Matching Theory

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Historically Matching Theory has enjoyed a close connection with algebra; this connection is particularly strong for the maximum cardinality matching problem. Indeed the early results on bipartite matching were motivated by and even stated in terms of matrices. Later, for general graphs, Tutte discovered a beautiful formulation of the matching problem as a matrix rank problem. Tutte was able to use this formulation to great effect in proving his famous matching theorem.

Despite this close connection to algebra much of the literature on Matching Theory focuses primarily on purely graphical methods. Of course, this position is easily justified as augmenting path algorithms provided the first algorithmic solutions to the matching problem and remain the most efficient solution technique. Nevertheless, we survey some the many applications of Tutte's algebraic formulation in Matching Theory. In particular, we will prove the Tutte–Berge Theorem and the Edmonds–Gallai Structure Theorem, and we develop randomized and deterministic algorithms for the maximum cardinality matching problem. The algebraic approach attains elegance and simplicity at the cost of computational efficiency.

These notes were developed for a minicourse on Matching Theory at the Euler Institute for Discrete Mathematics and its Applications, March 5–9, 2001.

1 Some definitions

A *matching* of a graph $G = (V, E)$ is a subset M of E with the property that no two edges in M share a common end. We are primarily interested in the *maximum matching problem*; that is, th problem of finding a matching of maximum cardinality. For simplicity, we refer to a matching of maximum cardinality as a *maximum matching*, and let $\nu(G)$ denote the size of a maximum matching in G.

Let M be a matching in G, and let v be a vertex of G. If v is the end of an edge in M, then we say that M *saturates* v. The set of all vertices saturated by a particular matching is called a *matchable set* of G. Note that, since each edge saturates two vertices, matchable sets have even cardinality. A matching that saturates every vertex is called *perfect*. The *perfect matching problem* is the problem of deciding whether a graph has a perfect matching. Obviously, G has a perfect matching if and only if $|V| = 2\nu(G)$, so the perfect matching problem is a special case of the maximum matching problem.

A vertex is said to be *avoided* by M if it is not saturated by M. The minimum number of vertices avoided by any matching is called the *deficiency* of G, and is denoted by $\text{def}(G)$. Evidently, $\text{def}(G) = |V| - 2\nu(G)$. Therefore, computing the deficiency is equivalent to computing the size of a maximum matching.

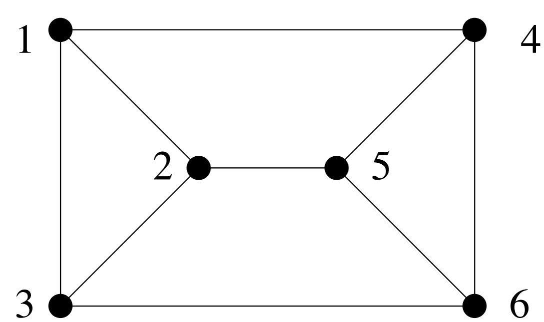

Figure 1: A graph

Consider the graph G in Figure 1. The edge set $\{13, 45\}$ is a matching and hence $\{1, 3, 4, 5\}$ is a matchable set. This is not a maximum matching, in fact, G has a perfect matching: $\{14, 25, 36\}$. Thus $\nu(G) = 3$ and $\text{def}(G) = 0$.

Lovász and Plummer [12] and Cook, Cunningham, Pulleyblank, and Schrijver [2] provide more comprehensive treatments of Matching Theory, particularly regarding weighted matching problems and augmenting path methods.

2 Algebraic formulations

Let $G = (V, E)$ be a bipartite graph with bipartition (V_r, V_c), and let $(z_e \,:\, e \in E)$ be algebraically independent commuting indeterminates. Now, define a V_r by V_c matrix X, such that $X_{i,j} = z_e$ if $ij = e \in E$ and $X_{i,j} = 0$ otherwise. We call X the *bipartite–matching matrix* of G.

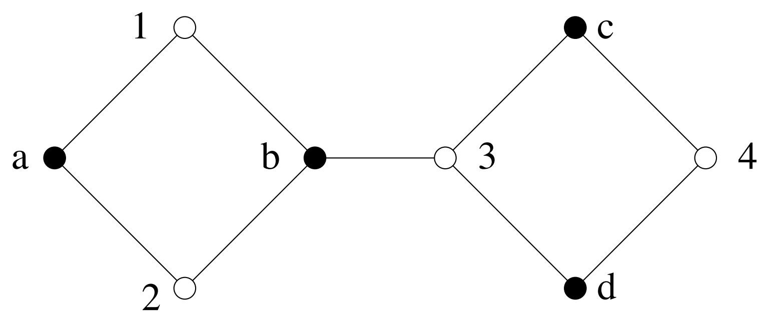

Figure 2: A bipartite graph

The following matrix is the bipartite matching matrix of the graph in Figure 2.

$$X = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} z_{a1} & z_{a2} & 0 & 0 \\ z_{b1} & z_{b2} & z_{b3} & 0 \\ 0 & 0 & z_{c3} & z_{c4} \\ 0 & 0 & z_{d3} & z_{d4} \end{pmatrix} \end{array}.$$

Note that,

$$\det X = z_{a1}z_{b2}z_{c3}z_{d4} - z_{a1}z_{b2}z_{c4}z_{d3} - z_{a2}z_{b1}z_{c3}z_{d4} + z_{a2}z_{b1}z_{c4}z_{d3}.$$

Thus, we see that $\det X$ enumerates the perfect matchings of G. This is clearly true more generally. Suppose that $|V_r| = |V_c|$, and let $\mathcal{M}$ be the set of perfect matchings of G. Then, by considering the permutation expansion of the determinate, we have

$$\det X = \sum_{M \in \mathcal{M}} \sigma_M \prod_{e \in M} z_e,$$

where $\sigma_M = \pm 1$.

Lemma 2.1 *Let X be the bipartite matching matrix of a bipartite graph G. Then G has a perfect matching if and only if X is square and nonsingular.* ∎

The submatrices of X are bipartite matching matrices of vertex induced subgraphs of G. Therefore, we obtain the following strengthening of the previous lemma.

Lemma 2.2 *If X is the bipartite–matching matrix of a bipartite graph G, then $\nu(G) = \text{rank } X$.* ∎

Thus, we have reformulated the bipartite matching problem as a matrix–rank problem. This formulation is not a panacea, as the matrix in question has indeterminate entries, so it is non–trivial to compute its rank. However, we shall see that such formulations easily provide efficient randomized algorithms, and may be used to obtain min–max theorems and efficient deterministic algorithms.

We now progress to matching in general graphs; for this we introduce "Pfaffians". Let $G = (V, E)$ be a simple graph, and let $(z_e : e \in E)$ be algebraically independent commuting indeterminates. We define a V by V skew–symmetric matrix T, called the *Tutte matrix* of G, such that $T_{ij} = \pm z_e$ if $ij = e \in E$, and $T_{ij} = 0$ otherwise. The Tutte matrix was introduced by Tutte, in 1947, in his seminal paper on matching.

The following matrix is the Tutte matrix of the graph in Figure 1. There is some choice for the signs of the entries, however the signing does not effect the rank.

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & z_{12} & z_{13} & -z_{14} & 0 & 0 \\ -z_{12} & 0 & -z_{23} & 0 & -z_{25} & 0 \\ -z_{13} & z_{23} & 0 & 0 & 0 & -z_{36} \\ z_{14} & 0 & 0 & 0 & z_{45} & -z_{46} \\ 0 & z_{25} & 0 & -z_{45} & 0 & z_{56} \\ 0 & 0 & z_{36} & z_{46} & -z_{56} & 0 \end{pmatrix} \end{array}.$$

One can check that,

$$\det T = (z_{12}z_{24}z_{36} - z_{12}z_{36}z_{45} + z_{13}z_{25}z_{36} + z_{14}z_{23}z_{56})^2.$$

Thus, we see the perfect matchings of G are enumerated by the square–root of $\det T$. This is true more generally. The determinant of any skew–symmetric matrix is always a perfect square, and its square–root is called the *Pfaffian*. The Pfaffian has an expansion somewhat like the permutation expansion of the determinant. Suppose that $V = \{1, \ldots, n\}$ and let $\mathcal{M}$ be the set of all perfect matchings of G. The *Pfaffian* of T, denoted *Pf*(T), is defined as follows:

$$\mathrm{Pf}(T) := \sum_{M \in \mathcal{M}} \sigma_M \prod_{\substack{uv \in M \\ u<v}} T_{uv},$$

where σ_M takes the value 1 or -1 as appropriate; see Godsil [5].

Lemma 2.3 *Let T be the Tutte matrix of a graph G. Then G has a perfect matching if and only if T is nonsingular.* ∎

For $A \subseteq V$, we will denote $T[A, A]$ by $T[A]$. Note that $T[A]$ is the Tutte matrix of $G[A]$. Now suppose that A is a maximum cardinality matchable set. By Lemma 2.3, rank $T \geq |A| = 2\nu(G)$. In fact, this inequality holds with equality. To see this we require the following elementary fact about matrices.

