Circle Graph Obstructions Under Pivoting

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Abstract: A circle graph is the intersection graph of a set of chords of a circle. The class of circle graphs is closed under pivot-minors. We determine the pivot-minor-minimal non-circle-graphs; there are 15 obstructions. These obstructions are found, by computer search, as a corollary to Bouchet's characterization of circle graphs under local complementation. Our characterization generalizes Kuratowski's Theorem. © 2009 Wiley Periodicals, Inc. J Graph Theory 61: 1–11, 2009

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1. INTRODUCTION

The class of circle graphs is closed with respect to vertex-minors and hence also pivot-minors. (Definitions are postponed until Section 2.) Bouchet [5] gave the following characterization of circle graphs; the graphs W_5, W_7, and BW_3 are defined in Figure 1. All graphs in this paper are simple; graphs have no loops and no parallel edges.

Theorem 1.1 (Bouchet)**.** *A graph is a circle graph if and only it has no vertex-minor that is isomorphic to W_5, W_7, or BW_3.*

As a corollary to Bouchet's theorem we prove the following result.

Theorem 1.2. *A graph is a circle graph if and only it has no pivot-minor that is isomorphic to any of the graphs depicted in Figure* 2*.*

In addition we prove the following related theorem.

Theorem 1.3. *Let $\mathcal{G}$ be a class of graphs closed under vertex-minors. If the excluded vertex-minors for $\mathcal{G}$ each have at most k vertices, then the excluded pivot-minors for $\mathcal{G}$ each have at most $2^k - 1$ vertices.*

The bounds in Theorem 1.3 are not tight enough to be of practical use in proving Theorem 1.2. We show that the excluded pivot-minors can be determined from the excluded vertex-minors by a simple inductive search. Before we discuss this method further, we will briefly discuss the motivation.

De Fraysseix [7] showed that bipartite circle graphs are fundamental graphs of planar graphs. It is then straightforward to show that Theorem 1.2 is a generalization of Kuratowski's Theorem. In fact, Theorem 1.2 applied to bipartite circle graphs is equivalent to the following result, initially due to Tutte [12]: *a binary matroid is the cycle matroid of a planar graph if and only if it does not contain a minor isomorphic to F_7, $M(K_5)$, $M(K_{3,3})$, or to the dual of any of these matroids.* The fundamental graphs of matroids are bipartite and it is straightforward to verify that a pivot-minor of a fundamental graph of a binary matroid (or graph) is a fundamental graph of a minor of the given matroid (or graph). Finally, the graphs H_1, H_2, and BW_3 are fundamental graphs of the matroids $M(K_{3,3})$, $M(K_5)$, and F_7 respectively. (See Figure 3 for drawings of H_1 and H_2.)

The primary motivation for Theorem 1.2 is as a step toward characterizing PU-orientable graphs (defined in Section 2). Bipartite PU-orientable graphs are the

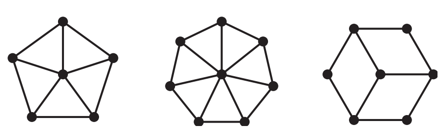

FIGURE 1. W_5, W_7, and BW_3: Excluded vertex-minors for circle graphs.

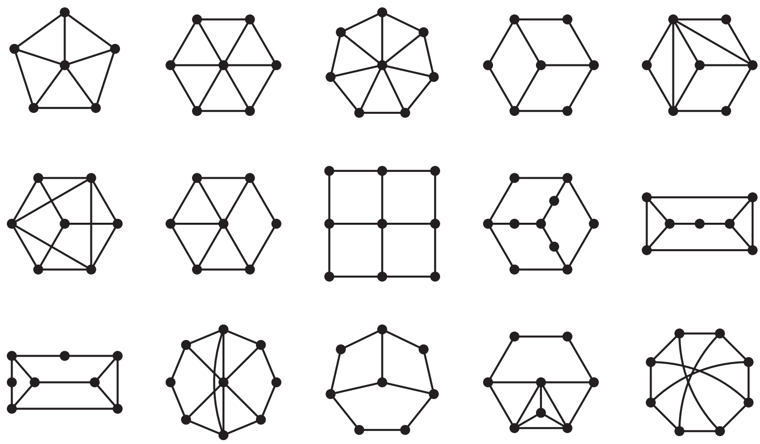

FIGURE 2. Excluded pivot-minors for circle graphs.

