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Embedding grids in surfaces

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Abstract

We show that if a very large grid is embedded in a surface, then a large subgrid is embedded in a disc in the surface. This readily implies that: (a) a minor-minimal graph that does not embed in a given surface has no very large grid; and (b) a minor-minimal k-representative embedding in the surface has no very large grid. Similar arguments show (c) that if G is minimal with respect to crossing number, then G has no very large grid. This work is a refinement of Thomassen (J. Combin. Theory Ser. B 70 (1997) 306).

1. Introduction

As part of their "Graph Minors" project, Robertson and Seymour [10] proved the following result.

Theorem 1. *For any surface Σ, there are only finitely many graphs that do not embed in Σ and that are minor-minimal with this property.*

The proof by Robertson and Seymour is long and difficult. However, there is now a remarkably accessible proof based on their original ideas. This proof is summarized in the following three results.

(1) *Let b be an integer and let $G_1, G_2, \ldots$ be an infinite sequence of graphs each with branch-width at most b. Then there exist $i < j$ such that G_i is a minor of G_j.*
(2) *For any positive integer k there is an integer $f(k)$ such that if G is a graph with branch-width at least $f(k)$, then G contains the k by k grid as a minor.*

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(3) *For any surface Σ there exists an integer k such that, if G is a graph that has a k by k grid as a minor, and G does not embed in Σ, then there exists an edge e of G such that $G - e$ does not embed in Σ.*

A graph has small branch-width (or, similarly, tree-width) if it can be decomposed across noncrossing separations into small pieces. (As we do not need these notions here we shall not give precise definitions.) The *k by k grid* is the graph on k^2 vertices $v_{i,j}$, $1 \leq i, j \leq k$, such that $v_{i,j}$ is adjacent to $v_{i\pm1,j}$ and $v_{i,j\pm1}$, whenever the subscripts are between 1 and k.

Robertson and Seymour's proof of (1), in [11], was stated in terms of tree-width and relied on a result of Thomas [13]. These results have simpler proofs for branch-width; see Geelen et al. [3]. A marvellously elementary proof of (2) is given by Diestel et al. [1]. (Actually, their proof is for tree-width, rather than branch-width. These two statements are equivalent, and their proof becomes slightly easier in the branch-width version.) Thomassen [14] gives an elegant short proof of (3), the final link in the chain, and we provide another short proof in this paper. The main distinctions between [14] and the current work are: (a) we highlight more directly the embeddings of grids in surfaces; and (b) we demonstrate that essentially the same arguments work for crossing number rather than genus.

A slightly different approach is given by Mohar [8] (see also, [9]).

For this work, a *surface* is a compact connected 2-manifold without boundary. The Euler characteristic of a surface Σ is denoted $\chi(\Sigma)$ and the *Euler genus* $\varepsilon(\Sigma)$ of Σ is $2 - \chi(\Sigma)$. We note that if Σ is obtained from the sphere by the addition of k crosscaps and h handles, then $\varepsilon(\Sigma) = k + 2h$. Thus, for example, the torus and Klein bottle both have Euler genus 2.

The main purpose of this paper is to prove that, when a large grid is embedded on a fixed surface, most of the grid is embedded in a plannar way.

Lemma 1. *Let G be a grid embedded in a surface Σ. Then, the number of noncontractible 4-cycles in the grid is at most $9\varepsilon(\Sigma)$.*

(We believe that the correct upper bound is actually $2\varepsilon(\Sigma) + 8$, which is the most we have been able to achieve.)

Consider a very large grid embedded on a fixed surface Σ. By Lemma 1, most of the 4-cycles in the grid are contractible. Contractible cycles bound discs, and in Σ these small discs are glued together along the edges of the grid. Thus, we see that much of the grid is embedded in a planar way on Σ. In particular, some large subgrid is embedded in a disc. These observations lead to an easy proof of (3). Moreover, we can also deduce that a minor-minimal r-representative embedding in a surface has only bounded sized grid minors.