Theorem 2.4 *Let Q be a V_r by V_c matrix, let Y_r index a maximal set of linearly independent rows of Q and let Y_c index a maximal set of linearly independent columns of Q. Then, $Q[Y_r, Y_c]$ is square and nonsingular.* ∎

Now consider a set $A \subseteq V$ that indexes a maximal set of linearly independent rows of T. By skew–symmetry, A also indexes a maximal set of linearly independent columns of T. Hence, $T[A]$ is nonsingular. Then, by Lemma 2.3, A is a matchable set of G. Then, rank $T = |A| \leq 2\nu(G)$. Therefore, we obtain the following strengthening of the previous lemma.

Lemma 2.5 *If T is the Tutte matrix of a graph G, then $2\nu(G) = \text{rank}\, T$.* ∎

Thus, we have reformulated the matching problem as a matrix–rank problem.

Exercise Set 2

2.1 Prove Lemma 2.3 without using Pfaffians. (Hint: Consider the permutation expansion of the determinant of T.)

2.2 Prove Theorem 2.4.

2.3 Without considering Pfaffians, prove that the rank of any skew–symmetric matrix is even.

3 Evaluations and randomized algorithms

The matrix–rank formulations do not immediately provide efficient algorithms, as we cannot efficiently perform basic operations on a matrix with indeterminate entries. For example, the determinant of a bipartite matching matrix is a polynomial that may have exponentially many terms. Lovász [11] overcomes this problem by replacing the indeterminates with rational values. If K is a matrix with indeterminate entries, then an *evaluation* of K is any matrix obtained from K by replacing the indeterminates with rational values.

Let $\tilde{K}$ be an evaluation of a matrix K. Note that rank $\tilde{K} \leq$ rank K. Consider the bipartite matching matrix X of the graph given in Figure 2. Now consider the evaluation $\tilde{X}$ obtained by replacing each indeterminate by 1. Now X has rank 4 and $\tilde{X}$ has rank 3. What went wrong? Recall that

$$\det X = z_{a1}z_{b2}z_{c3}z_{d4} - z_{a1}z_{b2}z_{c4}z_{d3} - z_{a2}z_{b1}z_{c3}z_{d4} + z_{a2}z_{b1}z_{c4}z_{d3}.$$

The determinant of X is a nonzero polynomial but we get a root of the polynomial by setting each of the variables to 1. Fortunately, as demonstrated by the following theorem of Zippel [18] and Schwartz [16], roots of polynomials are relatively scarce.

Theorem 3.1 *Let $p(x_1, \ldots, x_k)$ be a nonzero polynomial of degree at most d, and let S be a finite subset of* $\mathbf{R}$. *If $(\hat{x}_1, \ldots, \hat{x}_k)$ is a random element of S^k, then $p(\hat{x}_1, \ldots, \hat{x}_k) \neq 0$ with probability at least $1 - \frac{d}{|S|}$.*

The proof of Theorem 3.1 is left as an exercise. Now we can show that random evaluations of a bipartite matching matrix have the desired rank.

Theorem 3.2 *Let G be a bipartite graph with bipartition (V_r, V_c) and let X be the bipartite matching matrix of G. If $\tilde{X}$ is an evaluation of X with entries chosen independently and at random from $\{1, \ldots, 2|V_r|\}$, then rank $\tilde{X} = \nu(G)$ with probability at least $\frac{1}{2}$.*

Proof. Let $A_r \cup A_c$ be a maximum cardinality matchable set where $A_r \subseteq V_r$ and $A_c \subseteq V_c$. Thus $X[A_r, A_c]$ is nonsingular, and its determinant is a polynomial of degree at most $|V_r|$. So, by Theorem 3.1, $\tilde{X}[A_r, A_c]$ is nonsingular with probability at least $\frac{1}{2}$. Therefore rank $\tilde{X}$ = rank X = $\nu(G)$ with probability at least a $\frac{1}{2}$. ∎

By considering the Pfaffian of the Tutte matrix of a graph, we can prove the following theorem in a similar fashion.

Theorem 3.3 *Let T be the Tutte matrix of a graph $G = (V, E)$. If $\tilde{T}$ is an evaluation of T with entries chosen independently and at random from $\{1, \ldots, |V|\}$, then rank $\tilde{T} = 2\nu(G)$ with probability at least $\frac{1}{2}$.* ∎

These theorems provide efficient randomized algorithms for computing the size of a maximum matching. The reader may not be comfortable with a one in two chance of failure, but the odds improve significantly with repeated trials. If $n = |V|$ then among n independent evaluations of the Tutte matrix, one has the correct rank with probability at least $1 - 1/2^n$.

Let X be the bipartite matching matrix of a bipartite graph G and let $\tilde{X}$ be an evaluation of X. Note that G has a matching of size at least rank $\tilde{X}$. Unfortunately it seems difficult to use $\tilde{X}$ to efficiently obtain a matching of size rank $\tilde{X}$.

Exercise Set 3

3.1 Prove Theorem 3.1 (Hint: Use induction on the number of variables.)

3.2 Consider the Tutte matrix T of the graph in Figure 1. What is the minimum rank of an evaluation of T if none of the indeterminates is assigned to be zero?

3.3 Let X be the bipartite matching matrix of a bipartite graph G. Show that, for any distinct real numbers a and b, there is an evaluation $\tilde{X}$, with entries chosen from $\{a, b\}$, of X such that rank $(\tilde{X})$ = rank (X).

4 Matroids and matrices

We need not discuss matroids in our development of Matching Theory, however, there are very nice connections and it would be a shame for these to pass unnoticed. This section gives a brief introduction to elementary matroid concepts.

Let $M = (V, \mathcal{I})$ be a pair where V is a finite set and $\mathcal{I}$ is a collection of subsets of V; these sets are the *independent* sets of M. We call M a *matroid* if it satisfies the following axioms.

(I0) The empty set is independent.

(I1) Any subset of an independent set is independent.

(I2) If A is a subset of V then all maximal independent subsets of A have the same cardinality.

A *basis* of M is a maximal independent subset. As well as independent sets and bases, we also equip matroids with a rank function. For a subset A of V, we denote by $r_M(A)$, or just $r(A)$, the maximum size of an independent subset of A.

As is evident from the terminology, matroids are related to matrices. Consider a matrix Q whose columns are indexed by V. Call a subset of V independent if it indexes a set of linearly independent columns, and let $\mathcal{I}$ be the set of all independent sets. Evidently, $(V, \mathcal{I})$ is a matroid. This matroid is called the *column–matroid* of Q and is denoted by $M_c(Q)$. The column–matroid of the transpose of Q is called the *row–matroid* of Q and is denoted by $M_r(Q)$. With these new definitions, we can restate Theorem 2.4: *If Y_r is a basis of $M_r(Q)$ and Y_c is a basis of $M_c(Q)$ then $Q[Y_r, Y_c]$ is nonsingular.* Let T be the Tutte matrix of a graph $G = (V, E)$. Since T is skew–symmetric $M_r(T)$ and $M_c(T)$ are the same. Therefore, by Theorems 2.3 and 2.4, each basis $M_c(T)$ is a matchable set of G. Indeed, it is straightforward to show that, the bases of $M_c(T)$ are precisely the maximum cardinality matchable sets of G; see the exercises for this section. Thus, we call $M_c(T)$ the *matching matroid* of G.

Let $M = (V, \mathcal{I})$ be a matroid and let v be an element of M. Then define $M \setminus v = (V - \{v\}, \mathcal{I}')$, where $\mathcal{I}'$ is the set of independent subsets of $V - \{v\}$. Evidently, $M \setminus v$ is a matroid. We say that $M \setminus v$ is obtained from M by *deleting* v. The order in which elements are deleted is clearly not important, so, for a set of elements X, we let $M \setminus X$ denote the matroid obtained from M by deleting each of the elements in X.

We are interested in minimal sets of elements whose deletion decrease the rank of a matroid. In particular, an element whose deletion decreases the rank of a matroid is called a *coloop*. Equivalently, a coloop is an element that is contained in every basis of a matroid. The following lemma shows that, if X is the set of all coloops of a matroid M, then M is completely determined by $M \setminus X$. Indeed, the bases of M are obtained by appending X to the bases of $M \setminus X$.

Lemma 4.1 *Let X be the set of all coloops of the matroid M. Then, for any subset A of $V - X$ we have $r_M(A \cup X) = r_M(A) + |X|$.*

Proof. See Exercise 4.2. ∎

Let Q be a V_r by V_c matrix. A row of Q is *avoidable* if its deletion does not decrease the rank. That is, $v \in V_r$ indexes an avoidable row of Q if and only if v is not a coloop of the row–matroid of Q. The set of avoidable rows is denoted by $D_r(Q)$. We define *avoidable columns* analogously, and we let $D_c(Q)$ denote the set of avoidable columns.