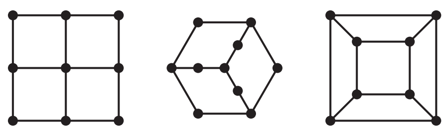

FIGURE 3. H_1, H_2, and Q_3.

fundamental graphs of regular matroids. Seymour's decomposition Theorem [11] provides a good characterization and a recognition algorithm for regular matroids and we hope to obtain similar results for PU-orientable graphs. Bouchet [2] proved that circle graphs admit PU-orientations and we hope that the class of circle graphs will play a central role in a decomposition theorem for PU-orientable graphs. The class of PU-orientable graphs is closed under pivot-minors but not under vertex-minors, and hence it is desirable to have the excluded pivot-minors for the class of circle graphs. Although the class of PU-orientable graphs is not closed under local complementation, Bouchet's theorem does imply the following curious connection between PU-orientability and circle graphs: *a graph is a circle graph if and only if every locally equivalent graph is PU-orientable.*

We prove Theorem 1.2 by studying the graphs that are pivot-minor-minimal while containing a vertex-minor isomorphic to one of W_5, W_7, or BW_3. We require the following two lemmas that are proved in Section 3. The proofs are direct but inelegant. These facts are transparent in the context of isotropic systems; see Bouchet [3]. However, the direct proofs are shorter than the requisite introduction to isotropic systems.

Lemma 1.4 *(*Bouchet [3, (9, 2)]*)*. *Let H be a vertex-minor of a graph G, let $v \in V(G) - V(H)$, and let w be a neighbor of v. Then H is a vertex-minor of one of the graphs $G - v$, $(G * v) - v$, and $(G \times vw) - v$.*

Note that the vertex w in Lemma 1.4 is an arbitrary neighbor of v. Indeed, if w_1 and w_2 are neighbors of v, then $G \times vw_1 = (G \times vw_2) \times w_1 w_2$; see [8, Proposition 2.5]. (This fact is elementary and has been known for more than 20 years, but we could not find an earlier reference.) Therefore, $(G \times vw_1) - v$ is pivot-equivalent to $(G \times vw_2) - v$. We let G/v denote the graph $(G \times vw) - v$ for some neighbor w of v; if v has no neighbors then we let G/v denote $G - v$. Thus, G/v is well defined up to pivot-equivalence and, hence, also up to local equivalence.

Let H be a graph. A graph G is called *H-unique* if G contains H as a vertex-minor and, for each vertex $v \in V(G)$, at most one of the graphs $G - v$, $(G * v) - v$, and G/v has a vertex-minor isomorphic to H. Note that if G is a graph that is pivot-minor-minimal with the property that it has a vertex-minor isomorphic to H, then G is H-unique.

Lemma 1.5. *Let G be an H-unique graph and let G' be a vertex-minor of G that contains H as a vertex-minor. Then G' is H-unique.*

As an immediate corollary to Lemma 1.5 we obtain the following result.

Lemma 1.6. *Let H be a graph and let $k > |V(H)|$. If there is no H-unique graph on k vertices, then every H-unique graph has at most $k - 1$ vertices.*

Using Lemma 1.6 and computer search we prove the following three results. The computation takes less than 3 minutes on a SUN Workstation; we use the package NAUTY for isomorphism-testing.

Lemma 1.7. *Every W_5-unique graph is locally equivalent to a graph that is isomorphic to one of the* 11 *graphs depicted in Figure* 4.

Lemma 1.8. *If G is W_7-unique then either G is locally equivalent to W_7 or G has a vertex-minor isomorphic to W_5.*

Lemma 1.9. *If G is BW_3-unique then either G is locally equivalent to BW_3 or Q_3, or G has a vertex-minor isomorphic to W_5.* (*The graph Q_3 is depicted in Figure* 3.)