This paper is an amalgamation of joint research of the first two authors with independent work of the third author. The methods in Section 2 are those of the first two authors; the third author proved analogues of Lemmas 4 and 5 with different techniques. The material on crossing numbers is essentially due to the third author; in particular, Theorem 4 and the "bounded path-width" conjecture are originally due to him.

2. Grids in surfaces

In this section, we prove Lemma 1. The main topological result we require is the following. (Here, and for the rest of work, if $\mathcal{C}$ is a set of sets, then we use $\cup\mathcal{C}$ to denote $\cup_{\gamma\in\mathcal{C}}\gamma$.)

Lemma 2. *Let Σ be a surface and let $\mathcal{C}$ be a set of pairwise disjoint simple closed noncontractible curves in Σ such that some component of $\Sigma\backslash(\cup\mathcal{C})$ has all the curves in $\mathcal{C}$ in its boundary. Then $|\mathcal{C}| \le \varepsilon(\Sigma)$.*

This result, perhaps with a different bound, also follows from [6] (see also p. 107 of [9]).

Proof. Let Γ be a component of $\Sigma\backslash(\cup\mathcal{C})$ such that all the curves of $\mathcal{C}$ are contained in the boundary of Γ. Suppose that $\mathcal{C}$ contains a separating curve C, and let Σ_1 and Σ_2 be the components of $\Sigma\backslash C$, where Γ is contained in Σ_1. Since C is noncontractible, Σ_2 is not a disc, and, hence, contains a nonseparating closed curve C'. Replacing C with C' in $\mathcal{C}$ reduces the number of separating curves in $\mathcal{C}$. Thus, we may assume that $\mathcal{C}$ contains only nonseparating curves.

Let n be the number of components of $\Sigma\backslash(\cup\mathcal{C})$, let h be the number of 2-sided curves in $\mathcal{C}$, and let k be the number of 1-sided curves in $\mathcal{C}$. Since every curve is incident with Γ and every 1-sided curve is incident only with Γ, some collection of $h-(n-1)$ 2-sided curves in $\mathcal{C}$ does not separate Σ. Each such curve contributes a handle to Σ. Since the 1-sided curves are pairwise disjoint, each contributes a cross-cap to Σ. Thus, $\varepsilon(\Sigma) \ge 2(h-(n-1))+k$.

If $n = 1$, then $\varepsilon(\Sigma) \ge 2h + k \ge h + k = |\mathcal{C}|$, as required. Thus, we may assume that $n > 1$.

As $\mathcal{C}$ does not contain a separating curve, every component of $\Sigma\backslash(\cup\mathcal{C})$ other than Γ has at least two curves in its boundary. Each curve in the boundary of such a component is 2-sided and has Γ on the other side, so $h \ge 2(n-1)$. Therefore, $\varepsilon(\Sigma) \ge 2(h-(n-1)) + k \ge h + k = |\mathcal{C}|$, as required. □

We obtain Lemma 1 as a consequence of our next result. A subgraph H of a graph G *separates* G if there exist proper subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $H = G_1 \cap G_2$. The point is, if H is a nonseparating subgraph of G and G is embedded in a surface Σ, then, relative to the induced embedding of H in Σ, the rest of G is contained in (the closure of) one face of H.

Lemma 3. *Let G be a connected graph embedded in a surface Σ and let $\mathcal{C}$ be a set of pairwise disjoint cycles in G such that $\cup\mathcal{C}$ does not separate G. If every cycle in $\mathcal{C}$ is noncontractible in Σ, then $|\mathcal{C}| \le \varepsilon(\Sigma)$.*

Proof. Let $H = \cup\mathcal{C}$. Since H does not separate G, there is one component of $\Sigma\backslash H$ in which the rest of G is embedded. Since G is connected, each cycle in H is in the boundary of this component. By Lemma 2, $|\mathcal{C}| \le \varepsilon(\Sigma)$. □

We are now ready for the proof of Lemma 1.