We now consider what happens to the set of avoidable columns of a matrix when we delete rows.

Lemma 4.2 *Let Q be a V_r by V_c matrix, and let Y_r be a subset of V_r. Then, $D_c(Q[Y_r, V_r]) \subseteq D_c(Q)$.*

Proof. Theorem 2.4 shows that deleting avoidable rows does not affect dependencies among the columns. Therefore, we may assume that Q has full row–rank. Now it is straightforward to see that each basis of $M_c(Q)$ contains a basis of $M_c(Q[Y_r, V_c])$. Thus each avoidable column of Q is also avoidable in $Q[Y_r, V_c]$. ∎

A vertex v of a graph G is called *avoidable* if $\nu(G - v) = \nu(G)$. The set of all avoidable vertices is denoted by $D(G)$. We will see that this set plays a very important role in Matching Theory. The following lemmas follow from these definitions.

Lemma 4.3 *Let G be a bipartite graph with bipartition (V_r, V_c) and let X be the bipartite matching matrix of G. Then $D(G) = D_r(X) \cup D_c(X)$.* ∎

Lemma 4.4 *Let T be the Tutte matrix of a graph $G = (V, E)$. Then, $D(G) = D_r(T)$.* ∎

Let x and y be elements of a matroid M. If neither x nor y is a coloop of M but $r_M(V - \{x, y\}) < r_M(V)$, then we call (x, y) a *series–pair*. Note that, if (x, y) is a series–pair then x is a coloop of $M \setminus y$. Series–pairs enjoy the property of being transitive.

Lemma 4.5 *If x, y, z are distinct elements of a matroid M such that (x, y) and (y, z) are series–pairs then (x, z) is a series pair.*

Proof. Suppose otherwise, and thus there exists a basis B of M that contains neither x nor z. Since (x, y) is a series pair, B must contain y. Now $B - \{y\}$ is an independent set of $M \setminus y$ and, since y is not a coloop of M, $r_M(V - \{y\}) = r_M(V) = |B|$. We can extend $B - \{y\}$ to a basis B' of $M \setminus y$. Now $|B'| = |B - \{y\}| + 1$ and B contains neither x nor y, so B' cannot contain both x and y. By symmetry we may assume that B' does not contain x. Thus B' is a basis of M that contains neither x nor y. This contradicts the assertion that (x, y) is a series pair. ∎

Exercise Set 4

4.1 Let T be the Tutte matrix of G. Show that the bases of $M_c(T)$ are exactly the maximum cardinality matchable sets of G.

4.2 Prove Lemma 4.1.

4.3 Do the set of matchings of a graph necessarily determine the independent sets of a matroid?

5 Kőnig's Theorem

The purpose of this section is to provide a good characterization for the size of a maximum matching in a bipartite graph. Let G be the bipartite graph in Figure 3, and let $M^* = \{2d, 1g, 5c, 3e, 7a, 4b\}$. Thus M^* is a matching of G; we claim that M^* is in fact a maximum matching of G. To establish this claim we need a succinct method to show that G has no matching with more than 6 edges. Let $C^* = \{1, 2, 5, a, b, e\}$. Thus C^* is a cover of G; that is, each edge of G is incident with at least one vertex in C^*. Now consider any matching M of G. Since C^* is a cover of G, each edge in M^* is incident with at least one vertex in C^*. Moreover, as M is a matching, no two edges of M are incident with a common vertex in C^*. We conclude that $|M| \leq |C^*| = 6 = |M^*|$. Hence M^* is indeed a maximum matching.

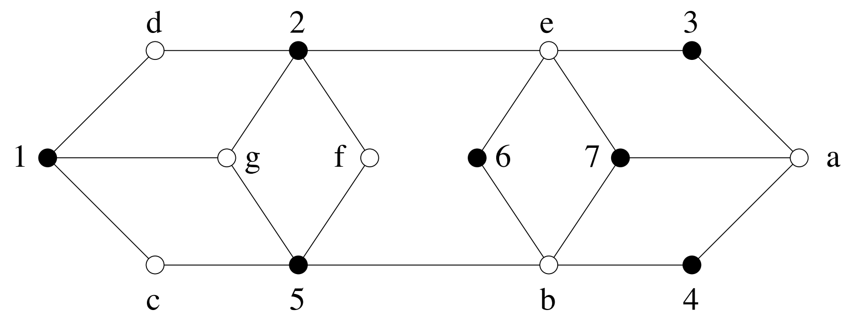

Figure 3: Another bipartite graph

More generally, if M is a matching and C is a cover, then, $|M| \leq |C|$. Furthermore, if $|M| = |C|$ then M is a maximum matching of G and C is a minimum cardinality cover. Unfortunately, it is not always the case that we can find a matching and a cover of the same size; indeed, consider K_3. The situation is, however, much nicer for bipartite graphs.

Theorem 5.1 (Kőnig [10]) *In a bipartite graph the size of a maximum matching is the minimum size of a cover.*

If C is a cover of a graph $G = (V, E)$ then, by definition, there are no edges having both ends in $V - C$. A subset Y of vertices is called *stable* if there is no edge having both ends in Y. That is, a stable set is the complement of a cover. The following lemma is the key to proving Kőnig's Theorem. (Recall that $D(G)$ is the set of avoidable vertices of G.)

Lemma 5.2 *If G is a bipartite graph then $D(G)$ is a stable set of G.*

Proof. Let (V_r, V_c) be the bipartition of G and let X be the matching matrix of G. Suppose the result is false, and thus there exists an edge uv of G where $u \in D_r(X)$ and $v \in D_c(X)$. Now, u is not a coloop of $M_r(X)$, so there exists a basis Y_r of $M_r(X)$ that does not contain u. Similarly, there exists a basis Y_c of $M_c(X)$ that does not contain v. By Theorem 2.4, $X[Y_r, Y_c]$ is nonsingular, and, hence, $Y_r \cup Y_c$ is a maximum cardinality matchable set of G. However this is a contradiction since $Y_r \cup Y_c \cup \{u, v\}$ must also be a matchable set. ∎

Proof of Kőnig's Theorem. Let $G = (V, E)$ be a bipartite graph, and choose $C \subseteq V$ maximal such that $\nu(G) = \nu(G - C) + |C|$. By our choice of C, every vertex of $G - C$ is avoidable. Then, by Lemma 5.2, $V - C$ is a stable set. Thus, C is a cover of G and $\nu(G) = \nu(G - C) + |C| = |C|$. ∎

Exercise Set 5

5.1 (Hall's Theorem) Let G be a bipartite graph with bipartition (V_r, V_c). Prove that there exists a matching of G covering each vertex in V_r if and only if for each subset A of V_r, $|N(A)| \geq |A|$. (Here $N(A)$ denotes the set of vertices that are adjacent to some vertex in A.)

5.2 Let G be a bipartite graph with bipartition (V_r, V_c), let X be the matching matrix of G, and let Q be a real V_r by V_c matrix. Consider the problem of determining the rank of $Q + X$; this is a generalization of the bipartite matching problem.

(a) Show that, if $Y_r \subseteq V_r$ and $Y_c \subseteq V_c$ such that $X[Y_r, Y_c] = 0$ then

$$\text{rank } (Q + X) \leq \text{rank } Q[Y_r, Y_c] + |V_r - Y_r| + |V_c - Y_c|.$$

(b) Prove that there exists $Y_r \subseteq V_r$ and $Y_c \subseteq V_c$ such that $X[Y_r, Y_c] = 0$ and

$$\text{rank } (Q + X) = \text{rank } Q[Y_r, Y_c] + |V_r - Y_r| + |V_c - Y_c|.$$

6 The Tutte–Berge Formula

We now progress to general graphs, but still we are looking for a good characterization for the size of a maximum matching. Consider the graph G in Figure 4 and the matching $M^* = \{15, 26, 39, 48\}$ of G. We claim that M^* is a maximum matching. Note that M^* leaves just two exposed vertices, so to show that M^* is a maximum matching it suffices to show that no matching leaves fewer than 2 exposed vertices. Recall that the *deficiency* of G, denoted $\text{def}(G)$, is the minimum number of vertices that are left exposed by any matching. Note that, when we delete a vertex the deficiency can increase by at most one. Therefore, $\text{def}(G) \geq \text{def}(G - \{6, 9\}) - 2$. Now $G - \{6, 9\}$ has 4 components and each of these components has an odd number of vertices. Therefore, for any matching of $G - \{6, 9\}$, there must be an exposed vertex in each component. Therefore, $\text{def}(G - \{6, 9\}) \geq 4$ and, hence, $\text{def}(G) \geq 2$. Finally, we conclude that $\text{def}(G) = 2$ and that M^* is, as claimed, a maximum matching.

