_1[V-\{x\}], K_2[V-\{x\}]$ is a rank-splitting decomposition of $K[V-\{x\}]$. Moreover, as x,y is a seriespair of $M(K_1)$, y is a coloop of $M(K_1[V-\{x\}])$. Thus, by 2.10, y is a coloop of $M(K[V-\{x\}])$, and, hence, x, y are in series in $M(K)$. ∎

If $T+K$ is a V by V mixed skew-symmetric matrix, then we call $T+K$ *critical* if

(1) $M(T+K)$ has no coloops, and
(2) for each series-class X of $M(T+K)$, $G[X]$ is complete.

From the discussion above, we can focus on critical matrices. In the next section we will prove the following theorem; below we show that this result implies Theorem 1.2.

2.16. *Let $T+K$ be a V by V mixed skew-symmetric matrix such that $T+K$ is critical, and let $X_1, \ldots, X_k$ be the series-classes of $M(T+K)$ that have at least two elements. Then there exists a rank-splitting decomposition $K_0, T_1+K_1, \ldots, T_k+K_k$ of $T+K$ such that*

i. *K_0 is a real V by V skew-symmetric matrix and $T_1+K_1,\dots,T_k+K_k$ are V by V mixed skew-symmetric matrices, and*
ii. *for $i=1,\dots,k$, X_i supports T_i and X_i is a cover of K_i.*

Let $K_0, T_1+K_1,\dots,T_k+K_k$ be the rank-splitting decomposition of $T+K$ given by 2.16. Now consider some part T_i+K_i of the decomposition. By 2.15 and the fact that $T+K$ is critical, X_i is a series class of $M(T_i+K_i)$. Therefore,

$$\begin{aligned}\text{rank } T_i + K_i &= |X_i| + \text{rank } (T_i + K_i)[V, V - X_i] - 1\\ &= |X_i| + \text{rank } (T_i + K_i)[X_i, V - X_i] - 1\\ &= |X_i| + \text{rank } K_i[X_i, V - X_i] - 1;\end{aligned}$$

where the last two equalities follow from the fact that X_i is a cover of K_i and that X_i supports T_i. Also note that, since $\text{rank}\, T_i+K_i$ is even, $|X_i|+\text{rank}\, K_i[X_i, V-X_i]$ is odd. Therefore, Theorem 2.16 implies Theorem 1.2.

3. Proof of Theorem 2.16

The next result provides a sufficient condition for finding a rank-splitting decomposition. The result following shows that the sufficient conditions are met if there are no single element series classes. We complete the proof of Theorem 2.16 by using congruence transformations to combine single element series classes with other series classes.

3.1. *Let $T+K$ be a V by V mixed skew-symmetric matrix such that $T+K$ is critical. Moreover, let X be a series-class of $M(T+K)$ such that* $\text{rank}\,(T+K)[V, V-X] = \text{rank}\,(T+K)[V-X]$, *and $|X|$ is odd. Then there exists a rank-splitting decomposition T_1+K_1, T_2+K_2 of $T+K$ such that*

i. *T_1+K_1 and T_2+K_2 are V by V mixed skew-symmetric matrices,*
ii. *X supports T_1+K_1 and* $\text{rank}\, T_1+K_1 = |X|-1$,
iii. *for each $x\in X$, $\{x\}$ is a series-class of $M(T_2+K_2)$, and*
iv. *if Y is a series-class of $M(T_2+K_2)$ that is disjoint from X, then Y is a series class of $M(T+K)$.*

Proof. Let T_1 and T_2 be the Tutte matrices such that X supports T_1, $V-X$ supports T_2 and $T=T_1+T_2$.

3.1.1. *There exists a unique skew-symmetric V by V matrix K' such that* $\text{rank}\, T_2+K' = \text{rank}\,(T+K)[V-X]$ *and $K'[V, V-X] = K[V, V-X]$. (The entries of $K'[X]$ may be rational functions of the indeterminates in T.)*

Let Y be a maximal subset of $V-X$ such that $(T+K)[Y]$ is nonsingular. Now, for distinct elements $x,y\in X$, define $A=(T+K)[Y\cup\{x,y\}]$. Now, let a be a variable, and let A' be the matrix obtained from A by replacing $A_{x,y}$ and $A_{y,x}$ with a and $-a$ respectively. By 2.5, there is a unique value a' for a that makes A' singular. Let $K'_{x,y}=a'$ and $K'_{y,x}=-a'$. By considering other values of x and y we can completely determine K'. Moreover, by definition, $(T_2+K')[Y]$ is a maximal nonsingular principal submatrix of T_2+K', so $\operatorname{rank} T_2+K'=|Y|=\operatorname{rank}(T+K)[V-X]$, as required.

3.1.2. *T_1+K-K', T_2+K' is a rank-splitting decomposition of $T+K$.*

Since X is a series-class of $M(T+K)$, we have $\operatorname{rank}(T+K)[V,V-X]=\operatorname{rank}(T+K)-(|X|-1)$. Moreover, by definition, T_1+K-K' is supported by X, and $|X|$ is odd. Hence $\operatorname{rank}(T_1+K-K')\le|X|-1$. Consequently,

$$\begin{aligned}\operatorname{rank} T+K &= \operatorname{rank}\,(T+K)[V,V-X]+|X|-1\\ &\ge \operatorname{rank}\,(T_2+K')+\operatorname{rank}\,(T_1+K-K').\end{aligned}$$

Hence, T_1+K-K', T_2+K' is a rank-splitting decomposition of $T+K$, as required.

3.1.3. *All entries in K' are real.*

Let Y be a maximal subset of $V-X$ such that $(T+K)[Y]$ is nonsingular. Recalling the proof of 3.1.1, we see that any indeterminate that occurs in $K'[X]$ also occurs in $T[Y]$. Now, since X is a series class of $M(T+K)$, $M((T+K)[V,V-X])$ has no coloops. Moreover, as $(T+K)[V,V-X]$ has the same rank as $(T+K)[V-X]$, $M((T+K)[V-X])$ has no coloops. Therefore, for any $y\in V-X$ there exists a maximal nonsingular principal submatrix $(T+K)[Y']$ of $(T+K)[V-X]$ such that $y\notin Y'$. Consequently, there is no indeterminate that can occur in each matrix $T[Y]$, where $(T+K)[Y]$ is a maximal nonsingular principal submatrix of $(T+K)[V-X]$. Therefore, K' must be a real matrix, as claimed.

Let $K_1=K-K'$ and $K_2=K'$. Then we have proved parts *i* and *ii*. Part *iii* follows from 2.12, 2.15, and the fact that $M(T_2+K_2)$ has a basis contained in $V-X$; part *iv* follows from 2.13, 2.15, and the fact that $T+K$ is critical. ∎

3.2. *Let $T+K$ be a V by V mixed skew-symmetric matrix such that $T+K$ is critical. Now let $\mathcal{S}$ be a maximal collection of series classes of $M(T+K)$ such that $(T+K)[\cup(S\in\mathcal{S})]$ is nonsingular. Then, for each series-class X not in $\mathcal{S}$, $|X|$ is odd and* $\operatorname{rank}(T+K)[V,V-X]=\operatorname{rank}(T+K)[V-X]$.

Proof. Let $Y=\cup(S\in\mathcal{S})$, and let G be the graph represented by T. Since $T+K$ is critical, $G[X]$ is complete. If $|X|$ is even, then $G[X]$ has a perfect matching, and, hence, $(T+K)[Y\cup X]$ is nonsingular. This contradicts the maximality of $\mathcal{S}$, so $|X|$ is odd.