Proof of Lemma 1. Let G be a k by k grid embedded in a surface Σ. We give the 4-cycles coordinates (i, j), where $1 \le i, j \le k-1$, in the natural way. Considering the coordinates modulo 3 partitions the 4-cycles of G into nine sets. Note that the union of cycles in any one of these sets is nonseparating in G. Therefore, by Lemma 3, each of these sets contains at most $\varepsilon(\Sigma)$ noncontractible cycles. Hence, there are at most $9\varepsilon(\Sigma)$ noncontractible 4-cycles in G. □

We now derive two easy consequences of Lemma 3. The first shows that, if a very large grid is embedded in a surface, then a large subgrid is embedded in a disc, while the second says that if a very large grid is embedded in a surface, then some ring of 4-cycles surrounding the centre of the grid is embedded in a cylinder. (These results follow slightly more easily from Lemma 1, but Lemma 3 gives sharper bounds.)

The *boundary cycle* of a k by k grid is the cycle that bounds the infinite face in the usual planar embedding of the grid (that is, consists of the subgraph induced by the $v_{i,j}$ such that $\{i, j\} \cap \{1, k\} \neq \emptyset$).

Lemma 4. *Let t, k, n be positive integers such that $n \geq t(k + 1)$ and let G be an n by n grid. If G is embedded in a surface Σ of Euler genus at most $t^2 - 1$, then some k by k subgrid of G is embedded in a closed disc in Σ such that the boundary cycle of the k by k grid is the boundary of the disc.*

Proof. Clearly G contains t^2 pairwise disjoint k by k subgrids such that no two vertices from distinct subgrids are adjacent in the grid. By Lemma 3, not all of these subgrids can have noncontractible 4-cycles, so one has only contractible 4-cycles. Each of these bounds a closed disc and the union of these closed discs is the required closed disc. □

(It is straightforward to embed the tk by tk grid in an orientable surface of genus at most $t^2 - 1$ so that no k by k subgrid is embedded in a closed disc. Lemma 4 shows this is not true of the $t(k + 1)$ by $t(k + 1)$ grid.)

For the second use, let G be a k by k grid. Let $1 \leq t \leq (k/2) - 1$ be a given integer. A *t-collar* is a subgraph of G induced by, for some positive integer $i \leq k/2 - t + 1$, the vertices of G at distance at least $i - 1$ and at most $i + t - 1$ from the boundary cycle of the grid. The *exterior cycle* and *interior cycle* of this t-collar are the cycles induced by the vertices at distance $i - 1$ and distance $i + t - 1$, respectively. We note that the exterior cycle is the boundary cycle of a $(k - 2i + 2)$ by $(k - 2i + 2)$ subgrid.

We will only use 2-collars.

Lemma 5. *Let Σ be a surface of Euler genus ε and let t be a positive integer. Let $k \geq 2(\varepsilon + 1)(t + 1)$. Let G be a k by k grid embedded in Σ. Then G contains a t-collar embedded in a cylinder in Σ.*

Proof. For each $i = 1, 2, \ldots, \lfloor k/2 \rfloor - t + 1$, let C_i be the t-collar consisting of the vertices at distance at least $i - 1$ and at most $i + t - 1$ form the boundary cycle of G.

By Lemma 3, one of the $\varepsilon + 1$ collars C_i, $i \in \{1, 1+(t+1), 1+2(t+1), \ldots, 1 + \varepsilon(t+1)\}$, contains only contractible 4-cycles. (Lemma 3 does not apply immediately, since C_i and C_{i+t} might have vertices that are adjacent in G. However, when we apply Lemma 3 on some set of 4-cycles that have adjacent vertices, we can delete the connecting edges and then apply Lemma 3.) For one in which all 4-cycles are contractible, the 4-cycles bound discs and the union of these discs is a cylinder. □

3. Applications

Our first application is to prove the following form of (3).