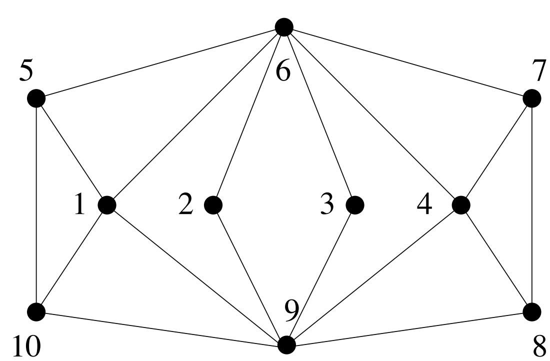

Figure 4: Another graph

More generally, if $G = (V, E)$ is a graph and $A \subseteq V$ then $\text{def}(G) \geq \text{odd}(G - A) - |A|$, where $\text{odd}(G - A)$ denotes the number of components of $G - A$ that have an odd number of vertices.

Therefore, if M is a matching and $A \subseteq V$ such that $|V| - 2|M| = \text{odd}(G - A) - |A|$ then M is a maximum matching.

Theorem 6.1 (Tutte–Berge Formula) *If $G = (V, E)$ is a graph then*

$$\textit{def}(G) = \min\left(\textit{odd}(G - A) - |A| \,:\, A \subseteq V\right).$$

Before proving the Tutte–Berge Formula, we observe an important special case: Tutte's Matching Theorem.

Theorem 6.2 (Tutte) *A graph $G = (V, E)$ has a perfect matching if and only if, for each subset A of V, we have $\textit{odd}(G - A) \le |A|$.*

To prove the Tutte–Berge Formula we require the following lemma. A graph $G = (V, E)$ is called *hypomatchable* if $G - v$ has a perfect matching for each vertex $v \in V$. Note that, if G is hypomatchable then G has an odd number of vertices.

Lemma 6.3 (Gallai) *If G is connected and every vertex of G is avoidable then G is hypomatchable.*

Proof. Let M be the matching matroid of G. That is, the bases of M are the maximum cardinality matchable sets of G. Since each vertex of G is avoidable, M has no coloops. Now consider an edge uv of G. There is no maximum cardinality matchable set of G that avoids both u and v. In the context of matroids, this says that (u, v) is a series pair of M. However, M is connected and series pairs are transitive. Therefore, each pair of vertices is a series pair. Thus any maximum cardinality matchable set can avoid at most 1 vertex. However, every vertex is avoidable, so $G - v$ must have a perfect matching for each $v \in V$. ∎

Proof of the Tutte–Berge Formula. We have already established that $\text{def}(G) \ge \text{odd}(G - A) - |A|$, for all $A \subseteq V$, so it suffices to show that this is attained with equality for some set A. Choose $A \subseteq V$ maximal such that $\text{def}(G) = \text{def}(G - A) - |A|$. By our choice of A, each vertex of $G - A$ is avoidable. Therefore, by Gallai's Lemma, each component of $G - A$ is hypomatchable. Thus, $\text{def}(G - A) = \text{odd}(G - A)$ and, hence, $\text{def}(G) = \text{odd}(G - A) - |A|$, as required. ∎

Exercise Set 6

6.1 Prove the Tutte–Berge Formula as a direct corollary of Tutte's Matching Theorem.

6.2 Let M be the matching matroid of a graph $G = (V, E)$. For a subset S of V, the rank, $r(S)$, of S in M is the maximum number of vertices in S that can be covered by a matching of G. Prove that,

$$r(S) = \min\left(|S| - (\text{odd}_S(G - A) - |A|) \,:\, A \subseteq V\right),$$

where $\text{odd}_S(G - A)$ denotes the number of odd components of $G - A$ whose vertices are all elements of S.

6.3 Let T be the Tutte matrix of a graph $G = (V, E)$ and let $Y_r, Y_c \subseteq V$. Consider the problem of determining the rank of $T[Y_r, Y_c]$; this is a generalization of the matching problem.

(a) If $X_r \subseteq Y_r$ and $X_c \subseteq Y_c$ such that $T[X_r - X_c, X_c] = 0$ and $T[X_r, X_c - X_r] = 0$ then we call (X_r, X_c) a *bi–stable pair.* (Equivalently, (X_r, X_c) is bi–stable if no indeterminate of T occurs exactly once in $T[X_r, X_c]$.) Show that, if (X_r, X_c) is a bi–stable pair then

$$\text{rank } T[Y_r, Y_c] \leq |X_r \cap X_c| + |Y_r - X_r| + |Y_c - X_c| - \text{odd}(G[X_r \cap X_c]).$$

(b) Prove that there exists a bi–stable pair such that

$$\text{rank } T[Y_r, Y_c] = |X_r \cap X_c| + |Y_r - X_r| + |Y_c - X_c| - \text{odd}(G[X_r \cap X_c]).$$

7 An algorithm for bipartite matching

In this section we present an efficient deterministic algorithm for finding an optimum evaluation of the bipartite matching matrix. This allows us to determine the size of a maximum matching in a bipartite graph.

Let G be the bipartite graph in Figure 2, let X be its bipartite matching matrix, and consider the following evaluation of X.

$$\tilde{X} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Note that X has rank 4, but $\tilde{X}$ has rank 3. A greedy approach is to try to change a single entry in a way that increases the rank. For an edge e of G and a real number a we let $\tilde{X}(z_e \to a)$ denote the evaluation of $\tilde{X}$ obtained by changing the value of z_e to a. We say that $\tilde{X}(z_e \to a)$ is obtained from $\tilde{X}$ by *perturbation.* We claim that we cannot increase the rank of the matrix above by a single perturbation. By way of contradiction, suppose that $\tilde{X}(z_e \to a)$ is nonsingular for some edge e and real number a. Note that the last two rows of $\tilde{X}$ are the same, so we must have perturbed some entry in these rows. Similarly, the first two columns are the same, so we must have perturbed some entry in these columns. However, we may not perturb entries in the submatrix $\tilde{X}[\{c, d\}, \{1, 2\}]$. This verifies the claim.

Since this greedy approach fails, we will have to accept more modest rewards from perturbation. However, before giving up on the greedy approach, it is helpful to understand when perturbations increase the rank of a matrix.

Theorem 7.1 *Let Q be a V_r by V_c matrix, let $i \in V_r$ and $j \in V_c$ and let $Q(a)$ be the matrix obtained from Q by replacing the (i, j)–entry of Q with the value a. If $a \neq Q_{ij}$, then rank $Q(a) >$ rank Q if and only if $i \in D_r(Q)$ and $j \in D_c(Q)$.*

Proof. First we suppose that $i \notin D_r(Q)$. Thus,

$$\begin{aligned} \text{rank } Q(a) &\leq \text{rank } (Q(a)[V_r - \{i\}, V_c]) - 1 \\ &= \text{rank } (Q[V_r - \{i\}, V_c]) - 1 \\ &= \text{rank } Q. \end{aligned}$$

That is, rank $Q(a) \leq$ rank Q. Similarly, if $j \notin D_c(Q)$ then rank $Q(a) \leq$ rank Q.

Conversely, suppose that $i \in D_r(Q)$ and that $j \in D_c(Q)$. Let Y_r be a basis of $M_r(Q)$ that does not contain i and let Y_c be a basis of $M_r(Q)$ that does not contain j. By Theorem 2.4, $Q[Y_r, Y_c]$ is nonsingular. Consider $p(a) = \det Q(a)[Y_r \cup \{i\}, Y_c \cup \{j\}]$. Now $p(a)$ is a nonzero linear function in a. However, as $Q[Y_r \cup \{i\}, Y_c \cup \{j\}]$ is singular, $p(Q_{ij}) = 0$. Thus $a = Q_{ij}$ is the only root of $p(a)$. Hence, for all $a \neq Q_{ij}$, the matrix $Q(a)[Y_r \cup \{i\}, Y_c \cup \{j\}]$ is nonsingular, and, hence, rank $Q(a) >$ rank Q. ∎

Since the greedy algorithm fails, we introduce a more refined ordering (actually, a quasi–ordering) on matrices than simply comparing the rank. Let Q_1 and Q_2 be V_r by V_c matrices. We write $Q_1 \succeq Q_2$ if rank $Q_1 >$ rank Q_2, or rank $Q_1 =$ rank Q_2 and $D^r(Q_2) \subseteq D^r(Q_1)$. Similarly, we write $Q_1 \approx Q_2$ if rank $Q_1 =$ rank Q_2 and $D^r(Q_1) = D^r(Q_2)$. If $Q_1 \succeq Q_2$ but $Q_1 \not\approx Q_2$ then we write $Q_1 \succ Q_2$. This gives a quasi–ordering of matrices; if $Q_1 \succ Q_2$ then we say that Q_1 is *more independent* than Q_2.

Consider the evaluation $\tilde{X}$ above. It is easy to check that $\tilde{X}(z_{d4} \to 2)$ is more independent than $\tilde{X}$. Our algorithm is now clear, we make perturbations if doing so increases the independence.