Suppose that $\operatorname{rank}(T+K)[X\cup Y, V-X] > \operatorname{rank}(T+K)[Y]$. Now $(T+K)[Y]$ is nonsingular, but it is not a maximal nonsingular submatrix of $(T+K)[X\cup Y, V-X]$. Therefore, there exists $x\in X$ and $x'\in V-(X\cup Y)$ such that $(T+K)[Y\cup\{x\}, Y\cup\{x'\}]$ is nonsingular. Now, $(T+K)[Y]$ and $(T+K)[Y\cup\{x,x'\}]$ are both skew-symmetric, and, hence, have even rank. Moreover $\operatorname{rank}(T+K)[Y\cup\{x,x'\}] > \operatorname{rank}(T+K)[Y]$ since $(T+K)[Y\cup\{x,x'\}]$ contains $(T+K)[Y\cup\{x\}, Y\cup\{x'\}]$ as a submatrix. Consequently, $(T+K)[Y\cup\{x,x'\}]$ is nonsingular. Now let X' be the series-class of $M(T+K)$ that contains x'. Recall that $|X|$ and $|X'|$ are both odd. Hence $G[X-\{x\}]$ and $G[X'-\{x'\}]$ both contain perfect matchings. Therefore, $(T+K)[(Y\cup\{x,x'\})\cup(X-\{x\})\cup(X'-\{x'\})]$ is nonsingular. That is, $(T+K)[Y\cup X\cup X']$ is nonsingular. This contradicts the maximality of $\mathcal{S}$, and, hence, $\operatorname{rank}(T+K)[X\cup Y, V-X] = \operatorname{rank}(T+K)[Y]$. Therefore, $\operatorname{rank}(T+K)[V, V-X] = \operatorname{rank}(T+K)[V-X]$. ∎

It may seem that Theorem 2.16 follows from 3.1 and 3.2. However, if $|X|=1$ in Theorem 3.1, then $T_1+K_1=0$, and we do not obtain a proper reduction of $T+K$. So, in the remainder of this section we consider singleton series classes; to obtain further reductions we use congruence transformations.

3.3. *Let $T+K$ be a V by V mixed skew-symmetric matrix such that $T+K$ is critical. Let $\{x\}$ and Y be series classes of $M(T+K)$ and let $y\in Y$ such that $K_{x,y}\neq 0$. Then there exists a real nonsingular matrix Q such that $QTQ^{\mathrm{t}}=T$, every series class of $M(Q(T+K)Q^{\mathrm{t}})$ that does not contain x is also a series class of $M(T+K)$, and either x is a coloop of $M(Q(T+K)Q^{\mathrm{t}})$ or $Y\cup\{x\}$ is a series-class of $M(Q(T+K)Q^{\mathrm{t}})$. In particular, if $|Y|=1$ then x is a coloop of $M(Q(T+K)Q^{\mathrm{t}})$.*

Proof. Construct a matrix K' from K by adding multiples of the row and column indexed by x to other rows and columns so that every entry in row and column indexed by y are zero except for $K'_{x,y}$ and $K'_{y,x}$. Now there exists a nonsingular matrix Q so that $K'=QKQ^{\mathrm{t}}$. By 2.13, $T=QTQ^{\mathrm{t}}$. For any $Z\subseteq V-\{x\}$, $\operatorname{rank}(T+K)[V,V-Z] = \operatorname{rank}(T+K')[V-Z]$ since $(T+K')[V,V-Z]$ is obtained from $(T+K)[V,V-Z]$ by elementary row and column operations. In particular, $M(T+K')$ has no coloops excect, possibly, x, and elements $i,j\in V-\{x\}$ are in series in $M(T+K)$ if and only if they are in series in $M(T+K')$. If x is a coloop of $M(T+K')$ then we are done; assume otherwise.

We claim that $Y \cup \{x\}$ is a series-class of $M(Q(T+K)Q^{\mathrm{t}})$. By the transitivity of series pairs, it suffices to prove that x and y are in series in $M(T+K')$. Suppose that x and y are not in series, then there exists $Y' \subseteq V-\{x,y\}$ such that $(T+K')[Y']$ is nonsingular and $|Y'| = \operatorname{rank} T+K$. Now, by 2.6, there exists a partition (Y_1, Y_2) of Y' such that $T[Y_1]$ and $K'[Y_2]$ are nonsingular. Consider $K'[Y_2 \cup \{x,y\}]$. $K'_{x,y} \neq 0$ but all other entries in the row indexed by y are zero. Hence, by 2.4, $K'[Y_2 \cup \{x,y\}]$ is nonsingular. Then, by 2.6, $(T+K')[Y' \cup \{x,y\}]$ is nonsingular. This contradicts that $|Y'| = \operatorname{rank} T+K$. Therefore, x and y are in series as required.

Now consider the particular case when $|Y| = 1$; that is, $Y = \{y\}$. Now the row and column of $(T+K')[V-\{x\}]$ indexed by y contain only zero entries. Consider a maximal subset Y' of $V-\{x\}$ such that $(T+K')[Y']$ is nonsingular. Clearly $y \notin Y'$. Now, by 2.4, $(T+K')[Y' \cup \{x,y\}]$ is nonsingular. Therefore, $|Y'| < \operatorname{rank} T+K'$, and, hence, by our choice of Y', x is a coloop of $M(T+K')$, as required. ∎

3.4. *Let $T+K$ be a V by V mixed skew-symmetric matrix such that $T+K$ is critical. Let $\{x\}$ be a singleton series-class of $M(T+K)$ and let Q be a real nonsingular matrix such that $QTQ^{\mathrm{t}} = T$ and x is a coloop of $M(Q(T+K)Q^{\mathrm{t}})$. Then there exists a real V by V skew-symmetric matrix K' of rank two such that $K', T+K-K'$ is a rank-splitting decomposition of $T+K$.*

Proof. Since x is a coloop of $M(Q(T+K)Q^{\mathrm{t}})$, there exists a rank-splitting decomposition $K_1, T+K_2$ of $Q(T+K)Q^{\mathrm{t}}$ such that K_1 is a real skew-symmetric matrix of rank two. Thus $Q^{-1}K_1(Q^{\mathrm{t}})^{-1}, Q^{-1}(T+K_2)(Q^{\mathrm{t}})^{-1}$ is a rank-splitting decomposition of $T+K$. Let $K' = Q^{-1}K_1(Q^{\mathrm{t}})^{-1}$. Thus K' is a real skew-symmetric matrix of rank two. Since $QTQ^{\mathrm{t}} = T$ we have $T = Q^{-1}T(Q^{\mathrm{t}})^{-1}$. Hence $K', T+K-K'$ is a rank-splitting decomposition of $T+K$. ∎

We prove Theorem 2.16 by double induction, first on $\operatorname{rank} T+K$, and then on the number of series classes of $M(T+K)$. In summary, the above lemmas leave us in one of the following cases.

(1) There exists a series class X of $M(T+K)$ such that $|X|$ is odd and contains at least 3 elements, and there is a rank-splitting decomposition of $T+K$ into mixed skew-symmetric matrices T_1+K_1 and T_2+K_2 such that T_1+K_1 is supported by X, $\operatorname{rank} T_1+K_1 = |X|-1$, for each $x \in X$, $\{x\}$ is a series class of T_2+K_2, and each series class of $M(T_2+K_2)$ that is disjoint from X is also a series class of $M(T+K)$.

(2) There exists a real V by V skew-symmetric matrix K', with positive rank, such that $K', T+K-K'$ is a rank-splitting decomposition of $T+K$.

(3) There exists a real nonsingular matrix Q, a singleton series class $\{x\}$ of $M(T+K)$ and a nonsingleton series class Y of $M(T+K)$ such that $QTQ^{\mathrm{t}}=T$, $Y\cup\{x\}$ is a series class of $M(Q(T+K)Q^{\mathrm{t}})$ and every other series class of $M(Q(T+K)Q^{\mathrm{t}})$ is a series class of $M(T+K)$.

In the first case we can readily apply induction to prove Theorem 2.16. Consider the second case. That is, K' is a rank two skew-symmetric matrix such that $K',T+K-K'$ is a rank-splitting decomposition of $T+K$. By 2.13, 2.15, and the fact that $T+K$ is critical, we see that $M(T+K-K_1)$ has the same series classes as $M(T+K)$. Again Theorem 2.16 follows easily by induction. Now consider the remaining case. Let $K'=QKQ^{\mathrm{t}}$; thus $Q(T+K)Q^{\mathrm{t}}=T+K'$. By 2.14, we can add indeterminates to T' such that $T'+K'$ is critical and $\operatorname{rank} T'+K'=\operatorname{rank} T+K'$. By 2.13 and the fact that $T+K$ is critical, $M(T'+K')$ and $M(T+K')$ have the same series classes. Let $X_1,\dots,X_k$ be the series classes of $M(T'+K')$ that have two or more elements; we may assume that $X_1=Y\cup\{x\}$.