Theorem 2. *Let Σ be a surface of Euler genus ε. Let $n \geq \lceil\sqrt{\varepsilon+3}\rceil\,(6\varepsilon+8)$ and let G be a graph containing an n by n grid as a minor. If, for every $e \in E(G)$, $G-e$ embeds in Σ, then G embeds in Σ.*

In the proof, we shall need the concept of a bridge. Let G be a graph and let H be a subgraph of G. A *bridge* of H is either an edge of G, not in H, together with its ends, if both ends are in H, or a component of $G-V(H)$, together with any edge of G incident with a vertex in that component, and the ends from H of such edges. We note that a subgraph H of G is separating if and only if it has at least two bridges.

An edge e of a k by k grid is *central* if either (1) $k = 2m-1$ is odd and e is incident with $v_{m,m}$ or (2) $k = 2m$ is even and e is in the 4-cycle induced by $v_{m,m}$, $v_{m,m+1}$, $v_{m+1,m}$, and $v_{m+1,m+1}$.

Proof. Let e be any edge of G. An embedding of $G-e$ in Σ can be used to obtain an embedding of G in Σ', obtained from Σ by adding a handle. Let M be a minimal subgraph of G that contracts to the n by n grid G_n.

Since $\varepsilon(\Sigma') = \varepsilon(\Sigma)+2 = \varepsilon+2$, Lemma 4 implies that some $(6\varepsilon+7)$ by $(6\varepsilon+7)$ subgrid G' of G_n is embedded in a disc in Σ'. Let K be a minimal subgraph of M that contracts to G'. By the minimality of K, the edges of K that are contracted do not contain a cycle; in general, if the contraction of a graph by an acyclic set is contained in a disc, then so is the original graph. Thus K is also embedded in a disc.

Let e be an edge of K that, after contraction of other edges, becomes a central edge of G'. Let K' be a minimal subgraph of $K-e$ that contracts to a $(6\varepsilon+6)$ by $(6\varepsilon+6)$ grid. Embed $G-e$ in Σ. By Lemma 5, there is a 2-collar J of K' that is embedded in a cylinder. Let I be the interior cycle of J, let E be the exterior cycle of J, and let C be the cycle in $J-V(I\cup E)$.

Let B_E be the bridge of C in G that contains E, let B_I be the bridge of C in G that contains I, and let $\mathcal{B}$ denote the set of all other bridges of C. We claim that we can arrange the embeddings of G in Σ' and $G-e$ in Σ so that if $B \in \mathcal{B}$, then B is in the cylinder bounded by $C \cup I$ either in both embeddings or in neither embedding.

Let $\mathcal{B}'$ denote the subset of $\mathcal{B}$ consisting of those bridges that are in the cylinder bounded by $C \cup I$ in one embedding but not the other. Let $\mathcal{O}$ denote the "overlap diagram" for the bridges in $\mathcal{B}'$: its vertices are the bridges in $\mathcal{B}'$ and two bridges are adjacent in $\mathcal{O}$ if they cannot be simultaneously embedded on the same side of C.

Since every bridge in $\mathcal{B}$ must be embedded in the cylinder bounded by $E \cup I$, $\mathcal{O}$ is bipartite. One side of the bipartition corresponds to those bridges that are inside the cylinder bounded by $C \cup I$ in one embedding and the other side of the bipartition corresponds to the bridges that are outside in the same embedding. Because we are in the cylinder, we can simply switch the embeddings of the bridges in $\mathcal{B}'$ in one of the two embeddings, so that each one is either in both cylinders or in neither cylinder.

Thus, the subgraphs of G that are contained in the cylinders bounded by $C \cup I$ are the same in both embeddings.

In order to obtain an embedding of G in Σ, let Δ_1 be the disc in Σ' bounded by C and let Δ_2 be a small closed disc in Δ_1, disjoint from G. Let Γ be the cylinder in Σ' obtained by deleting the interior of Δ_2 from Δ_1. It is bounded by C and some other simple closed curve.

