COLOURING GRAPHS WITH NO ODD-K_n MINOR.

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Abstract. We prove that if G is a simple graph such that the signed-graph odd-G contains no odd-K_{n+1}-minor, then G is 2^{n-1}-colourable.

1. Introduction

We prove the following theorem.

Theorem 1.1. *For any simple graph G and $n \in \mathbb{N}$, if odd-G has no odd-K_{n+1}-minor, then G is 2^{n-1}-colourable.*

We assume that the reader is familiar with sign-graphs, however, the basic definitions can be found in Section 2. Theorem 1.1 generalizes the following result of Wagner [5]: *For any simple graph G and $n \in \mathbb{N}$, if G has no K_{n+1}-minor, then G is 2^{n-1}-colourable.*

Theorem 1.1 provides evidence for the following strengthening of Hadwiger's Conjecture [4]; this strengthening was conjectured by Gerards and Seymour.

Conjecture 1.2. *For any simple graph G and $n \in \mathbb{N}$, if odd-G has no odd-K_{n+1}-minor, then G is n-colourable.*

Conjecture 1.2 has been verified for $n = 3$, by Catlin [1], and for $n = 4$, by Guenin [3].

Geelen et al. [2] strengthen Theorem 1.1 by reducing 2^{n-1} to $O(n\sqrt{\log n})$. The proof in this paper is considerably simpler and is a straightforward extension of Wagner's original proof.

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2. Signed-graphs

A *signed-graph* is a pair (G, Σ) consisting of a graph G with a *signature* $\Sigma \subseteq E(G)$; the edges in Σ are *odd* and the other edges are *even.* For a graph G we let *odd-G* denote the signed-graph $(G, E(G))$.

For $X \subseteq V(G)$, we let $\delta_G(X)$ denote the *cut* induced by X; that is, the set of all edges with exactly one end in X. Two signatures $\Sigma_1, \Sigma_2 \subseteq E(G)$ are *equivalent* if the symmetric difference of Σ_1 and Σ_2 is a cut in G. The operation of replacing a signature in a signed graph with an equivalent signature is called *re-signing*.

A *minor* of a signed graph (G, Σ) is any signed graph that can be obtained from (G, Σ) by any sequence of the following operations: re-signing, deleting vertices or edges, and contracting even edges.

In this paper we are primarily interested in identifying odd-K_n-minors in odd-graphs. The following lemma provides a more tangible description of an odd-K_n-minor; the result is well-known so we skip the elimentary proof.

Lemma 2.1. *Let G be a simple graph. Then, odd-G has an odd-K_n-minor if and only if there exist vertex disjoint trees $(T_1, \ldots, T_n)$ in G and a set $X \subseteq V(G)$ such that*

- *$E(T_i) \subseteq \delta_G(X)$ for each $i \in \{1, \ldots, n\}$, and*
- *for each $1 \le i < j \le n$ there exists an edge $uv \in E(G) - \delta_G(X)$ with $u \in V(T_i)$ and $v \in V(T_j)$.*

3. The main theorem

An *apex vertex* in a graph is a vertex that is adjacent to all other vertices. If v is an apex vertex of a graph G and $G - v$ has a K_n-minor, then G clearly has a K_{n+1}-minor. The analogous result for signed graphs is less obvious.

Lemma 3.1. *Let G be a graph, let v be an apex vertex of G, and let $H = G - v$. If odd-H has an odd-K_n-minor, then odd-G has an odd-K_{n+1}.*

Proof. By Lemma 2.1, There exist vertex disjoint trees $(T_1, \ldots, T_n)$ in H and a set $X \subseteq V(H)$ such that

(1) $E(T_i) \subseteq \delta_G(X)$ for each $i \in \{1, \ldots, n\}$, and
(2) for each $1 \le i < j \le n$ there exists an edge $uv \in E(G) - \delta_G(X)$ with $u \in V(T_i)$ and $v \in V(T_j)$.

Consider distinct $i, j \in \{1, \ldots, n\}$. By (2), we can not have $V(T_i) \subseteq X$ and $V(T_j) - X = \emptyset$. Therefore, by possibly replacing X with $V(H) - X$, we may assume that $V(T_i) - X \neq \emptyset$ for each $i \in \{1, \ldots, n\}$. Let T_{n+1}

be the tree in G consisting of the single vertex v. Now, by Lemma 2.1, odd-G has an odd-K_{n+1}-minor. □

We are now ready to prove Theorem 1.1; for convenience we restate it here in the contrapositive.

Theorem 3.2. *For any simple graph G and $n \in \mathbb{N}$, if G is not 2^{n-1}-colourable, then odd-G has an odd-K_{n+1}-minor,*

Proof. The result is immediate when $n = 1$. We assume that the result holds for $n = k - 1 \geq 1$ and consider the case that $n = k$.

Let G be a simple graph that is not 2^{n-1}-colourable. We lose no generality in assuming that G is connected. Let $v \in V(G)$ and let T be a breadth-first tree of G grown from v. Now, for each $i \in \mathbb{N}$ let $V_i \subseteq V(G)$ be the set of vertices at distance i from v in T and let H_i be the subgraph of G induced by V_i. Let $C^* = E(G) - (E(H_1) \cup E(H_2) \cup \cdots)$. Since C^* is a cut, the restriction of G to C^* is 2-colourable. Then, since G is not 2^{n-1}-colourable, $G - C^*$ is not 2^{n-2}-colourable. The components of $G - C^*$ are $(H_0, H_1, \ldots)$, so there exists $i \in \mathbb{N}$ such that H_i is not 2^{n-2}-colourable. By the induction hypothesis odd-H_i has an odd-K_n-minor. Let G' be obtained by adding an apex vertex to H_i; note that odd-G' is a minor of odd-G. By Lemma 3.1, odd-G' has an odd-K_{n+1}-minor, and, hence, so does odd-G. □

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