Since $M(T'+K')$ has fewer series classes than $M(T+K)$, we may apply the induction hypothesis to $T'+K'$. Let $K'_0,T'_1+K'_1,\dots,T'_k+K'_k$ be the rank-splitting decomposition of $T'+K'$ given by Theorem 2.16. We may assume that rank $K'_0=0$ since otherwise we could apply case (2). Now, for each $i\in\{1,\dots,k\}$, let T_i be the matrix obtained from T'_i by setting each of the new indeterminates to zero, and let $K_i=Q^{-1}K'_i(Q^{\mathrm{t}})^{-1}$. Thus, $T_1+K'_1,\dots,T_k+K'_k$ is a rank-splitting decomposition of $T+K'$, and, hence, $T_1+K_1,\dots,T_k+K_k$ is a rank-splitting decomposition of $T+K$. It remains to show that the matrices T_i+K_i are of the appropriate form.

For each $i\in\{1,\dots,k\}$, X_i supports T'_i and X_i is a cover of $T'_i+K'_i$. Then, for each $i\in\{2,\dots,k\}$, X_i supports T_i and, since $QT_iQ^t=T_i$ and since X_i is a cover of K'_i, X_i is a cover of T_i+K_i. Recall that, $X_1=Y\cup\{x\}$, and, by 3.3, $|Y|>1$. Moreover, Y is a series-class of $M(T+K)$ and, since X_i supports T'_1, Y supports T_1. If Y is not a cover of T_1+K_1, then it is straightforward to find a V by V skew-symmetric matrix L, with positive rank, such that L,T_1+K_1-L is a rank-splitting decomposition of T_1+K_1. But, then $L,T+K-L$ is a rank-splitting decomposition of $T+K$ and we can apply case (2). Therefore, we may assume that Y is a cover of T_1+K_1, as required.

4. Matroid parity

In this section we derive a min-max theorem for the linear matroid parity problem.

Matroid parity problem *Given a matroid M on the ground set V, and a partition $\varPi=(\pi_1,\dots,\pi_m)$ of V into pairs, find a maximum size collection $(\pi_{i_1},\dots,\pi_{i_k})$ of these pairs such that $\pi_{i_1}\cup\cdots\cup\pi_{i_k}$ is independent in M.*

Let $\nu_\varPi(M)$ denote the maximum number of pairs whose union is independent in M. A subset S of V is called a *parity set* if each pair in $\varPi$ is either contained in S or is disjoint from S.

The matroid parity problem is intractible (using the usual oracle based approach to matroid algorithms) [8,9] and NP-hard [13]. More surprisingly, Lovász [11] showed that $\nu_\varPi(M)$ can be computed efficiently if M is linear (that is, M is *represented* by a matrix). We shall see that, for a linear matroid, computing $\nu_\varPi(M)$ can be formulated as matrix rank problem, from which we derive a min-max theorem. A different formulation is given by Lovász and Plummer [13, Theorem 11.1.2].

4.1 (Matrix formulation). *Let A be a matrix with rows and columns indexed by R and V respectively, and let $\varPi$ be a partition of V into pairs. Now let T be the Tutte matrix of the graph with vertex set $R\cup V$ and edge set $\varPi$, and let*

$$K := \begin{array}{c} \\ R \\ V \end{array}\begin{array}{c} \begin{array}{cc} R & V \end{array} \\ \begin{pmatrix} 0 & A \\ -A^{\mathrm t} & 0 \end{pmatrix} \end{array}.$$

Then, $2\nu_\varPi(M(A))=\operatorname{rank}(T+K)-|V|$.

Proof. Note that $T[V]$ is nonsingular. Therefore, V is an independent set of $M(T+K)$. Let $V\cup R'$ be a basis of $M(T+K)$. Therefore, $(T+K)[V\cup R']$ is nonsingular. By 2.6, there exists $X\subseteq V$ such that $K[(V-X)\cup R']$ and $T[X]$ are both nonsingular. Since $T[X]$ is nonsingular, X is a parity set. Thus $V-X$ is also a parity set. Now, since $K[(V-X)\cup R']$ is nonsingular, $|R'|=|V-X|$ and $A[R',V-X]$ is nonsingular. Hence $V-X$ is an independent set of $M(A)$. Thus,

$$2\nu_\varPi(M(A)) \ge |V-X| = |R'| = |R'\cup V| - |V| = \operatorname{rank}\,(T+K) - |V|.$$

Now suppose that $Y\subseteq V$ is a parity set of size $2\nu_\varPi(M(A))$ that is independent in $M(A)$. Then, there exists $R'\subseteq R$ such that $|R'|=|Y|$ and $A[R',Y]$ is nonsingular. Hence, $K[R'\cup Y]$ is nonsingular. Since Y is a parity set, so is $V-Y$. Therefore, $T[V-Y]$ is nonsingular. By 2.6, $(T+K)[V\cup R']$ is nonsingular. Hence,

$$\operatorname{rank}\,(T+K) - |V| \ge |V\cup R'| - |V| = |R'| = |Y| = 2\nu_\varPi(M(A)).$$

The result follows from the two inequalities above. ∎

Using the formulation above, we will derive a min-max theorem for linear matroid parity. The following two lemmas provide the desired upper bound on $\nu_\Pi(M)$. (The reader is left to check the validity of these bounds.)

4.2. *Let $A = A_1 + A_2$, where A, A_1, A_2 are matrices whose columns are indexed by V, then*

$$\nu_\Pi(M(A)) \leq \text{rank } A_1 + \nu_\Pi(M(A_2)).$$

4.3. *Let $(X_1, \ldots, X_k)$ be a partition of V into parity sets, then*

$$\nu_\Pi(M(A)) \leq \sum_{i=1}^{k} \left\lfloor \frac{\text{rank } A[R, X_i]}{2} \right\rfloor.$$

The following theorem is a slight variation on Lovász' min-max theorem for linear matroid parity [13]. The essential difference is that Lovász finds a decomposition $A = A_1 + A_2$ in which $\text{rank}\, A = \text{rank}\, A_1 + \text{rank}\, A_2$.

4.4 (Lovász' min-max theorem). *Let A be a matrix whose rows and columns are indexed by R and V respectively, and let Π be a partition of V into pairs. Now, if $A = A_1 + A_2$ and $(X_1, \ldots, X_k)$ is a partition of V into parity sets, then*

$$\nu_\Pi(M(A)) \leq \text{rank } A_1 + \sum_{i=1}^{k} \left\lfloor \frac{\text{rank } A_2[R, X_i]}{2} \right\rfloor.$$

Moreover, equality is attained by some such matrices A_1 and A_2 and some partition $(X_1, \ldots, X_k)$ of V into parity sets.

Proof. The upper bound on $\nu_\Pi(M(A))$ follows immediately from 4.2 and 4.3; thus it remains to show that equality can be attained. We make the following two assumptions without loss of generality.

4.4.1. *If $A = A_1 + A_2$ and $\nu_\Pi(M(A)) = \text{rank}\, A_1 + \nu_\Pi(M(A_2))$, then $A_1 = 0$.*

4.4.2. *For any pair π in Π, $\text{rank}\, A[R, \pi] = 2$.*

Note that the result is invariant under elementary row operations on A. Define K and T as in Theorem 4.1.

4.4.3. *No element of R is a coloop of $M(T + K)$.*

Suppose otherwise, and let $r\in R$ be a coloop of $M(T{+}K)$. Define matrices A_1 and A_2 such that $A=A_1+A_2$, A_1 is only nonzero in the row indexed by r, and A_2 is zero in the row indexed by r. Since r is a coloop of $M(T+K)$, $\operatorname{rank}(T+K)[(R\cup V)-\{r\}]=\operatorname{rank}(T+K)-2$. Then, by 4.1, we deduce that $\nu_\Pi(M(A))=\operatorname{rank} A_1+\nu_\Pi(M(A_2))$. This contradicts 4.4.1, which proves the claim.