Theorem 7.2 *Let X be the bipartite matching matrix of a bipartite graph G, with bipartition (V_r, V_c), and let $\tilde{X}$ be an evaluation of X. Then either rank $\tilde{X} = \nu(G)$, or there exists an edge e of G and $a \in \{1, \ldots, |V_r| + 1\}$ such that $\tilde{X}(z_e \to a) \succ \tilde{X}$.*

This theorem clearly provides a polynomial–time deterministic algorithm for computing the size of a maximum matching in a graph. Let us briefly consider the running time of a naive implementation of our algorithm. Firstly the rank of an evaluation is at most $|V_r|$ and there are at most $|V_r|$ avoidable rows. Therefore, our algorithm may require as many as $(|V_r| + 1)^2$ steps. At each step we may have to consider each edge and each value of $a \in \{1, \ldots, |V_r| + 1\}$. Thus, in a step we may have to compare as many as $(|V_r| + 1)|E|$ evaluations with $\tilde{X}$. For each of these evaluations we must determine the rank and the set of avoidable rows; this can be done with $|V_r| + 1$ rank computations. Each of these rank computations requires $\mathcal{O}(n^3)$ time, where n is the number of vertices of G. Combining the numbers above, we reluctantly concede that, in the worst case, the algorithm may require as much as $\mathcal{O}(n^9)$ time. With some modifications the running time of this algorithm can be reduced to $\mathcal{O}(n^5)$.

Using augmenting path methods, the size of a maximum matching can be easily determined in $\mathcal{O}(n^3)$, and, with work, this can be improved to $\mathcal{O}(n^{2.5})$; see Hopcroft and Karp [9]. Moreover, the augmenting path algorithms actually find a maximum matching, whereas our evaluation algorithms do not.

Now we commence the proof of Theorem 7.2, for this we require the following easy lemmas.

Lemma 7.3 *If Q is a V_r by V_c matrix, $Y_r = D_r(Q)$, and $Y_c = D_c(Q[Y_r, V_c])$ then*

i. *rank $Q[Y_r, V_c] =$ rank $Q - |V_r - Y_r|$ and each row of $Q[Y_r, V_c]$ is avoidable, and*

ii. *rank $Q[Y_r, Y_c] =$ rank $Q - |V_r - Y_c| - |V_c - Y_c|$.*

Proof. Part i is an immediate consequence of Lemma 4.1, and part ii follows from part i and Lemma 4.1. ∎

Lemma 7.4 *Let Q_1 and Q_2 be V_r by V_c matrices such that $Q_1 \approx Q_2$, and let $Y_r = D_r(Q_1)$. Then rank $Q_1[Y_r, V_c] =$ rank $Q_2[Y_r, V_c]$.*

Proof. This follows immediately from Lemma 7.3, part i. ∎

Lemma 7.5 *Let X be the bipartite matching matrix of a bipartite graph G, with bipartition (V_r, V_c), and let $\tilde{X}$ be an evaluation of X. For any edge e of G and indeterminate a, there exists $a' \in \{1, \ldots, |V_r| + 1\}$ such that $\tilde{X}(z_e \to a) \approx \tilde{X}(z_e \to a')$.*

Proof. Let $\tilde{X}(a)$ denote $\tilde{X}(z_e \to a)$. Clearly, $\tilde{X}(a) \succeq \tilde{X}(a')$. Moreover, $\tilde{X}(a) \approx \tilde{X}(a')$ if and only if

i. rank $\tilde{X}(a) =$ rank $\tilde{X}(a')$ and

ii. For each $v \in V_r$, rank $\tilde{X}(a)[V_r - \{v\}, V_c] =$ rank $\tilde{X}(a')[V_r - \{v\}, V_c]$.

That is, we are seeking an evaluation that preserves the rank of $|V_r| + 1$ matrices. Now, a is in some row, say u. Then $\tilde{X}(a)[V_r - \{u\}, V_c] = \tilde{X}(a')[V_r - \{u\}, V_c]$, so we need not consider this matrix. For each of the other $|V_r|$ matrices there is at most one choice for a' that decreases the rank of the matrix; see Exercise 7.1. Therefore, there remains some choice for $a' \in \{1, \ldots, |V_r| + 1\}$ such that $\tilde{X}(a) \approx \tilde{X}(a')$. ∎

What have we gained in the previous results? Let $\tilde{X}$ be an evaluation of X, let e be an edge of G, and let a be an indeterminate. Clearly $\tilde{X}(z_e \to a) \succeq \tilde{X}$. If $\tilde{X}(z_e \to a) \succ \tilde{X}$ then, by Lemma 7.5 we can an evaluation of X that is more independent than $\tilde{X}$. So we may assume that $\tilde{X}(z_e \to a) \approx \tilde{X}$. Let $Y_r = D_r(\tilde{X})$. Then, by Lemma 7.4, rank $\tilde{X}[Y_r, V_c] =$ rank $\tilde{X}(z_e \to a)[Y_r, V_c]$. Thus, increasing the the independence of $\tilde{X}$ is tantamount to increasing the rank of $\tilde{X}[Y_r, V_c]$; the following lemma makes this more precise.

Lemma 7.6 *Let X be the bipartite matching matrix of a bipartite graph G, with bipartition (V_r, V_c), let $\tilde{X}$ be an evaluation of X, and let $Y_r = D_r(\tilde{X})$. If there exists any edge e of G and and indeterminate a such that rank $\tilde{X}(z_e \to a)[Y_r, V_c] >$ rank $\tilde{X}[Y_r, V_c]$ then there exists $a' \in \{1, \ldots, |V_r| + 1\}$ such that $\tilde{X}(z_e \to a') \succ \tilde{X}$.* ∎

To complete the proof of Theorem 7.2 we need a sufficient condition for determining when rank $\tilde{X} = \nu(G)$.

Lemma 7.7 *Let X be the bipartite matching matrix of a bipartite graph G, with bipartition (V_r, V_c), let $\tilde{X}$ be an evaluation of X, let $Y_r = D_r(\tilde{X})$, and let $Y_c = D_c(\tilde{X}[Y_r, V_c])$. If $Y_r \cup Y_c$ is a stable set of G then rank $\tilde{X} = \nu(G)$.*

Proof. Firstly, rank $\tilde{X} \leq \nu(G)$, so it suffices to prove the reverse inequality. If $Y_r \cup Y_c$ is a stable set of G then $(V_r - Y_r) \cup (V_c - Y_c)$ is a cover, and, hence, $\nu(G) \leq |V_r - Y_r| + |V_c - Y_c|$. Moreover, as $Y_r \cup Y_c$ is stable, $\tilde{X}[Y_r, Y_c] = 0$. Then, by Lemma 7.3 part *ii*, rank $\tilde{X} = |V_r - Y_r| + |V_c - Y_c|$, as required. ∎

Proof of Theorem 7.2. Let $Y_r = D_r(\tilde{X})$ and $Y_c = D_c(\tilde{X}[Y_r, V_c])$. By Lemma 7.7, we may assume that $Y_r \cup Y_c$ is not a stable set of G. Thus, there exists an edge $e = uv$ of G such that $u \in Y_r$ and $v \in Y_c$. By definition, v is an avoidable column of $\tilde{X}[Y_r, V_c]$, and, by Lemma 7.3 part *i*, v is an avoidable row of $\tilde{X}[Y_r, V_c]$. Therefore, by Lemma 7.1, rank $\tilde{X}(z_e \to a)[Y_r, V_c] >$ rank $\tilde{X}[Y_r, V_c]$, where a is an indeterminate. Therefore, by Lemma 7.6, there exists $a' \in \{1, \ldots, |V_r| + 1\}$ such that $\tilde{X}(z_e \to a')$ is more independent than $\tilde{X}$. ∎

Exercise Set 7

7.1 Let $Q(a)$ be a matrix that contains an indeterminate a in exactly one entry. Prove that there is at most one real number a' such that rank $Q(a') <$ rank $Q(a)$.

7.2 Consider ways of improving the running time of the evaluation algorithm.

(a) Show that rank $\tilde{X}$ and $D_r(\tilde{X})$ can be computed in $\mathcal{O}(n^{2.38})$. (For n by n matrices, multiplication and inversion require $\mathcal{O}(n^{2.38})$ time.)

(b) Show that, in $\mathcal{O}(n^{2.38})$ time you can either prove that rank $\tilde{X} = \nu(G)$ or you can find an edge e of G such that, for an indeterminate a, $\tilde{X}(z_e \to a)$ is more independent than $\tilde{X}$.

(c) Suppose that, for some edge e of G and indeterminate a, $\tilde{X}(z_e \to a)$ is more independent than $\tilde{X}$. How efficiently can you find $a' \in \{1, \ldots, |V_r| + 1\}$ such that $\tilde{X}(z_e \to a')$ is more independent than $\tilde{X}$.