We obtain an embedding of G in Σ as follows. From the embedding of $G - e$ in Σ, we use the embedding of $B_E \cup C$, together with everything embedded in the cylinder in Σ bounded by $C \cup E$. Complete the embedding of G by replacing the cylinder in Σ that is bounded by $C \cup I$ with the cylinder Γ, so that the two copies of C are identified. $\square$

An embedding of a graph G in a surface Σ is *r-representative* if every noncontractible closed curve in Σ intersects the graph at least r times. The embedding is *minor-minimal r-representative* if it is r-representative and the deletion or contraction (in the surface) of any edge produces an embedding which is not r-representative. (An introduction to representativity is given by Robertson and Vitray [12].) We note the following result.

Theorem 3. *Let G be a minor-minimal r-representative embedding in a surface Σ of Euler genus ε. Let $n \geq \lceil\sqrt{\varepsilon + 1}\rceil(r + 2)$. Then G has no n by n grid minor.*

Proof. If G has an n by n grid as a minor, then by Lemma 4 G has a subgraph M that contracts to an $r+1$ by $r+1$ grid such that M is embedded in a disc with the outer boundary of M being the boundary of the disc. Delete any central edge e of such a subgraph. Suppose there is a noncontractible curve γ having fewer than r intersections with $G - e$. Then γ must intersect e and, therefore, must come into and leave the disc containing M. But then γ must cross the remaining r by r grid at least r times, a contradiction. $\square$

We note that (using (1) and (2)) Theorem 2 implies Theorem 1. It is not clear that Theorem 3 implies the number of minor-minimal r-representative embeddings in a fixed surface is finite (up to a homeomorphism of the surface to itself). The problem is that it is possible that the graph G can be a minor of the graph H and both have minor-minimal r-representative embeddings. This is because the same graph can have two embeddings in the same surface so that the embeddings have different representatives. That the number of minor-minimal r-representative embeddings is finite is proved by other means in [2, 5, 7]. (Alternatively, one could trundle out the machinery for the bounded branch-width result (1) and apply it in the surface, but it is a different theorem.)

In a slightly different direction, let $\mathrm{cr}(G)$ denote the crossing number of G, i.e. the minimum number of pairwise crossings of edges in a drawing of G in the plane.

Theorem 4. *Let $k \geq 1$ be an integer. Let G be a graph such that $\mathrm{cr}(G) \leq k$ and, for every edge e of G, $\mathrm{cr}(G - e) < k$. Let $n \geq \lceil\sqrt{2k + 1}\rceil(12k - 5)$. If G contains an n by n grid as a minor, then $\mathrm{cr}(G) < k$.*

This result is proved in much the same manner as Theorem 2. We start by finding a large grid with no crossings at all, delete a central edge e, draw $G - e$ with fewer than k crossings and then use a collar of the large grid that has no crossings in the second drawing. This allows us to draw G with fewer than k crossings. Alternatively, one could adapt the methods of [14].

There is obviously a version of this last result that also applies to the crossing number of a graph drawn on some surface, not just the plane. This does not seem to be of independent interest at the moment.

Theorem 4 does not imply a useful finiteness result, since the property of having crossing number $\leq k$ is not closed under contraction of an edge. Even for $k = 2$, the number of graphs that have crossing number 2 and all proper subgraphs have crossing number at most 1 is infinite.

One example of an infinite class is obtained by taking three paths with common ends, but otherwise disjoint, each of length at least 2, doubling their edges and adding a new vertex adjacent to exactly one internal vertex of each of the paths. It is easy to see that this graph has crossing number 2 and that the deletion of any edge reduces the crossing number to at most 1. One of these graphs gives another as a minor by deleting one of two parallel edges (reducing the crossing number to 1) and contracting the second of the two parallel edges (raising the crossing number back to 2). As all known examples of infinite crossing-critical families have some "repetitive structure", we put forward the following.

Conjecture 1. *Let k be a positive integer. Then there is an integer $f(k)$ such that if G is a graph for which $\mathrm{cr}(G) = k$ and $\mathrm{cr}(G - e) < k$ for all edges e of G, then the path-width of G is at most $f(k)$.*

This conjecture has recently been proved [4].

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