4.4.4. *Let (T_1+K_1,T_2+K_2) be a rank-splitting decomposition of $T+K$ where T_1 and T_2 are Tutte matrices supported by V, K_1 and K_2 are real skew-symmetric matrices, and $K_1[R]=0$ and $K_1[V]=0$. If $v\in V$ is a loop of $M(T_1)$, then v is a loop of $M(K_1)$.*

Suppose otherwise. Thus, there exists $v\in V$ such that v is a loop of $M(T_1)$ but not of $M(K_1)$. Then, for some $r\in R$, $(K_1)_{r,v}\neq 0$. Define A_1 and A_2 such that

$$K_1 := \begin{array}{c} \\ R \\ V \end{array}\!\!\begin{array}{c} \begin{array}{cc} R & V \end{array} \\ \begin{pmatrix} 0 & A_1 \\ -A_1^{\mathrm t} & 0 \end{pmatrix}\end{array} \text{ and } K_2 := \begin{array}{c} \\ R \\ V \end{array}\!\!\begin{array}{c} \begin{array}{cc} R & V \end{array} \\ \begin{pmatrix} 0 & A_2 \\ -A_2^{\mathrm t} & 0 \end{pmatrix}\end{array}.$$

Applying row operations simultaneously to A_1, A_2 and A, we may assume that $(A_1)_{r,v}$ is the only nonzero element in the column of A_1 indexed by v. Thus, $(T_1{+}K_1)_{v,r}$ is the only nonzero element in the row of $T_1{+}K_1$ indexed by v. Hence, r is a coloop of $M(T_1+K_1)$. Thus, by 2.10, r is a coloop of $M(T+K)$. This contradicts 4.4.3, which completes the proof of the claim.

4.4.5. *$M(T+K)$ has no coloops.*

Suppose otherwise, and let v be a coloop of $M(T+K)$. By 4.4.3, $v\in V$. Suppose that $\{v,w\}$ is the pair in Π containing v. Define matrices K_1 and T_1 such that $(V\cup R)-\{v\}$ supports $T_1{+}K_1$, $T_1[(V\cup R)-\{v\}]=T[(V\cup R)-\{v\}]$ and $K_1[(V\cup R)-\{v\}]=K[(V\cup R)-\{v\}]$. Now let $T_2:=T-T_1$ and $K_2=K-K_1$. Since v is a coloop of $M(T{+}K)$, $(T_1{+}K_1,T_2{+}K_2)$ is a rank-splitting decomposition of $T+K$. Now w is a loop of $M(T_1)$, so w is a loop of $M(K_1)$. But, then, w is a loop of $M(A)$. This contradicts 4.4.2, which proves the claim.

By applying elementary row operations to A, we may assume that, for some basis B of $M(A)$, A has the form

$$A := R\ \begin{array}{c} \begin{array}{cc} B & V-B \end{array} \\ \begin{pmatrix} I & A' \end{pmatrix}\end{array}.$$

Now, by 2.14, we can extend T to a Tutte matrix $\tilde T$ such that $\tilde T+K$ is critical.

4.4.6. *$M(K+\tilde T)$ has no singleton series-class.*

Note that, if $\{v,w\}$ is a pair in Π, then v and w are in the same series-class of $M(\tilde{T}+K)$. Therefore, if $M(\tilde{T}+K)$ has a singleton series-class, then there exists $r\in R$ such that $\{r\}$ is a series-class of $M(\tilde{T}+K)$. There exists some $v\in B$ such that $A_{r,v}$ is the only nonzero entry in the column of A indexed by v. Now, since r and v are not in series in $M(K+\tilde{T})$, there exists disjoint subsets X,Y of $(V\cup R)-\{r,v\}$ such that $|X|+|Y|=\text{rank}(\tilde{T}+K)$, and $\tilde{T}[X]$ and $K[Y]$ are nonsingular. However, it is easy to see that $K[Y\cup\{r,v\}]$ is nonsingular, so, by 2.6, $(\tilde{T}+K)[X\cup Y\cup\{r,v\}]$ is nonsingular. This contradicts that $\text{rank}(\tilde{T}+K)=|X|+|Y|$; which proves the claim.

By 3.2, there exists a series-class X of $M(\tilde{T}+K)$ such that $|X|$ is odd, and $\text{rank}(\tilde{T}+K)[V\cup R,(V\cup R)-X]=\text{rank}(\tilde{T}+K)[(V\cup R)-X]$. By 3.1, there exists a rank splitting decomposition of $\tilde{T}+K$ into mixed skew-symmetric matrices $\tilde{T}_1+K_1$ and $\tilde{T}_2+K_2$ such that X supports $\tilde{T}_1+K_1$, and $\text{rank}(\tilde{T}_1+K_1)=|X|-1$. Note that $\text{rank}(\tilde{T}_2+K_2)=\text{rank}(\tilde{T}+K)-|X|+1$, and, since X is a series-class of $M(\tilde{T}+K)$, $\text{rank}(\tilde{T}+K)[R\cup V,(R\cup V)-X]=\text{rank}(\tilde{T}+K)-|X|+1$. Therefore, $\text{rank}(\tilde{T}_2+K_2)=\text{rank}(\tilde{T}+K)[R\cup V,(R\cup V)-X]=\text{rank}(\tilde{T}+K)[(V\cup R)-X]$.

4.4.7. $K_2[R]=0$ *and* $K_2[V]=0$.

Consider any pair $a,b\in R\cap X$. Recall that $\text{rank}(\tilde{T}_2+K_2)=\text{rank}(\tilde{T}_2+K_2)[(V\cup R)-X]$, and that $(\tilde{T}+K)[V\cup R,(V\cup R)-X]=(\tilde{T}_2+K_2)[V\cup R,(V\cup R)-X]$. By 2.6, there exist disjoint subsets A,B of $(V\cup R)-X$ such that $\tilde{T}[A]$ is nonsingular, $K[B]$ is nonsingular, and $|A|+|B|=\text{rank}(\tilde{T}+K)[(V\cup R)-X]$. By 2.6 and since $\text{rank}(\tilde{T}_2+K_2)=\text{rank}(\tilde{T}_2+K_2)[(V\cup R)-X]$, it must be the case that $\text{rank}\,K_2[B\cup\{a,b\}]=\text{rank}\,K_2[B]$. By 2.5, there is a unique choice for $(K_2)_{a,b}$ such that $\text{rank}\,K_2[B\cup\{a,b\}]=\text{rank}\,K_2[B]$. However, since V is a cover for K, $\text{rank}\,K[B\cup\{a,b\}]=\text{rank}\,K[B]$. Hence $(K_2)_{a,b}=K_{a,b}=0$. Thus, $K_2[R\cap X]=0$. Since $K[R]=0$ and X supports K_1, $K_2[R,R-X]=0$. Therefore, $K_2[R]=0$ as claimed. A similar argument proves that $K_2[V]=0$.

Note that, since $K=K_1+K_2$ we also have $K_1[R]=0$ and $K_1[V]=0$. Since $\text{rank}(\tilde{T}+K)=\text{rank}(T+K)$, there exist Tutte matrices T_1 and T_2 such that (T_1+K_1,T_2+K_2) is a rank-splitting decomposition of $T+K$. By 4.4.4, $V\cap X$ is a cover of K_1 and $V-X$ is a cover of K_2. Let $A_1:=A[R,X\cap V]$ and $A_2:=A[R,V-X]$. Now let Π_1 and Π_2 be the pairs of Π in $X\cap V$ and $V-X$ respectively. By 4.1, we see that

$$\nu_{\Pi}(M(A))=\nu_{\Pi_1}(M(A_1))+\nu_{\Pi_2}(M(A_2)).$$

Now $\text{rank}(T_1+K_1)=\text{rank}(\tilde{T}_1+K_1)=|X|-1$. Thus, by 4.1,

$$\nu_{\Pi_1}(M(A_1))=\frac{1}{2}(|X-V|-1).$$

Moreover, as X supports K_1, $\operatorname{rank} A[R, X\cap V] = \operatorname{rank} A_1 \le |X - V|$. Then, from 4.3, we see that

$$\nu_{\Pi_1}(M(A_1)) = \left\lfloor \frac{\operatorname{rank} A[R, X\cap V]}{2} \right\rfloor.$$

Now the result follows inductively by considering $\nu_{\Pi_2}(M(A_2))$. ∎

5. Delta-matroids

Let K be a V by V skew-symmetric matrix, and let $\mathcal{F}_K = \{X \subseteq V : \operatorname{rank} K[X] = |X|\}$. Bouchet [1] observed that the setsystem $(V, \mathcal{F}_K)$ satisfies the following axiom:

delta-matroid exchange axiom If $X, Y \in \mathcal{F}_K$ and $x \in X \Delta Y$, then there exists $y \in X\Delta Y$ such that $X \Delta \{x, y\} \in \mathcal{F}$.