Theorem 1.1 and the above lemmas imply that every pivot-minor-minimal non-circle-graph is locally equivalent to W_7, BW_3, Q_3, or to one of the 11 graphs depicted in Figure 4. The number below each of the graphs is the number of pair-wise non-isomorphic graphs that are locally equivalent to it; in total there are 4,239 such graphs. In addition, there are $9+22+4$ graphs locally equivalent to BW_3, W_7, and Q_3. To prove Theorem 1.2, it suffices to check which of these 4,274 graphs is a pivot-minor-minimal non-circle-graph. This is also done by computer and takes less than 3 minutes. This includes 2.5 minutes to generate the 4,274 graphs, 3 seconds to generate all circle graphs up to 9 vertices, and 2 seconds to test which of the 4,274 graphs is a pivot-minor-minimal non-circle-graph.

In the context of delta-matroids, Theorem 1.2 is an excluded-minor characterization for the class of *even* Eulerian delta-matroids. Using Lemmas 1.7, 1.8, and 1.9 one can

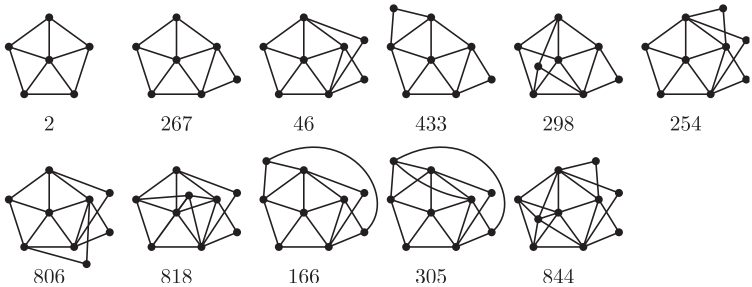

FIGURE 4. W_5-unique graphs.

prove that all excluded-minors for the class of Eulerian delta-matroids have at most 10 elements. We discuss this further in Section 4.

We conclude the introduction by proving the following theorem that immediately implies Theorem 1.3.

Theorem 1.10. *Let H be a graph with $|V(H)|=k$. Then every H-unique graph has at most 2^k-1 vertices.*

Proof. Let G be an H-unique graph. Up to local equivalence we may assume that H is an induced subgraph of G.

Consider any vertex $v \in V(G)-V(H)$. Let G_v denote the subgraph of G induced by the vertex set $V(H)\cup\{v\}$. By Lemma 1.5, G_v is H-unique. Note that $G_v - v = H$ and, hence, $(G_v * v)-v \neq H$. Therefore, v has at least two neighbors in $V(H)$.

Now consider any two distinct vertices $u, v \in V(G)-V(H)$. Let G_{uv} denote the subgraph of G induced by the vertex set $V(H)\cup\{u,v\}$. By Lemma 1.5, G_{uv} is H-unique. Note that $G_{uv}-u-v=H$. Suppose that u and v both have the same neighbors among $V(H)$. If u and v are adjacent, then $G_{uv}\times uv = G_{uv}$ and, hence, both $G_{uv}-u$ and G_{uv}/u have H as a vertex-minor. If u and v are not adjacent, then $G_{uv}*u*v=G_{uv}$ and, hence, both $G_{uv}-u$ and $(G_{uv}*u)-u$ have H as a vertex-minor. In either case we contradict the fact that G_{uv} is H-unique, and hence u and v have distinct neighbors among $V(H)$.

In summary, each vertex in $V(G)-V(H)$ has at least two neighbors in $V(H)$ and no two vertices in $V(G)-V(H)$ have the same neighbors in $V(H)$. Therefore, $|V(G)| \leq |V(H)|+2^k-(k+1)=2^k-1$. ■

We remark that we can slightly improve the above bound to 2^k-2k-1 when the graph H has minimum degree at least 2 and H has no "twin" vertices. Two distinct vertices $u, v \in V(H)$ are *twins* if $N_H(u)-\{v\}=N_H(v)-\{u\}$; here $N_H(v)$ denotes the set of all neighbors of v.

Theorem 1.10 has an interesting consequence.

Theorem 1.11. *Let G_1, G_2 be graphs. If G is a vertex-minor-minimal graph containing both G_1 and G_2 as vertex-minors, then*

$$|V(G)| \leq 2^{|V(G_1)|}+2^{|V(G_2)|}-2.$$

Proof. We claim that if a graph G has a vertex-minor isomorphic to H, then there exists a set X of at most $2^{|V(H)|}-1$ vertices of G such that, for each vertex $v \in V(G)-X$, at least two of the graphs $G-v$, $(G*v)-v$, and G/v have vertex-minors isomorphic to H.