(d) How efficiently can you implement the evaluation algorithm?

7.3 Given an integer k find an example of a bipartite matching matrix X and an evaluation $\tilde{X}$ of X such that rank $\tilde{X} <$ rank X and the rank of $\tilde{X}$ cannot be increased by perturbing any k variables.

7.4 Let G be a bipartite graph with bipartition (V_r, V_c), let X be the matching matrix of G, and let Q be a real V_r by V_c matrix. Prove an analogue of Theorem 7.2 for $Q + X$.

8 An algorithm for general matching

Motivated by the results in the previous section, a natural algorithm for determining the size of a maximum matching in a graph is to take an evaluation of the Tutte matrix, and repeatedly apply perturbations in a way that increases the independence. The main theorem of this section shows that this algorithm works; see [6].

Theorem 8.1 *Let T be the Tutte matrix of a graph $G = (V, E)$, and let $\tilde{T}$ be an evaluation of T. Then either rank $\tilde{T} = 2\nu(G)$, or there exists an edge e of G and $a \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a) \succ \tilde{T}$.*

A naive implementation of the evaluation algorithm requires $\mathcal{O}(n^9)$ time, whereas, Edmonds' Algorithm [4] can be implemented to run in $\mathcal{O}(n^3)$, and more sophisticated augmenting path algorithms have running time as low as $\mathcal{O}(n^{2.5})$; see Micali and Vazirani [13]. Nevertheless, there is a case to be made for evaluation algorithms. For general graphs, augmenting path algorithms become complicated, while the evaluation algorithm remains essentially trivial.

We can use some of the results from the previous section to prove Theorem 8.1. However, some of the lemmas in that section refer specifically to bipartite matching matrices, and we will have to develop analogues of these. Suppose that $\tilde{T}(z_e \to a)$ is a nonsingular matrix, where $\tilde{T}$ is an evaluation of the Tutte matrix of a graph $G = (V, E)$, $e \in E$, and a is an indeterminate. The indeterminate a occurs twice in $\tilde{T}(z_e \to a)$, so the determinant of $\tilde{T}(z_e \to a)$ is quadratic in a. Therefore, the Pfaffian of $\tilde{T}(z_e \to a)$ (which is the square–root of the determinant) is a linear function in a. Therefore, there is at most one real number a' for which $\tilde{T}(z_e \to a')$ is singular. (This can also be deduced without using Pfaffians; see Exercise 8.2.) This observation allows us to prove the appropriate analogue of Lemma 7.5.

Lemma 8.2 *Let T be the Tutte matrix of a graph $G = (V, E)$, and let $\tilde{T}$ be an evaluation of T. For any edge e of G and indeterminate a, there exists $a' \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a) \approx \tilde{T}(z_e \to a')$.*

Proof. See Exercise 8.3. ∎

Lemma 8.3 *Let T be the Tutte matrix of a graph $G = (V, E)$, let $\tilde{T}$ be an evaluation of T, and let $Y_r = D_r(\tilde{T})$. If there exists and edge e of G and and indeterminate a such that rank $\tilde{T}(z_e \to a)[Y_r, V] >$ rank $\tilde{T}[Y_r, V]$ then there exists $a' \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a') \succ \tilde{T}$.*

Proof. If $\tilde{T}(z_e \to a)$ is more independent than $\tilde{T}$ then, by Lemma 8.2, there exists $a' \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a')$ is more independent than $\tilde{T}$. Moreover, it is clear that $\tilde{T}(z_e \to a) \succeq \tilde{T}$. Therefore, we may assume that $\tilde{T}(z_e \to a) \approx \tilde{T}$. Therefore, by Lemma 7.4, rank $\tilde{T}(z_e \to a)[Y_r, V] =$ rank $\tilde{T}[Y_r, V]$. This contradiction completes the proof. ∎

We also require an analogue of Lemma 7.7. That is, given an evaluation $\tilde{T}$ of T, we require a necessary condition that allows us to conclude that $\tilde{T}$ has the desired rank.

Lemma 8.4 *Let T be the Tutte matrix of a graph $G = (V, E)$, let $\tilde{T}$ be an evaluation of T, let $Y_r = D_r(\tilde{T})$, and let $Y_c = D_c(\tilde{T}[Y_r, V])$. If*

i. There is no edge having one end in Y_r and the other end in $Y_c - Y_r$, and *ii. If u and v are adjacent vertices of G where $u, v \in Y_r$ then (u, v) is a series–pair of $M_c(\tilde{T}[Y_r, V])$,*

then rank $\tilde{T} = 2\nu(G)$.

Proof. Certainly, rank $\tilde{T} \leq 2\nu(G)$, so it suffices to prove the reverse inequality. By Lemma 4.2 we observe that $Y_r \subseteq Y_c$. Thus,

$$\begin{aligned} 2\nu(G) &= \text{rank } T \\ &\leq \text{rank } T[Y_r, Y_c] + |V - Y_r| + |V - Y_c| \\ &\leq \text{rank } T[Y_r] + |V - Y_r| + |V - Y_c| \\ &= |V_r| - \text{def}(G[Y_r]) + |V - Y_r| + |V - Y_c| \\ &\leq |V_r| - \text{odd}(G[Y_r]) + |V - Y_r| + |V - Y_c|. \end{aligned}$$

Now, by Lemma 7.3,

$$\text{rank } \tilde{T} = \text{rank } \tilde{T}[Y_r, Y_c] + |V - Y_r| + |V - Y_c|.$$

Since there is no edge with an end in Y_r and the other end in $Y_c - Y_r$, we have $T[Y_r, Y_c - Y_r] = 0$. Therefore,

$$\text{rank } \tilde{T} = \text{rank } \tilde{T}[Y_r] + |V - Y_r| + |V - Y_c|.$$

Now let $G[S]$ be a connected component of $G[Y_r]$, and let uv be an edge of $G[S]$. We are given that (u, v) is a series–pair of $M_c(\tilde{T}[Y_r, V])$. Then, by Lemma 4.1, (u, v) is a series–pair of $M_c(\tilde{T}[Y_r, Y_c])$. Now, since $\tilde{T}[Y_r, Y_c - Y_r] = 0$, (u, v) is a series–pair of $M_c(\tilde{T}[Y_r])$. Finally, considering the block diagonal structure of $G[Y_r]$, we see that (u, v) is a series–pair of $M_c(\tilde{T}[S])$. That is, for each edge uv of $G[S]$, (u, v) is a series–pair of $M_c(\tilde{T}[S])$. Then, by the transitivity of series–pairs, any two vertices of $G[Y_r]$ are a series–pair of $M_c(\tilde{T}[S])$. Therefore, each basis of $M_c(\tilde{T}[S])$ avoids exactly one vertex in each component of $G[Y_r]$. Thus, rank $\tilde{T}[S] = |S| - 1$, and, since $\tilde{T}[S]$ is skew–symmetric it has even rank, so $|S|$ is odd. Then, rank $\tilde{T}[Y_r] = |Y_r| - \text{odd}(G[Y_r])$. Therefore,

$$\begin{aligned} \text{rank } \tilde{T} &= |V_r| - \text{odd}(G[Y_r]) + |V - Y_r| + |V - Y_c| \\ &= 2\nu(G), \end{aligned}$$

as required. ∎

Proof of Theorem 8.1. Suppose that rank $\tilde{T} \neq 2\nu(G)$. Let $Y_r = D_r(\tilde{T})$ and let $Y_c = D_c(\tilde{T}[Y_r, V])$. Note that, by Lemma 4.2, $Y_r \subseteq Y_c$. Now, by Lemma 8.4, either

i. There exists an edge uv of G such that $u \in Y_r$ and $v \in Y_c - Y_r$, or

ii. There exits an edge uv of G such that $u, v \in Y_r$ and (u, v) is not a series–pair of $M_c(\tilde{T}[Y_r, V])$.

Consider the case that $e = uv$ is an edge of G such that $u \in Y_r$ and $v \in Y_c - Y_r$. Let a be an indeterminate. Since $v \notin Y_r$, the indeterminate a occurs in just one entry of $\tilde{T}(z_e \to a)[Y_r, V]$.

Moreover, by definition v is an avoidable column of $\tilde{T}[Y_r, V]$, and, by Lemma 4.2, u is an avoidable row of $\tilde{T}[Y_r, V]$. Therefore, by Lemma 7.1, rank $\tilde{T}(z_e \to a)[Y_r, V] >$ rank $\tilde{T}[Y_r, V]$. Then, by Lemma 8.3, there exists $a' \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a')$ is more independent than $\tilde{T}$, as required.