Here $X\Delta Y$ denotes the symmetric difference of X and Y. A setsystem $(V, \mathcal{F})$ satisfying the delta-matroid exchange axiom is called a *delta-matroid*. Thus, $DM(K) := (V, \mathcal{F}_K)$ is a delta-matroid. The sets in $\mathcal{F}$ are called the *feasible sets* of $(V, \mathcal{F})$. Note that the feasible sets in $DM(K)$ all have even cardinality. A delta-matroid whose feasible sets all have even cardinality or all have odd cardinality are called *even delta-matroids*. By 2.7, the maximal sets in $(V, \mathcal{F}_K)$ are the bases of $M(K)$. In fact, the maximal feasible sets in any delta-matroid always form the bases of a matroid. Moreover, a collection of equicardinal sets is a delta-matroid if and only if they are the bases of a matroid.

Note that, if T is the Tutte matrix of a graph G, then $\mathcal{F}_T$ is the set of matchable sets of G. We call $DM(T)$ the *matching delta-matroid* of G.

For a delta-matroid $M = (V, \mathcal{F})$ and $X \subseteq V$, we denote $M\dot{\Delta}X = (V, \mathcal{F}\dot{\Delta}X)$, where $\mathcal{F}\dot{\Delta}X = \{F\Delta X \mid F \in \mathcal{F}\}$. It is easy to see that $M\dot{\Delta}X$ is a delta-matroid. This operation is referred to as *twisting* by X, and $M\dot{\Delta}X$ is said to be *equivalent* to M. For any V by V skew-symmetric matrix K, and any subset X of V, we call $DM(K)\dot{\Delta}X$ a *linear delta-matroid*. The delta-matroid $M^* := M\dot{\Delta}V$ is the *dual* of M. It is also easy to see that $M\backslash X = (V\backslash X, \mathcal{F}\backslash X)$ defined by $\mathcal{F}\setminus X = \{F \mid F \in \mathcal{F}, F \subseteq V \setminus X\}$ is a delta-matroid. This operation is referred to as the *deletion* of X. The *contraction* of M by X means $(M\dot{\Delta}X)\backslash X$, and is denoted by M/X. Note that evenness is invariant under these operations.

Suppose the skew-symmetric matrix K has the form:

$$K = \begin{array}{c} \\ Y \\ X \end{array} \begin{array}{c} \begin{array}{cc} Y & X \end{array} \\ \begin{pmatrix} \alpha & \beta \\ -\beta^{\mathrm{t}} & \gamma \end{pmatrix} \end{array},$$

where $\alpha = K[Y]$ is nonsingular. We define a matrix $K * Y$ by

$$K * Y := \begin{array}{c} \\ Y \\ X \end{array} \begin{array}{c} \begin{array}{cc} Y & X \end{array} \\ \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ \beta^{\mathrm{t}}\alpha^{-1} & \gamma + \beta^{\mathrm{t}}\alpha^{-1}\beta \end{pmatrix} \end{array}.$$

This operation converting K to $K * Y$ is called a *pivoting*. The following theorem is fundamental to linear delta-matroids.

5.1 (Tucker [16]). *Let $K[Y]$ be a nonsingular principal submatrix of a skew-symmetric matrix K. Then, for all $S \subseteq V$,*

$$\det(K * Y)[S] = \det K[Y \varDelta S] / \det K[Y].$$

The following is an immediate corollary of 5.1.

5.2. *If K is a V by V skew-symmetric matrix, and F is a feasible set of $M(K)$, then $M(K)\dot{\varDelta}F = M(K * F)$.*

Bouchet and Cunningham [3] introduced jump systems, which are a nice generalization of delta-matroids. See also Lovász [12] for the “membership problem” in jump systems.

6. Delta-cover problem

Consider the following problems for delta-matroids.

Intersection problem Given two delta-matroids M_1 and M_2 on a common ground set V, does there exist a common feasible set?

Partition problem Given two delta-matroids M_1 and M_2 on a common ground set V, does there exist a partition (F_1, F_2) of V such that F_1 is feasible in M_1 and F_2 is feasible in M_2?

Parity problem Given a delta-matroid M on a ground set V, and a partition $\varPi$ of V into pairs, does there exist a feasible set F that is the union of pairs in $\varPi$?

The intersection problem on M_1 and M_2 is equivalent to the partition problem on M_1 and $M_2 \dot{\varDelta} V$. Now consider the parity problem on M and $\varPi$. A subset of V is called a *parity set* if it is the union of pairs in $\varPi$. The parity sets are in fact the feasible sets of a (linear) delta-matroid $M_\varPi$. Thus the parity problem is a special case of the intersection problem. Conversely, consider the intersection problem on M_1 and M_2. Suppose that $V := \{1, \ldots, n\}$ and define $V' := \{1', \ldots, n'\}$ and $\varPi := (\{1, 1'\}, \ldots, \{n, n'\})$. Let M_2' be a copy of M_2

on the ground set V'. Now $M_1 \oplus M_2'$ is the delta-matroid on ground set $V \cup V'$ whose feasible sets are the union of each feasible set of M_1 and each feasible set of M_2'. It is straightforward to check that the intersection problem on M_1 and M_2 is equivalent to the parity problem on $M_1 \oplus M_2'$ and Π. Therefore, the three problems mentioned above are all equivalent. As they contain matroid parity, they are intractible in general. However, they can be solved for linear delta-matroids; see [6]. Bouchet and Jackson [4] have extended many of Lovász's results on linear matroid matching to the linear delta-matroid parity problem. Their methods are elegant and provide much insight into the problem, but they fall short of providing a good characterization.

It is often more convenient to work with optimization problems than decision problems. A natural generalization of the parity problem is considered in [6]. The following natural generalizations of the partition problem were proposed by Bouchet [2].

Delta-cover problem Given two delta-matroids $M_1 = (V, \mathcal{F}_1)$ and $M_2 = (V, \mathcal{F}_2)$, find $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ maximizing $|F_1 \Delta F_2|$.

Disjoint union problem Given two delta-matroids $M_1 = (V, \mathcal{F}_1)$ and $M_2 = (V, \mathcal{F}_2)$, find disjoint sets $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ maximizing $|F_1 \cup F_2|$.

Let $\mathcal{F}_1 \ddot{\Delta} \mathcal{F}_2$ denote $\{F_1 \Delta F_2 \ : \ F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2\}$, and $\mathcal{F}_1 \ddot{\cup} \mathcal{F}_2$ denote $\{F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, F_1 \cap F_2 = \emptyset\}$. Now let $M_1 \ddot{\Delta} M_2$ denote $(V, \mathcal{F}_1 \ddot{\Delta} \mathcal{F}_2)$, and let $M_1 \ddot{\cup} M_2$ denote $(V, \mathcal{F}_1 \ddot{\cup} \mathcal{F}_2)$. Bouchet and Cunningham [3] proved that $M_1 \ddot{\Delta} M_2$ and $M_1 \ddot{\cup} M_2$ are delta-matroids. If M_1 and M_2 do not contain disjoint feasible sets, then the disjoint union problem is infeasible. The following result shows that, if M_1 and M_2 have disjoint feasible sets, then the disjoint union problem and the delta-cover problem are equivalent. See Murota [14] for a simple direct proof.

6.1. *Let $M_1 = (V, \mathcal{F}_1)$ and $M_2 = (V, \mathcal{F}_2)$ be delta-matroids that contain disjoint feasible sets. Then ,*

$$\max(|X| \ : \ X \in \mathcal{F}_1 \ddot{\Delta} \mathcal{F}_2) = \max(|X| \ : \ X \in \mathcal{F}_1 \ddot{\cup} \mathcal{F}_2).$$

Let K be a V by V skew-symmetric matrix, and let T be a Tutte matrix. By 2.6,

$$\text{rank}\,(T + K) = \max(|X| \ : \ X \in \mathcal{F}_K \ddot{\cup} \mathcal{F}_T).$$

Therefore, computing $\text{rank}\,(T+K)$ is a disjoint union problem (or, equivalently, a delta-cover problem). Now consider the delta-cover problem for a pair of linear delta-matroids $DM(K_1)\dot{\Delta}X_1$ and $DM(K_2)\dot{\Delta}X_2$. We will show how this can be formulated as computing $\text{rank}\,(T+K)$ for an appropriate choice of K and T. First, let $X = X_1 \Delta X_2$ and note that

$$(DM(K_1)\dot{\Delta}X_1)\ddot{\Delta}(DM(K_2)\dot{\Delta}X_2) = DM(K_1)\ddot{\Delta}(DM(K_2)\dot{\Delta}X).$$

Thus it suffices to consider the delta-cover problem for $M_1 := DK(K_1)$ and $M_2 := DM(K_2)\dot{\Delta}X$.