To prove the claim, we may assume that H is an induced subgraph of G. Let X be the vertices of G such that at most one of the graphs $G-v$, $(G*v)-v$, and G/v has vertex-minors isomorphic to H. Let G' be the subgraph of G induced on $V(H)\cup X$. Obviously G' has a vertex-minor isomorphic to H and for each vertex v in X, at most one of the graphs $G'-v$, $(G'*v)-v$, and G'/v has vertex-minors isomorphic to H. Therefore, G' has a vertex-minor G'' such that $X \subseteq V(G'')$, and G'' is H-unique. By Theorem 1.10, $|V(G'')| \le 2^{|V(H)|}-1$ and therefore $|X| \le 2^{|V(H)|}-1$.

By the claim, for $i \in \{1,2\}$, there exists a set X_i of at most $2^{|V(G_i)|}-1$ vertices such that for each vertex v in $V(G)-X_i$, at least two of the graphs $G-v$, $(G*v)-v$, and G/v have vertex-minors isomorphic to G_i. If $|V(G)| > 2^{|V(G_1)|}+2^{|V(G_2)|}-2$, then there exists a vertex $v \notin X_1 \cup X_2$. So at least one of the graphs $G-v$, $(G*v)-v$, and G/v has both G_1 and G_2 as vertex-minors. ∎

An analogous statement for graph minors was conjectured by Lovász and Milgram (see Ungar [13]) and the only known proofs are highly non-trivial and depend upon the graph minors structure theorem of Robertson and Seymour [10].

2. DEFINITIONS

We assume that readers are familiar with elementary definitions in matroid theory including cycle matroids, binary matroids, regular matroids, duality, and minors; see Oxley [9]. However, all references to matroids are peripheral to the main results in the paper.

All graphs in this paper are finite. The following definitions are mostly well known.

Circle graphs. A *chord* of a circle is a straight line segment whose two ends lie on the circle. Let V be a finite set of chords of a circle; the *intersection graph* of V is the graph $G=(V,E)$ where $uv \in E$ if and only if the chords u and v intersect. A *circle graph* is the intersection graph of a set of chords of a circle.

PU-orientable graphs. A *principally unimodular matrix* is a square matrix over the reals such that each non-singular principal submatrix has determinant ± 1. Let $G=(V,E)$ be an orientation of a graph. The *signed adjacency matrix* of G is the $V \times V$ matrix (a_{uv}) where $a_{uv}=1$ when $uv \in E$, $a_{uv}=-1$ when $vu \in E$, and $a_{uv}=0$ otherwise. A graph G is *PU-orientable* if it admits an orientation whose signed adjacency matrix is principally unimodular.

Local complementation and vertex-minors. Let v be a vertex of a graph G. The graph $G*v$ is the graph obtained from G by applying *local complementation* at v; that is, to replace the subgraph induced on the neighbors of v in G with its complement graph.

If G' can be obtained by a sequence of local complementations from G, then we say that G and G' are *locally equivalent*. A *vertex-minor* of G is an induced subgraph of any graph that is locally equivalent to G. (An *induced* subgraph is one that is obtained by vertex deletion.)

Pivot-minors. Let uv be an edge of a graph G. Let $G\times uv = G*u*v*u$; this operation is referred to as *pivoting*. It is straightforward to verify that $G*u*v*u = G*v*u*v$ and, hence, that pivoting is well defined. If G' can be obtained by a sequence of pivots from G, the we say that G and G' are *pivot equivalent*. A *pivot-minor* of G is an induced subgraph of any graph that is pivot equivalent to G.

Fundamental graphs. Let B be a basis of a matroid M. The *fundamental graph* of M with respect to B is the graph with vertex set $E(M)$ and edges uv where $u\in B$, $v\in E(M)-B$, and $(B-\{u\})\cup\{v\}$ is a basis of M. Note that the fundamental graph is bipartite. A *fundamental graph* of a graph G is a fundamental graph of the cycle matroid of G.