Now consider the case that $e = uv$ is an edge of G such that $u, v \in Y_r$ and (u, v) is not a series–pair of $M_c(\tilde{T}[Y_r, V])$. Let a be an indeterminate. By Lemma 4.1 and definition, each row of $\tilde{T}[Y_r, V]$ is avoidable. Also, by Lemma 4.2, each row of $\tilde{T}[Y_r, V - \{u\}]$ is avoidable. Moreover, as (u, v) is a series–pair of $M_c(\tilde{T}[Y_r, V])$, v is an avoidable column of $\tilde{T}[Y_r, V - \{u\}]$. Therefore, by Lemma 7.1, rank $\tilde{T}(z_e \to a)[Y_r, V - \{u\}] >$ rank $\tilde{T}[Y_r, V - \{u\}]$. However, since u is an avoidable row of $\tilde{T}[Y_r, V]$, rank $\tilde{T}(z_e \to a)[Y_r, V] >$ rank $\tilde{T}[Y_r, V]$. Then, by Lemma 8.3, there exists $a' \in \{1, \ldots, |V|\}$ such that $\tilde{T}(z_e \to a')$ is more independent than $\tilde{T}$, as required. ∎

Exercise Set 8

8.1 Let $\tilde{T}$ be an evaluation of the Tutte matrix of a graph $G = (V, E)$, let e be an edge of G, and let a be an indeterminate. Prove that, for any real number a', rank $\tilde{T}(z_e \to a') =$ rank $\tilde{T}(z_e \to a)$ if and only if, for any $v \in V$, rank $\tilde{T}(z_e \to a')[V - \{v\}, V] \geq$ rank $\tilde{T}(z_e \to a) - 1$.

8.2 Let $\tilde{T}$ be an evaluation of the Tutte matrix of a graph $G = (V, E)$, let e be an edge of G, and let a be an indeterminate. Without using Pfaffians, prove that there exists at most one real number a' such that rank $\tilde{T}(z_e \to a') <$ rank $\tilde{T}(z_e \to a)$.

8.3 Prove Lemma 8.2.

9 Structural results

In this section we explore some surprising properties of the set of avoidable vertices of a graph. In particular, if we are told $D(G)$ then we can easily determine $\nu(G)$ and we can say a lot about the structure of maximum matchings. Let A be the neighbour–set of $D(G)$. The most important point of this section is that the set A achieves equality in the Tutte–Berge Formula; that is, $\text{def}(G) = \text{odd}(G - A) - |A|$.

Theorem 9.1 (Edmonds–Gallai Structure Theorem) *Let $G = (V, E)$ be a graph and let (D, A, C) be a partition of V such that D is the set of avoidable vertices, A is the neighbour–set of D, and $C = V - (D \cup A)$. Then*

i. def$(G) =$ odd$(G - A) - |A|$,

ii. each component of $G[D]$ is hypomatchable, and

iii. $G[C]$ has a perfect matching.

We require the following lemma.

Lemma 9.2 *Let K be a V by V skew–symmetric matrix, let $Y_r = D_r(K)$, and let $Y_c = D_c(K[Y_r, V])$. Then, rank $K =$ rank $K[Y_c] + 2|V - Y_c|$, and $D_r(K[Y_c]) = D_r(K)$.*

Proof. Note that, by Lemma 4.2, $Y_r \subseteq Y_c$. Let $A = V - Y_c$. Now, A is a set of coloops of $M_r(K)$, so, since K is skew–symmetric, A is a set of coloops of $M_c(K)$. Therefore, by Lemma 4.1, rank $K[V, Y_c] =$ rank $K - |A|$. Now, by Lemma 4.2, $Y_r \subseteq D_r(K[V, Y_c])$. However, by Lemma 7.3,

$$\text{rank } K = \text{rank } K[Y_r, Y_c] + |V - Y_r| + |A|.$$

Thus, rank $K[Y_r, Y_c] =$ rank $K[V, Y_c] - |V - Y_r|$. Therefore, each element of $V - Y_r$ is a coloop of $M_r(K[V, Y_c])$, and, hence, $D_r(K[V, Y_c]) = Y_r$. Therefore, by Lemma 4.1, rank $K[Y_c] =$ rank $K[V, Y_c] - |A| =$ rank $K - 2|A|$, and $D_r(K[Y_c]) = Y_r$, as required. ∎

Lemma 9.3 *Let T be the Tutte matrix of a graph $G = (V, E)$, let $Y_r = D_r(T)$, and let $Y_c = D_c(T[Y_r, V])$. Then, $V - Y_c$ is the neighbour–set of Y_r.*

Proof. Let $A' = V - Y_c$, and let A be the neighbour–set of Y_r in G. By Theorem 7.3, each row of $T[Y_r, V]$ is avoidable, and, by definition, Y_c is the set of avoidable columns of $T[Y_r, V]$. Moreover, by Lemma 4.2, $Y_r \subseteq Y_c$. Suppose that there exists and edge $e = uv$ where $u \in Y_r$ and $v \in Y_c - Y_r$. Then, the indeterminate z_e occurs just once in $T[Y_r, V]$, and both the row and column containing z_e are avoidable. This contradicts Lemma 7.1. Therefore, $A \subseteq A'$. Now consider an element $v \in A'$. By definition, v is a coloop of $M_c(T[Y_r, V])$. Since v is in some basis of $M_c(T[Y_r, V])$, the column of $T[Y_r, V]$ indexed by v must contain some nonzero entry. Thus, $v \in A$. Therefore, $A = A'$, as required. ∎

Proof of Theorem 9.1. Let T be the Tutte–matrix of G, let $Y_r = D_r(T)$, and let $Y_c = D_c(T[Y_r, V])$. Clearly $D = D_r(T)$, and, by Lemma 9.3, $A = V - Y_c$. Now, by Lemma 9.2, rank $T =$ rank $T[V - A] + 2|A|$ and $D_r(T[Y_c]) = D$. Translating this back to graphs, we see that $\text{def}(G) = \text{def}(G - A) - |A|$, and $D(G - A) = D$. By definition, there are no edges from D to C. Therefore, $D(G[D]) = D$ and $D(G[C]) = \emptyset$. Since $G[C]$ has no avoidable vertices it must have a perfect matching. Now, every vertex of $G[D]$ is avoidable, so, by Lemma 6.3, each component of $G[D]$ is hypomatchable. ∎

A subset A' of the vertices of a graph G is called a *Tutte–set* if $\text{def}(G) = \text{odd}(G - A') - |A'|$. Thus, the neighbour–set, A, of $D(G)$ is a Tutte–set. We claimed that we can determine structural information about the maximum matchings from $D(G)$; this structure follows easily from the fact that A is a Tutte–set. A matching that saturates all but one vertex of a graph is called *near–perfect.*

Theorem 9.4 *Let $G = (V, E)$ be a graph, and let (D', A', C') be a partition of V where A' is a Tutte–set, D' is the set of vertices in odd components of $G - A'$, and C' is the set of vertices in even components of $G - A'$. If M is a maximum matching of G then*

- *M contains a perfect matching of $G[C']$,*

- *M contains a near–perfect matching of each component of $G[D']$, and*
- *each vertex of A' is matched to a vertex in D'.*

Proof. See Exercise 9.1. ∎

The Edmonds–Gallai Structure Theorem provided the original motivation for the evaluation algorithm. Let $\tilde{T}$ be an evaluation of the Tutte matrix of a graph G. If we are lucky rank $\tilde{T} = 2\nu(G)$, and $D_r(\tilde{T}) = D(G)$. The Edmonds–Gallai Structure Theorem lets us check whether this is the case, since we can determine $\nu(G)$ from $D(G)$.

Lemma 9.5 *Let $\tilde{T}$ be an evaluation of the Tutte matrix of a graph $G = (V, E)$, and let A be the neighbour–set of $D_r(\tilde{T})$ in G. If rank $\tilde{T} = |V| - (odd(G - A) - |A|)$ then rank $\tilde{T} = 2\nu(G)$.*

Proof. See Exercise 9.2. ∎

We can use Lemma 9.5 to obtain a more satisfactory randomized algorithm for determining the size of a maximum matching. Our randomized algorithms presented earlier are "Monte Carlo" algorithms; they guess $\nu(G)$ with high probability. However, when we are given the guess, we do not actually know whether or not the guess is correct. Suppose that $\tilde{T}$ is an evaluation of the Tutte matrix of the graph $G = (V, E)$. Lemma 9.5 provides sufficient condition to check whether $\tilde{T}$ is optimal. This leads to a randomized algorithm that will either terminate with the correct value of $\nu(G)$ or, with some limited probability, will terminate without guessing a value of $\nu(G)$. This "Las Vegas" algorithm was proposed by Cheriyan [1]. It remains to show that a random evaluation is likely to satisfy the optimality condition in Lemma 9.5.