Let $V = \{1,\dots,n\}$, $V' = \{1',\dots,n'\}$ and $V'' := \{1'',\dots,n''\}$. For $A \subseteq V$, let $A' := \{x' : x \in A\}$ and $A'' := \{x'' : x \in A\}$. Now, let K_2' be a copy of K_2 on V', then define

$$K := \begin{array}{c} \\ V \\ V' \\ V''-X'' \end{array} \begin{array}{c} \begin{array}{ccc} V & V' & V''-X'' \end{array} \\ \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2' & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Now let G be a graph with vertex set $V \cup V' \cup (V'' - X'')$ and edge set $\{ii' : i \in X\} \cup \{ii'' : i \in V - X\} \cup \{i'i'' : i \in V - X\}$, and let T be the Tutte matrix of G. We leave the proof of the following result as an exercise.

6.2. $\max(|F_1 \Delta F_2| : F_1 \in \mathcal{F}_{K_1}, F_2 \in (\mathcal{F}_{K_2}\dot{\Delta}X)) = \text{rank}\,(T+K) - 2|V| + |X|.$

A min-max theorem for the delta-cover problem is given in [6]. This theorem can be proved using the matrix formulation above, and the techniques developed in Sections 2 and 3. We have chosen to omit the derivation as it is long and technical. Instead, in the next section, we will consider a special case of the delta-cover problem, for which, we shall derive a new min-max theorem.

7. The diameter of a delta-matroid

Let $M = (V, \mathcal{F})$ be a delta-matroid. We define the *diameter* of M, denoted $\text{diam}(M)$ or $\text{diam}(\mathcal{F})$, to be $\max(|F_1 \Delta F_2| : F_1, F_2 \in \mathcal{F})$. Determining the diameter of M is obviously a delta-cover problem. Note that, for any $X \subseteq V$, $\text{diam}(M) = \text{diam}(M\dot{\Delta}X)$. That is, the diameter is invariant under twisting. For a general delta-matroid, determining the diameter is intractible; see [5]. However, there is an efficient algorithm [6] for computing the diameter of a linear delta-matroid. It is not known whether the diameter can be efficiently computed for even delta-matroids, but we conjecture otherwise. We prove a new min-max theorem for the diameter of a linear delta-matroid.

7.1. *Let $K = K_1 + K_2$ where K_1, K_2 and K are skew-symmetric matrices. Then,*

$$\text{diam}(\mathcal{F}_K) \leq \text{diam}(\mathcal{F}_{K_1}) + \text{diam}(\mathcal{F}_{K_2}).$$

Proof. By 6.1, there exist disjoint sets $F_1, F_2 \in \mathcal{F}_K$ such that $\mathrm{diam}(\mathcal{F}_K) = |F_1| + |F_2|$. Therefore,

$$\begin{aligned}\mathrm{diam}(\mathcal{F}_K) &= \mathrm{rank}\ K[F_1] + \mathrm{rank}\ K[F_2]\\ &\le (\mathrm{rank}\ K_1[F_1] + \mathrm{rank}\ K_2[F_1]) + (\mathrm{rank}\ K_1[F_2] + \mathrm{rank}\ K_2[F_2])\\ &= (\mathrm{rank}\ K_1[F_1] + \mathrm{rank}\ K_1[F_2]) + (\mathrm{rank}\ K_2[F_1] + \mathrm{rank}\ K_2[F_2])\\ &\le \mathrm{diam}(\mathcal{F}_{K_1}) + \mathrm{diam}(\mathcal{F}_{K_2}),\end{aligned}$$

as required. ∎

7.2. *If K is a V by V skew-symmetric matrix, X is a cover of K, and $|X|$ is odd, then*

$$\mathrm{diam}(\mathcal{F}_K) \le |X| + 2\mathrm{rank}\ K[X, V - X] - 1.$$

Proof. By 6.1, there exist disjoint sets $F_1, F_2 \in \mathcal{F}_K$ such that $\mathrm{diam}(\mathcal{F}_K) = |F_1| + |F_2|$. Therefore,

$$\begin{aligned}\mathrm{diam}(\mathcal{F}_K) &= \mathrm{rank}\ K[F_1] + \mathrm{rank}\ K[F_2]\\ &\le (\mathrm{rank}\ K[F_1, F_1 \cap X] + \mathrm{rank}\ K[F_1, F_1 - X])\\ &\quad + (\mathrm{rank}\ K[F_2, F_2 \cap X] + \mathrm{rank}\ K[F_2, F_2 - X])\\ &\le (|F_1 \cap X| + \mathrm{rank}\ K[X, V - X])\\ &\quad +(|F_2 \cap X| + \mathrm{rank}\ K[X, V - X])\\ &\le |X| + 2\mathrm{rank}\ K[X, V - X]).\end{aligned}$$

However $\mathrm{diam}(\mathcal{F}_K)$ is even and $|X| + 2\mathrm{rank}\, K[X, V - X]$ is odd. Therefore, $\mathrm{diam}(\mathcal{F}_K) \le |X| + 2\mathrm{rank}\, K[X, V - X] - 1$, as required. ∎

The following result is the main theorem of this section.

7.3. *Let K be a V by V skew-symmetric matrix. For any set $F \in \mathcal{F}_K$, disjoint odd subsets $X_1, \ldots, X_k$ of V, and V by V skew-symmetric matrices $K_1, \ldots, K_k$ such that $K * F = K_1 + \cdots + K_k$, and, for $i = 1, \ldots, k$, X_i is a cover of K_i, we have*

$$\mathrm{diam}(\mathcal{F}_K) \le \sum_{i=1}^{k} (|X_i| + 2\mathrm{rank}\ K_i[X_i, V - X_i] - 1).$$

Moreover, this bound is attained for some choice of F, $X_1, \ldots, X_k$, and $K_1, \ldots, K_k$.

We now outline the proof of Theorem 7.3. The inequality follows easily from 7.1 and 7.2. Thus, it remains to prove that equality is attained; the proof is by induction on $\mathrm{diam}(\mathcal{F}_K)$. Firstly, suppose that we can reduce the diameter by deleting or contracting a single element. By possibly pivoting, we may assume that $\mathrm{diam}(DM(K)\backslash\{x\})=\mathrm{diam}(DM(K))-2$. Now let K_1 be the matrix obtained from K by changing the entries in the row and column indexed by x to zero, and let $K_2 := K - K_1$. Note that, $\mathrm{diam}(DM(K_2))=2$ and that $\mathrm{diam}(DM(K))=\mathrm{diam}(DM(K_1))+\mathrm{diam}(DM(K_2))$. Now let $X_2 := \{x\}$. Thus, X_2 is a cover of K_2 and $\mathrm{diam}(DM(K_2))=|X_2|+2\,\mathrm{rank}\,K_2[X_2, V-X_2]-1$. Now Theorem 7.3 follows inductively. Henceforth, we assume that we cannot reduce the diameter by deleting or contracting a single element. This gives the following conditions.

7.3.1. *For any $x \in V$ there exist $F_1, F_2 \in \mathcal{F}_K$ such that $x \notin F_1 \cup F_2$ and* $\mathrm{diam}(\mathcal{F}_K)=|F_1 \Delta F_2|$.

7.3.2. *For any $x \in V$ there exist $F_1, F_2 \in \mathcal{F}_K$ such that $x \in F_1 \cap F_2$ and* $\mathrm{diam}(\mathcal{F}_K)=|F_1 \Delta F_2|$.