3. VERTEX-MINORS

In this section we prove Lemmas 1.4 and 1.5. As noted in the introduction, these results are easy in the context of isotropic systems [3], but the direct proofs given here avoid a lengthy introduction to isotropic systems. We start by proving the following key lemma.

Lemma 3.1. *Let $G=(V,E)$ be a graph and let $v,w\in V$.*

1. *If $v\neq w$ and $vw\notin E$, then $(G*w)-v$, $(G*w*v)-v$, and $(G*w)/v$ are locally equivalent to $G-v$, $(G*v)-v$, and G/v, respectively.*
2. *If $v\neq w$ and $uv\in E$, then $(G*w)-v$, $(G*w*v)-v$, and $(G*w)/v$ are locally equivalent to $G-v$, G/v, and $(G*v)-v$, respectively.*
3. *If $v=w$, then $(G*w)-v$, $((G*w)*v)-v$, and $(G*w)/v$ are locally equivalent to $(G*v)-v$, $G-v$, and G/v, respectively.*

Proof. We first consider the case that $v\neq w$. It is obvious that $(G*w)-v=(G-v)*w$ and hence that $(G*w)-v$ is locally equivalent to $G-v$.

Suppose that $vw\in E$. Note that $(G*w*v)-v=(G*w*v*w*w)-v=((G\times vw)-v)*w=(G/v)*w$ and hence $(G*w*v)-v$ is locally equivalent to G/v. Similarly, $(G*w)/v=((G*w)\times vw)-v=(G*w*w*v*w)-v=((G*v)-v)*w$ and hence $(G*w)/v$ is locally equivalent to $(G*v)-v$.

Now suppose that $vw\notin E$. Note that $(G*w*v)-v=(G*v*w)-v=((G*v)-v)*w$ and hence $(G*w*v)-v$ is locally equivalent to $(G*v)-v$. Let u be a neighbor of v. If $uw\notin E$, then $((G*w)\times uv)-v=((G\times uv)*w)-v=(G/v)*w$ and hence $((G*w)/v$ is locally equivalent to G/v. Hence, we may assume that $uw\in E$. Now $(G*w)/v=(G*w*u*v*u)-v$ and $(G*w*u*v*u)-v$ is locally equivalent to $(G*w*u*v*w)-v=(G*w*u*w*w*v*w)-v=(G\times uw\times vw)-v=(G\times uv)-v=G/v$, as required.

Now suppose that $v=w$. Then $(G*w)-v=(G*v)-v$ and $(G*w*v)-v=G-v$. Moreover, if $uv\in E$, then $(G*w)/v=((G*v)\times uv)-v=(G*v*v*u*v)-v=(G*u*v)-v=((G\times uv)-v)*u$ and hence $(G*w)-v$ is locally equivalent to G/v. ■

We now prove Lemma 1.4 which we restate here for convenience. This lemma appeared in [3, (9.2)].

Lemma 3.2. *Let H be a vertex-minor of a graph G and let $v \in V(G) - V(H)$. Then H is a vertex-minor of one of the graphs $G - v$, $(G * v) - v$, and G/v.*

Proof. If H is a vertex-minor of G, then there is a graph G' that is locally equivalent to G such that H is an induced subgraph of G. Now $G' - v$ contains H as a vertex-minor. Since G is locally equivalent to G' the result follows by Lemma 3.1. ■

Finally we now prove Lemma 1.5 which again we restate for convenience.

Lemma 3.3. *Let G be an H-unique graph and let G' be a vertex-minor of G that contains H as a vertex-minor. Then G' is H-unique.*

Proof. By Lemma 3.1 every graph that is locally equivalent to G is H-unique. Then, inductively, it suffices to consider the case that $G' = G - v$ for some vertex v. If $G - v$ is not H-unique, then there is a vertex $w \neq v$ such that at least two of $(G - v) - w$, $((G - v) * w) - w$, and $(G - v)/w$ contain H as a vertex-minor. But then at least two of $G - w$, $(G * w) - w$, and G/w contain H as a vertex-minor, contradicting the fact that G is H-unique. ■

4. EULERIAN DELTA-MATROIDS

In this section we prove the following theorem.