Lemma 9.6 *Let T be the Tutte matrix of a graph $G = (V, E)$. If $\tilde{T}$ is an evaluation of T with entries chosen independently and at random from $\{1, \ldots, |V|^2\}$, and A is the neighbour–set of $D_r(\tilde{T})$, then rank $\tilde{T} = |V| - (odd(G - A) - |A|)$ with probability at least $\frac{1}{2}$.*

Proof. See Exercise 9.3. ∎

Exercise Set 9

9.1 Prove Theorem 9.4.

9.2 Prove Theorem 9.5.

9.3 Prove Theorem 9.6.

10 Matroid intersection

Given two matroids M_1, M_2 on a common ground set S, we would like to find a common independent set of maximum cardinality; this is the *matroid intersection problem.* We let $\lambda(M_1, M_2)$ denote the maximum size of a common independent set of M_1 and M_2. The matroid intersection problem is a generalization of the maximum matching problem for bipartite graphs, and can be solved by augmenting path methods; see Edmonds [3]. In this section we consider the case that M_1 and M_2 are linear matroids. (A matroid is *linear* if it is "given" to us as a column–matroid of a matrix.) We formulate the linear matroid intersection problem as a matrix rank problem.

Consider a bipartite graph $G = (V, E)$ with bipartition (V_r, V_c). Now define a matroid M_r with ground set E such that a subset A of E is declared independent if no two edges in A are incident with a common vertex in V_r. (It is easy to check that M_r is indeed a matroid.) We define M_c analogously. Evidently, a set of edges is a matching if and only if it is a common independent set of M_r and M_c. Therefore, the matroid intersection problem is indeed a generalization of the maximum matching problem for bipartite graphs.

As well as finding an efficient algorithm for solving the matroid intersection theorem, Edmonds [3] also proved a remarkable min–max theorem.

Theorem 10.1 (Edmonds) *If $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ are matroids with rank functions $r_1(\cdot)$ and $r_2(\cdot)$ respectively, then*

$$\max(|A| \,:\, A \in \mathcal{I}_1 \cap \mathcal{I}_2) = \min(r_1(X) + r_2(S - X) \,:\, X \subseteq S).$$

Note that, if A is a common independent set of M_1 and M_2, and $X \subseteq S$, then

$$|A| = |A \cap X| + |A - X| \leq r_1(X) + r_2(S - X).$$

The hard part of the theorem is showing that there exist sets A and X that satisfy this inequality with equality.

If X is a V_r by V_c bipartite–matching matrix and Q is a V_r by V_c matrix over the rationals, then we call $Q + X$ a *mixed matrix*. Murota [15] studies mixed matrices extensively, and shows that computing their rank is in fact equivalent to the linear matroid intersection problem. The following lemma does not provide a good characterization for the rank of a mixed matrix, but it is useful in formulations.

Lemma 10.2 (Murota) *If $Q + X$ is a V_r by V_c mixed matrix then*

$$\textit{rank}\ (Q + X) = \max(\textit{rank}\ Q[A_r, A_c] + \textit{rank}\ X[V_r - A_r, V_c - A_c] \,:\, A_r \subseteq V_r,\ A_c \subseteq V_c).$$

Let Q_1 and Q_2 be matrices whose columns are indexed by a set $V := \{1, \ldots, n\}$. Consider the intersection problem for the linear matroids $M_1 := M_c(Q_1)$ and $M_2 := M_c(Q_2)$. We will formulate this as a matrix rank problem for a mixed matrix. Let $(z_1, \ldots, z_n)$ be algebraically independent commuting indeterminates, and consider the following matrix:

$$Z := \begin{array}{c} \\ c_1 \\ \vdots \\ c_n \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} r_1 & \cdots & r_n \end{array} \\ \left(\begin{array}{cccc} z_1 & & & \\ & \ddots & & Q_1^t \\ & & z_n & \\ & Q_2 & & 0 \end{array} \right) \end{array}.$$

Let $r_1(\cdot)$ and $r_2(\cdot)$ denote the rank functions of M_1 and M_2 respectively. By Lemma 10.2, we see that

$$\text{rank } Z = \max(r_1(A) + r_2(A) + |V - A| \, : \, A \subseteq V).$$

By considering the case that A is a maximum common independent set of M_1 and M_2 we see that rank $Z \geq \lambda(M_1, M_2) + |V|$. Moreover, if A is a minimal subset of V such that rank $Z = r_1(A) + r_2(A) + |V - A|$ then A is a common independent set of M_1 and M_2. Therefore, rank $Z = \lambda(M_1, M_2) + |V|$.

Using the above formulation, we will sketch a proof of the matroid intersection theorem for linear matroids. First consider any submatrix Z' of Z. If Z' contains the indeterminate z_i then either $r_i \notin D_r(Z')$ or $c_i \notin D_c(Z')$. That is we can reduce the rank of Z' by either deleting row r_i or column c_i. Therefore, there exists a partition (A, B) of V such that deleting the rows $(r_i \, : \, i \in A)$ and the columns $(c_i \, : \, i \in B)$ from Z reduces the rank of Z by $|V|$. The resulting matrix contains no indeterminates and has rank $r_1(B) + r_2(A)$. Therefore, $\lambda(M_1, M_2) + |V| = |V| + r_1(B) + r_2(A)$. That is, $\lambda(M_1, M_2) = r_1(B) + r_2(V - B)$, as required.

Exercise Set 10

10.1 Let $G = (V, E)$ be a bipartite graph with bipartition (V_r, V_c), and let M_r be a matroid with ground set E such that a subset A of E is declared independent if no two edges in A are incident with a common vertex in V_r. Find a matrix N_r such that M_r is the column–matroid of N_r.

10.2 Prove Kőnigs theorem as a corollary of Edmonds' matroid intersection theorem.

10.3 Show that the rank of a mixed matrix can be computed using matroid intersection.

10.4 Prove Lemma 10.2.

11 Matroid parity

Let $M = (S, \mathcal{I})$ be a matroid and let Π be a partition of S into pairs. A subset X of S is *matched* if each pair in Π is either contained in S or is disjoint from S. We are interested in finding a maximum cardinality matched independent set in M; this is the *matroid parity problem.* Let $\nu(M, \Pi)$ be the maximum size of a matched independent set in M. The matroid parity problem has been extensively studied by Lovász; see [12]; it is very general problem that contains both matching and matroid intersection. Unfortunately, the matroid parity problem is NP–hard and is intractable using the usual oracle approach to matroid algorithms. Nevertheless, Lovász was able to solve the matroid parity problem for linear matroids.

Consider a graph $G = (V, E)$. From an edge $e = uv$ of G we define two *half–edges* e_u and e_v. The half edge e_v is *incident* with v. Let S be the set of all half–edges, and define a matroid $M = (S, \mathcal{I})$ such that a subset A of S is declared to be independent if each vertex of G is incident with at most one half–edge in A. (It is easy to see that M is a matroid, moreover there exists a matrix Q such that $M = M_c(Q)$.) Now let $\Pi = (\{e_u, e_v\} : uv = e \in E)$. Then there is a natural bijection between matchings of G and matched independent sets of M. Hence, $\nu(M, \Pi) = \nu(G)$.

If T is the Tutte matrix of a graph $G = (V, E)$ and K is a V by V skew–symmetric matrix then we call $T + K$ a *mixed skew–symmetric matrix*. We will formulate the linear matroid parity problem as a matrix rank problem for a mixed skew–symmetric matrix. We require the following analogue of Lemma 10.2.

Lemma 11.1 (Murota) *If $T + K$ is a V by V mixed skew–symmetric matrix then*

$$\textit{rank}\ T + K = \max(\textit{rank}\ T[A] + \textit{rank}\ K[V - A] : A \subseteq V).$$

Let Q be a matrix whose columns are indexed by $S = \{1, \ldots, 2n\}$ and let $\Pi = (\{1, 2\}, \ldots, \{2n-1, 2n\})$. Consider the matroid parity problem for $M = M_c(Q)$. Let $(z_1, \ldots, z_n)$ be algebraically independent commuting indeterminates, and let

$$Z = \left(\begin{array}{cc|c|cc|c} 0 & z_1 & & & & \\ -z_1 & 0 & & & & \\ \hline & & \ddots & & & Q^t \\ \hline & & & 0 & z_n & \\ & & & -z_n & 0 & \\ \hline & & Q & & & 0 \end{array}\right).$$

Let $r(\cdot)$ denote the rank function of M. By Lemma 11.1,

$$\text{rank}\ Z = \max(2r(A) + |S - A| : A \text{ a matched subset of } S).$$

Considering a matched independent set A of M, we see that rank $Z \geq \nu(M, \Pi) + |S|$. Moreover, if A is a minimal matched subset of A such that rank $Z = 2r(A) + |S - A|$ then A is independent. Hence, rank $Z = \nu(M, \Pi) + |S|$.

Algorithms for solving the linear matroid parity problem are quite complicated. However, the matrix rank formulation above provides us with a trivial randomized algorithm. The matrix rank formulation is also useful in proving a minmax theorem, but this is somewhat more complicated, so we do not include the details here.

Exercise Set 11

11.1 Formulate a matroid intersection problem as a matroid parity problem.

11.2 Prove Lemma 11.1.

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