Since $\emptyset \in \mathcal{F}_K$, we have $\mathrm{diam}(\mathcal{F}_K) \geq \mathrm{rank}\,K$. Suppose that $\mathrm{diam}(\mathcal{F}_K) = |F_1 \Delta F_2|$, where $F_1, F_2 \in \mathcal{F}_K$. Now, since $F_1 \Delta F_2 \in \mathcal{F}_{K*F_1}$, we have $\mathrm{diam}(\mathcal{F}_{K*F_1})=\mathrm{rank}\,K * F_1$. Consider replacing K by $K * F_1$.

7.3.3. *We assume that* $\mathrm{diam}(\mathcal{F}_K)=\mathrm{rank}\,K$.

We now formulate the problem of computing $\mathrm{diam}(\mathcal{F}_K)$ as a matrix rank problem. This formulation is a special case of the one given in the previous section, but we restate it for clarity.

Let $V = \{1, \ldots, n\}$, $V' = \{1', \ldots, n'\}$, $V'' := \{1'', \ldots, n''\}$, and let $\tilde{V} := V \cup V' \cup V''$. For $A \subseteq V$, let $A' := \{x' : x \in A\}$ and $A'' := \{x'' : x \in A\}$. Now, let K' be a copy of K on V' and let K'' be a copy of K on V''. Define

$$\tilde{K} := \begin{array}{c} \\ V \\ V' \\ V'' \end{array} \begin{array}{c} \begin{array}{ccc} V & V' & V'' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & K' & 0 \\ 0 & 0 & K'' \end{pmatrix} \end{array}.$$

Now let G be a graph with vertex set $\tilde{V}$ and edge set $\{ii' : i \in V\} \cup \{ii'' : i \in V\}$, and let T be the Tutte matrix of G.

7.3.4. $\mathrm{diam}(\mathcal{F}_K)=\mathrm{rank}\,(T+\tilde{K})-2|V|$. *Therefore,* $\mathrm{rank}\,(T+\tilde{K})=\mathrm{rank}\,K + 2|V|$.

The next claim follows easily from 7.3.1 and 7.3.2 and from the construction of $T+\tilde{K}$.

7.3.5. *$M(T+\tilde{K})$ has no coloops. Moreover, for any $i\in V$, i ,i' and i'' are all in the same series-class of $M(T+\tilde{K})$.*

We do not see how to obtain Theorem 7.3 as a direct corollary of Theorem 1.2, but rather, we prove the result using the methods of Section 3. To do this we need to show that the block diagonal structure of $\tilde{K}$ is preserved in the decomposition. Toward this end, we require the following extension of 3.1. The proof is a minor variation of the proof of 3.1, and is left to the reader.

7.3.6. *Let $T+K$ be a V by V mixed skew-symmetric matrix such that $T+K$ is critical. Moreover, let X be a series-class of $M(T+K)$ such that* $\text{rank}\,(T+K)[V,V-X]=\text{rank}\,(T+K)[V-X]$, *and $|X|$ is odd. Then there exists a rank-splitting decomposition T_1+K_1, T_2+K_2 of $T+K$ such that*

i. T_1+K_1 and T_2+K_2 are V by V mixed skew-symmetric matrices and, for each set $S\subseteq V$ such that $K[S,V-S]=0$, we have $K_1[S,V-S]=K_2[S,V-S]=0$,
ii. X supports T_1+K_1 and $\text{rank}\,T_1+K_1=|X|-1$,
iii. for each $x\in X$, $\{x\}$ is a series-class of $M(T_2+K_2)$, and
iv. if Y is a series class of $M(T_2+K_2)$ that is disjoint from X, then Y is a series-class of $M(T+K)$.

It is also necessary to maintain the symmetry between V' and V'' in the decomposition of $T+\tilde{K}$, however this is quite straightforward.

7.3.7. *Let $\tilde{K}_1$ be a $\tilde{V}$ by $\tilde{V}$ skew-symmetric matrix such that the nonzero entries of $\tilde{K}_1$ are all in $\tilde{K}_1[V']$ or in $\tilde{K}_1[V'']$. If $\tilde{K}_1\neq 0$ then* $\text{rank}\,T+\tilde{K}<\text{rank}\,\tilde{K}_1+\text{rank}\,(T+\tilde{K}-\tilde{K}_1)$.

Proof. Suppose to the contrary that $\tilde{K}_1, T+\tilde{K}-\tilde{K}_1$ is a rank-splitting decomposition of $T+\tilde{K}$. We can find a rank-splitting decomposition $\tilde{K}_1', \tilde{K}_1''$ of $\tilde{K}_1$ such that $\tilde{K}_1'[V'']=0$ and $\tilde{K}_1''[V']=0$. By possibly swapping V' and V'' we may assume that $\tilde{K}_1'[V']\neq 0$. Now, $\tilde{K}_1', T+\tilde{K}-\tilde{K}_1'$ is a rank-splitting decomposition of $T+\tilde{K}$. However, $\text{rank}\,(T+\tilde{K}-\tilde{K}_1')\geq \text{rank}\,T[V\cup V']+\text{rank}\,(\tilde{K}-\tilde{K}_1')[V'']=\text{rank}\,T[V\cup V']+\text{rank}\,K=\text{rank}\,T+\tilde{K}$. This contradiction proves the result. ∎

In order to use induction to prove 7.3, we need to consider a slightly broader class of matrices. We now describe the type of matrices that will be used in the decomposition.

Let L be a V by V skew-symmetric matrix, and let L' and L'' be copies of L with row and column labels V' and V''. Now define

$$\tilde{L} := \begin{array}{c} \\ V \\ V' \\ V'' \end{array} \begin{array}{c} \begin{array}{ccc} V & V' & V'' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & L' & 0 \\ 0 & 0 & L'' \end{pmatrix} \end{array}.$$

Now, for $X \subseteq V$, let G_X be a graph with vertex set $V \cup V' \cup V''$ and edge set $\{ii' : i \in X\} \cup \{ii'' : i \in X\}$, and let T_X be the Tutte matrix of G_X. We are interested in mixed skew-symmetric matrices of the form $T_X + \tilde{L}$ such that

i. $\operatorname{rank}(T_X + \tilde{L}) = \operatorname{rank} T_X + \operatorname{rank} L$,
ii. $M(T_X + \tilde{L})$ has no coloops, and, for each $x \in V - X$, $\{x\}$, $\{x'\}$, and $\{x''\}$ are all series-classes.

Note that our matrix $T + \tilde{K}$ has this form. Suppose that we have a rank-splitting decomposition of $T + \tilde{K}$ into such matrices, and let $T_X + \tilde{L}$ be one such matrix in the decomposition. Note that, by 7.3.7, $T_X \neq 0$. Moreover, by 7.3.7 and 2.2 it is easy to prove the following result.

7.3.8. *X is a cover of L.*

7.3.9. *If $X \cup X' \cup X''$ is a series-class of $M(T_X + \tilde{L})$, then $|X|$ is odd and* $\operatorname{rank} T_X + \tilde{L} = 3|X| + 2\operatorname{rank} L[X, V - X] - 1$.

Proof. If $X \cup X' \cup X''$ is a series-class of $M(T_X + \tilde{L})$, then $\operatorname{rank}(T_X + \tilde{L}) = \operatorname{rank}(T_X + \tilde{L})[\tilde{V}, \tilde{V} - (X \cup X' \cup X'')] + 3|X| - 1 = 2\operatorname{rank} L[X, V - X] + 3|X| - 1$. ∎

If each of the matrices in the rank-splitting decomposition have the form described in 7.3.9, then 7.3 follows easily. Thus we need to consider the case that $X \cup X' \cup X''$ contains more than one series-class of $M(T_X + \tilde{L})$. In this case $T_X + \tilde{L}$ can be further decomposed by 3.3 and 7.3.6; the details are left to the reader. This completes our outline of the proof of 7.3.

8. A conjecture of Bouchet and Jackson

In this section we use 7.3 to prove a conjecture of Bouchet and Jackson. This conjecture provides an alternative min-max theorem for $\operatorname{diam}(\mathcal{F}_K)$.