Theorem 4.1. *The excluded minors for the class of Eulerian delta-matroids have at most* 10 *elements.*

The class of Eulerian delta-matroids is contained in the class of binary delta-matroids. Bouchet and Duchamp [6] determined the excluded minors for the class of binary delta-matroids; the largest of these has four elements. Then to prove Theorem 4.1, it suffices to consider binary delta-matroids. We give a terse introduction to binary delta-matroids and to Eulerian delta matroids, for more detail the reader is referred to Bouchet [1, 4].

Delta-matroids and minors. For sets X and Y, we let $X \Delta Y$ denote the symmetric difference of X and Y. A *delta-matroid* is a pair $M = (V, \mathcal{F})$ of a finite set V and a non-empty set $\mathcal{F}$ of subsets of V, satisfying the *symmetric exchange axiom*: if $A, B \in \mathcal{F}$ and $x \in A \Delta B$, then there is $y \in A \Delta B$ such that $A \Delta \{x, y\} \in \mathcal{F}$. The sets in $\mathcal{F}$ are called *feasible sets* of M. For $X \subseteq V$, we abuse notation be letting $M \Delta X$ denote the set-system $(V, \mathcal{F}')$ where $\mathcal{F}' = \{F \Delta X : F \in \mathcal{F}\}$. It is straightforward to verify that $M \Delta X$ is a delta-matroid. Now let $M \setminus X$ denote the set-system $(V - X, \mathcal{F}'')$ where $\mathcal{F}'' = \{F \subseteq V - X : F \in \mathcal{F}\}$. If $M \setminus X$ has at lease one feasible set, then $M \setminus X$ is a delta-matroid. For any sets $X, Y \subseteq V$, if $(M \Delta X) \setminus Y$ has a feasible set, then we call it a *minor* of M. Two delta-matroids M_1, M_2 are *equivalent* if $M_1 = M_2 \Delta X$ for some set X. A delta-matroid is *even* if its feasible sets either all have even cardinality or all have odd cardinality.

Binary delta-matroids. Let A be a symmetric $V \times V$ matrix over the binary field GF(2). For $X \subseteq V$, we let $A[X]$ denote the principal submatrix of A induced by X. A subset X of V is called *feasible* if $A[X]$ is non-singular. By convention, $A[\emptyset]$ is non-singular. We let $\mathcal{F}_A$ denote the set of all feasible sets and let $\mathrm{DM}(A)=(V,\mathcal{F}_A)$. Bouchet [4] proved that $\mathrm{DM}(A)$ is indeed a delta-matroid. A delta-matroid is *binary* if it is equivalent to $\mathrm{DM}(A)$ for some symmetric matrix A over GF(2). We remark that $\mathrm{DM}(A)$ is even if and only if the diagonal of A is zero.

Eulerian delta-matroids. Let $G=(V,E)$ be a graph and let $X\subseteq V$. Let $\mathrm{A}(G,X)$ denote the symmetric $V\times V$ matrix obtained from the adjacency matrix of G by changing the diagonal entries indexed by X from 0 to 1. Thus, any symmetric binary matrix can be written as $\mathrm{A}(G,X)$ for the appropriate choice of G and X. The binary delta-matroid $\mathrm{DM}(\mathrm{A}(G,X))\Delta Y$ is *Eulerian* if and only if G is a circle graph. This is the most convenient definition for the purpose of proving Theorem 4.1, but Eulerian delta-matroids arise more naturally in relation to euler tours in a connected 4-regular graph; see Bouchet [1].

Bouchet and Duchamp [6] proved that the class of binary delta-matroids is minor-closed. The class of Eulerian delta-matroids is also minor-closed, because the class of circle graphs is closed under local complementation.

If $v\in X$, then it is straightforward to prove that

$$\mathrm{DM}(\mathrm{A}(G,X))\Delta\{v\}=\mathrm{DM}(\mathrm{A}(G*v,X\Delta N_G(v))).$$

Similarly, if $uv\in E$ and $u,v\notin X$, then

$$\mathrm{DM}(\mathrm{A}(G,X))\Delta\{u,v\}=\mathrm{DM}(\mathrm{A}(G\times uv,X)).$$

The operations $\mathrm{A}(G,X)\to\mathrm{A}(G*x,X\Delta N_G(v))$, for $v\in X$, and $\mathrm{A}(G,X)\to\mathrm{A}(G\times uv,X)$, for $uv\in E$ and $u,v\notin X$, are referred to as *elementary pivots*. If $\mathrm{DM}(\mathrm{A}(G_1,X_1))=\mathrm{DM}(\mathrm{A}(G_2,X_2))\Delta Y$, then we can obtain $\mathrm{A}(G_2,X_2)$ from $\mathrm{A}(G_1,X_1)$ via a sequence of elementary pivots, implied by the uniqueness of binary representation for binary delta-matroids; see Bouchet and Duchamp [6, Property 3.1].

Lemma 4.2. *Let $G=(V,E)$ be a graph, let $X\subseteq V$, and let $v\in V$. If* $\mathrm{DM}(\mathrm{A}(G,X))$ *is an excluded minor for the class of Eulerian delta-matroids, then at least two of the graphs $G-v$, $(G*v)-v$, and G/v are circle graphs.*

Proof. Suppose that $v\in X$. Then both $\mathrm{DM}(\mathrm{A}(G,X))\setminus\{v\}$ and $(\mathrm{DM}(\mathrm{A}(G,X))\Delta\{v\})\setminus\{v\}$ are Eulerian. Thus, $G-v$ and $(G*v)-v$ are both circle graphs, as required. Now suppose that $v\notin X$. Since $G-v$ is a circle graph but G is not, $N_G(v)\neq\emptyset$; let $w\in N_G(v)$. Now suppose that $w\notin X$. Then $\mathrm{DM}(\mathrm{A}(G,X))\setminus\{v\}$ and $(\mathrm{DM}(\mathrm{A}(G,X))\Delta\{v,w\})\setminus\{v\}$ are both Eulerian. Thus, $G-v$ and G/v are both circle graphs, as required. Finally suppose that $w\in X$. Now $\mathrm{DM}(\mathrm{A}(G,X))\Delta\{w\}=\mathrm{DM}(\mathrm{A}(G*w,X\Delta N_G(w)))$ is an excluded minor for the class of Eulerian delta-matroids and $v\in X\Delta N_G(w)$. Then, by the first case in the proof, $(G*w)-v$ and $((G*w)*v)-v$ are both circle graphs. So, by Lemma 3.1, $G-v$ and G/v are both circle graphs. ■

Lemma 4.2 and Theorem 1.1 imply that, if $\mathrm{DM}(\mathrm{A}(G,X))$ is an excluded minor for the class of Eulerian delta-matroids, then G is W_5-, W_7-, or BW_3-unique. Then Theorem 4.1 follows immediately from Lemmas 1.7, 1.8 and 1.9.