Let $M_1 = (V_1, \mathcal{F}_1)$ and $M_2 = (V_2, \mathcal{F}_2)$ be delta-matroids on disjoint ground sets. The *direct sum* of M_1 and M_2 is the delta-matroid $M_1 \oplus M_2$ on ground set $V_1 \cup V_2$ and with feasible sets $\{F_1 \cup F_2 : F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2\}$. Given any delta-matroid $M = (V, \mathcal{F})$ there is a unique maximal partition $(V_1, \ldots, V_k)$

of V and delta-matroids $M_1,\ldots,M_k$ with ground sets $V_1,\ldots,V_k$ respectively such that $M = M_1 \oplus \cdots \oplus M_k$. (For a delta-matroid represented by a skew-symmetric matrix K the sets $V_1,\ldots,V_k$ are the vertex sets of the connected components of $G(K)$.) We let $\mathrm{odd}(M)$ denote the number of odd sets among $V_1,\ldots,V_k$. The following results are straightforward.

8.1. *If M is an even delta-matroid, then* $\mathrm{diam}(M) \le |V| - \mathrm{odd}(M)$.

8.2. *If $N=(V,\mathcal{F})$ is a delta-matroid and $Y \subseteq V$ such that $N \setminus Y$ and N/Y both contain feasible sets, then* $\mathrm{diam}(N\setminus Y) \le \mathrm{diam}(N/Y) + 2|Y|$.

These bounds on the diameter combine to give the following bound.

8.3. *If M and N are even delta-matroids such that $M = N \setminus Y$, then* $\mathrm{diam}(M) \le |V| - (\mathrm{odd}(N/Y) - 2|Y|)$.

We will prove the conjecture of Bouchet and Jackson, personal communication, that the bound given by 8.3 is attained for a linear delta-matroid.

8.4. *If K is a V by V skew-symmetric matrix, then there exists a skew-symmetric matrix L with rows and columns indexed by $V \cup Y$ such that $DM(K) = DM(L)\setminus Y$ and* $\mathrm{diam}(DM(K)) = |V| - (\mathrm{odd}(DM(L)/Y) - 2|Y|)$.

Proof. If M and N are delta-matroids such that $M = N\setminus Y$ and S is a set of elements of M, then $(M \Delta S) = (N \Delta S)\setminus Y$. Hence the conclusion of the theorem is invariant under twisting in $DM(K)$. Therefore, by Theorem 7.3 and possibly twisting, we may assume that there exist disjoint odd subsets $X_1,\ldots,X_k$ of V and V by V skew-symmetric matrices $K_1,\ldots,K_k$ such that $K = K_1 + \cdots + K_k$, X_i is a cover of K_i for each i, and

$$\mathrm{diam}(\mathcal{F}_K) = \sum_{i=1}^{k} (|X_i| + 2\mathrm{rank}\ K_i[X_i, V - X_i] - 1).$$

If there exists some element $x \in V - (X_1 \cup \cdots \cup X_k)$, then we can define $X_{k+1} = \{x\}$ and $K_{k+1} = 0$. Whence, we may assume that $(X_1,\ldots,X_k)$ is a partition of V. Let $Y_1,\ldots,Y_k$ be pairwise disjoint sets such that $|Y_i| = \mathrm{rank}\, K_i[X_i, V - X_i]$ for each i. Now, for each i, let B_i be an X_i by Y_i matrix whose columns span the column space of $K_i[X_i, V - X_i]$. Now let $Y = Y_1 \cup \cdots \cup Y_k$ and let L be the skew-symmetric matrix with rows and columns indexed by $V \cup Y$ such that $L[V] = K$, $L[X_i, Y_i] = B_i$ for each i, and all other entries are zero; see Figure 1.

For each i let L_i be the skew-symmetric matrix with rows and columns indexed by $V \cup Y$ such that $L_i[V] = K_i$, $L_i[X_i, Y_i] = B_i$, and all other entries are zero. Note that $L = L_1 + \cdots + L_k$, X_i is a cover of L_i and $\mathrm{rank}\, L_i[X_i, (V \cup$

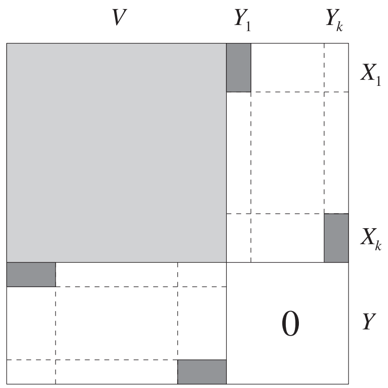

Figure 1. L

$Y) - X_i] = \operatorname{rank} K_i[X_i, V - X_i]$. Therefore, it follows that $\operatorname{diam}(DM(L)) = \operatorname{diam}(DM(K))$. We claim that $\operatorname{diam}(DM(K)) = |V| - (\operatorname{odd}(DM(L)/Y) - 2|Y|)$. If Y is empty then $L = K$ and $K_i[X_i, V - X_i] = 0$ for each i, so the claim follows. We prove the claim inductively by contracting the elements of Y one at a time. Suppose that $y \in Y_1$, and let $x \in X_1$ such that $L_{xy} \neq 0$. By scaling, we may assume that $L_{xy} = 1$. Note that $\operatorname{diam}(DM(L)/\{y\}) = \operatorname{diam}(DM(L * \{x,y\}) \setminus \{y\})$. Let a denote the vector $L[(V \cup Y) - \{x,y\}, x]$, let b denote the vector $L[(V \cup Y) - \{x,y\}, y]$, and let $D = L[(V \cup X) - \{x,y\}]$. Thus we have

$$L = \begin{array}{c} \\ x \\ y \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{pmatrix} 0 & 1 & -a^{\mathrm{t}} \\ -1 & 0 & -b^{\mathrm{t}} \\ a & b & D \end{pmatrix} \end{array}$$

and

$$L * \{x,y\} = \begin{array}{c} \\ x \\ y \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{pmatrix} 0 & -1 & b^{\mathrm{t}} \\ 1 & 0 & -a^{\mathrm{t}} \\ -b & a & D - ab^{\mathrm{t}} + ba^{\mathrm{t}} \end{pmatrix} \end{array}.$$

Let W denote $L * \{x,y\}[(V \cup X) - \{y\}]$. Now, for each $i \in \{1, \ldots, k\}$ let a_i denote the vector $L_i[(V \cup Y) - \{x,y\}, x]$, let

$$W_1 := \begin{array}{c} x \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} x & \phantom{L_1[(V \cup Y) - \{x,y\}] - a_1 b^t + ba_1^t} \end{array} \\ \begin{pmatrix} 0 & b^{\mathrm{t}} \\ -b & L_1[(V \cup Y) - \{x,y\}] - a_1 b^{\mathrm{t}} + b a_1^{\mathrm{t}} \end{pmatrix} \end{array},$$

and, for $i \neq 1$, let

$$W_i := \begin{matrix} & x & \\ x & \begin{pmatrix} 0 & 0 \\ 0 & L_i[(V \cup Y) - \{x, y\}] - a_i b^{\mathrm{t}} + b a_i^{\mathrm{t}} \end{pmatrix} \end{matrix}.$$

Thus $W = W_1 + \cdots + W_k$. Note that, for $i \neq 1$, all nonzero entries of a_i are indexed by elements of X_i, and all nonzero entries of b are indexed by elements of X_1. Then, it is straightforward to show that, for each $i \neq 1$, X_i is a cover of W_i and $\operatorname{rank} W_i[X_i, (V \cup (Y - \{y\})) - X_i] = \operatorname{rank} L_i[X_i, (V \cup Y) - X_i]$. Moreover, X_1 is a cover of W_1 and, by 2.1, $\operatorname{rank} W_1[X_1, (V \cup (Y - \{y\})) - X_1] = \operatorname{rank} L_1[X_1, (V \cup Y) - X_1] - 1$. Consequently, $\operatorname{diam}(DM(W)) = \operatorname{diam}(DM(L)) - 2$. Repeating this for the other elements of Y, we obtain $\operatorname{diam}(DM(L)/Y) = \operatorname{diam}(DM(K)) - 2|Y|$. Moreover, $\operatorname{diam}(DM(L)/Y) = |V| - \operatorname{odd}(DM(L)/Y)$. Thus, $\operatorname{diam}(DM(K)) = |V| - (\operatorname{odd}(DM(L)/Y) - 2|Y|)$, as required. ∎

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James Geelen

Department of Combinatorics and Optimization
University of Waterloo
Waterloo
Ontario, N2L 3G1
Canada
jfgeelen@uwaterloo.ca

Satoru Iwata

Department of Mathematical Informatics
University of Tokyo
Tokyo 1138656
Japan
iwata@sr3.tu-tokyo.ac.jp