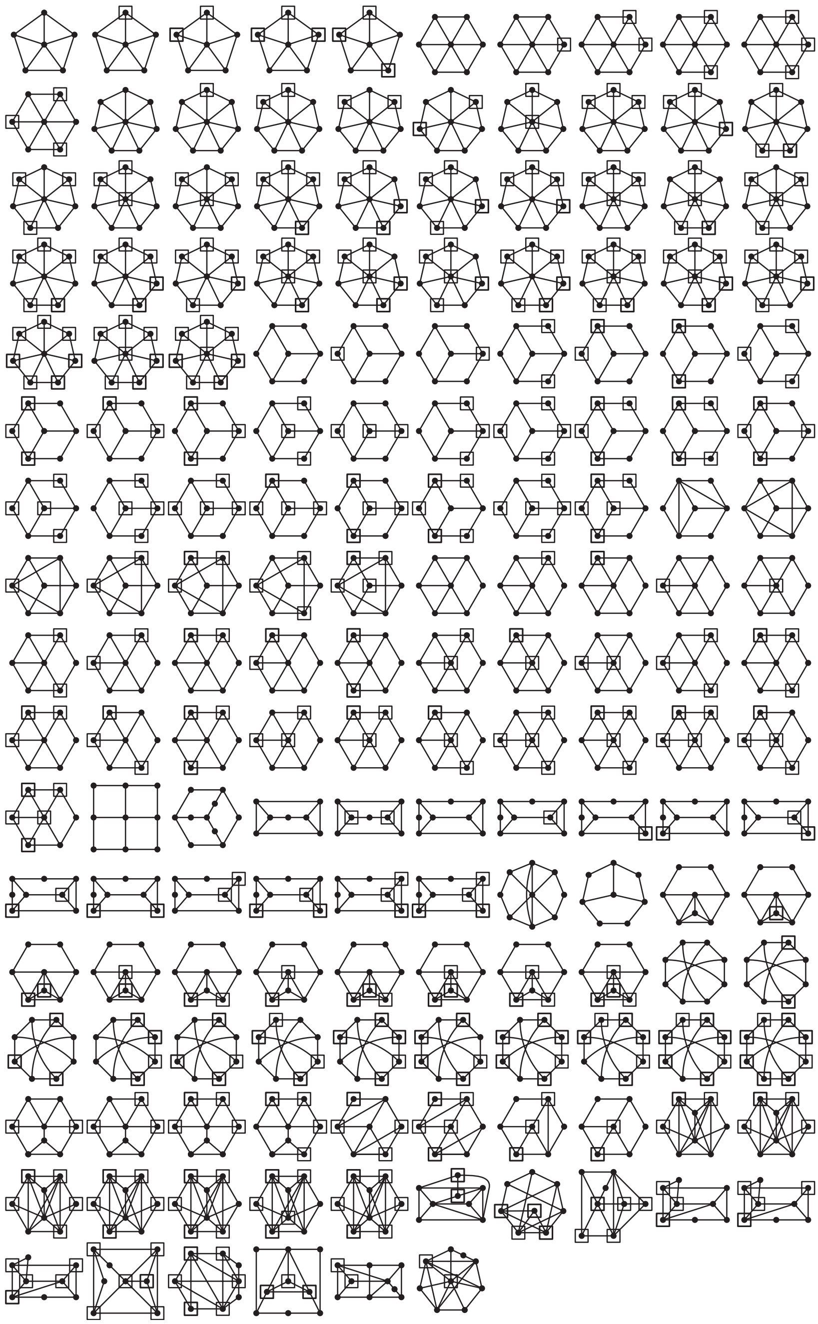

FIGURE 5. Pairwise non-equivalent binary excluded minors $\mathrm{DM}(\mathrm{A}(G, X))$ for the class of Eulerian delta-matroids (vertices in X are denoted by squares).

By computer search, we found 166 non-equivalent binary excluded minors for the class of Eulerian delta-matroids. We list them in Figure 5. Combined with the 5 excluded minors for the class of binary delta-matroids (see Bouchet and Duchamp [6]), we conclude that there are exactly 171 excluded minors for the class of Eulerian delta-matroids. This computation takes 14 minutes if the list of all W_5-unique graphs is given.

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REFERENCES

[1] A. Bouchet, Greedy algorithm and symmetric matroids, Math Programming 38(2) (1987), 147–159.

[2] A. Bouchet, Unimodularity and circle graphs, Discrete Math 66(1–2) (1987), 203–208.

[3] A. Bouchet, Graphic presentations of isotropic systems, J Combin Theory Ser B 45(1) (1988), 58–76.

[4] A. Bouchet, Representability of Δ-matroids, *Combinatorics* (*Eger*, 1987), North-Holland, Amsterdam, Colloq Math Soc János Bolyai 52 (1988), 167–182.

[5] A. Bouchet, Circle graph obstructions, J Combin Theory Ser B 60(1) (1994), 107–144.

[6] A. Bouchet and A. Duchamp, Representability of Δ-matroids over GF(2), Linear Algebra Appl 146 (1991), 67–78.

[7] H. de Fraysseix, A characterization of circle graphs, European J Combin 5(3) (1984), 223–238.

[8] S. Oum, Rank-width and vertex-minors, J Combin Theory Ser B 95(1) (2005), 79–100.

[9] J. G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

[10] N. Robertson and P. Seymour, Graph minors. XVI. Excluding a non-planar graph, J Combin Theory Ser B 89(1) (2003), 43–76.

[11] P. Seymour, Decomposition of regular matroids, J Combin Theory Ser B 28(3) (1980), 305–359.

[12] W. T. Tutte, Matroids and graphs, Trans Amer Math Soc 90 (1959), 527–552.

[13] P. Ungar, Research problems: dissections and intertwinings of graphs, Amer Math Monthly 85(8) (1978), 664–666.