Circle Graph Obstructions Under Pivoting

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Abstract: A circle graph is the intersection graph of a set of chords of a circle. The class of circle graphs is closed under pivot-minors. We determine the pivot-minor-minimal non-circle-graphs; there are 15 obstructions. These obstructions are found, by computer search, as a corollary to Bouchet’s characterization of circle graphs under local complementation. Our characterization generalizes Kuratowski’s Theorem. © 2009 Wiley Periodicals, Inc. J Graph Theory 61: 1–11, 2009

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1. INTRODUCTION

The class of circle graphs is closed with respect to vertex-minors and hence also pivot-minors. (Definitions are postponed until Section 2.) Bouchet [5] gave the following characterization of circle graphs; the graphs $W_5$, $W_7$, and $BW_3$ are defined in Figure 1. All graphs in this paper are simple; graphs have no loops and no parallel edges.

**Theorem 1.1** (Bouchet). A graph is a circle graph if and only if it has no vertex-minor that is isomorphic to $W_5$, $W_7$, or $BW_3$.

As a corollary to Bouchet’s theorem we prove the following result.

**Theorem 1.2.** A graph is a circle graph if and only if it has no pivot-minor that is isomorphic to any of the graphs depicted in Figure 2.

In addition we prove the following related theorem.

**Theorem 1.3.** Let $G$ be a class of graphs closed under vertex-minors. If the excluded vertex-minors for $G$ each have at most $k$ vertices, then the excluded pivot-minors for $G$ each have at most $2^k - 1$ vertices.

The bounds in Theorem 1.3 are not tight enough to be of practical use in proving Theorem 1.2. We show that the excluded pivot-minors can be determined from the excluded vertex-minors by a simple inductive search. Before we discuss this method further, we will briefly discuss the motivation.

De Fraysseix [7] showed that bipartite circle graphs are fundamental graphs of planar graphs. It is then straightforward to show that Theorem 1.2 is a generalization of Kuratowski’s Theorem. In fact, Theorem 1.2 applied to bipartite circle graphs is equivalent to the following result, initially due to Tutte [12]: a binary matroid is the cycle matroid of a planar graph if and only if it does not contain a minor isomorphic to $F_7$, $M(K_5)$, $M(K_{3,3})$, or to the dual of any of these matroids. The fundamental graphs of matroids are bipartite and it is straightforward to verify that a pivot-minor of a fundamental graph of a binary matroid (or graph) is a fundamental graph of a minor of the given matroid (or graph). Finally, the graphs $H_1$, $H_2$, and $BW_3$ are fundamental graphs of the matroids $M(K_{3,3})$, $M(K_5)$, and $F_7$ respectively. (See Figure 3 for drawings of $H_1$ and $H_2$.)

The primary motivation for Theorem 1.2 is as a step toward characterizing PU-orientable graphs (defined in Section 2). Bipartite PU-orientable graphs are the

![Figure 1. $W_5$, $W_7$, and $BW_3$: Excluded vertex-minors for circle graphs.](image)

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fundamental graphs of regular matroids. Seymour’s decomposition Theorem [11] provides a good characterization and a recognition algorithm for regular matroids and we hope to obtain similar results for PU-orientable graphs. Bouchet [2] proved that circle graphs admit PU-orientations and we hope that the class of circle graphs will play a central role in a decomposition theorem for PU-orientable graphs. The class of PU-orientable graphs is closed under pivot-minors but not under vertex-minors, and hence it is desirable to have the excluded pivot-minors for the class of circle graphs. Although the class of PU-orientable graphs is not closed under local complementation, Bouchet’s theorem does imply the following curious connection between PU-orientability and circle graphs: a graph is a circle graph if and only if every locally equivalent graph is PU-orientable.

We prove Theorem 1.2 by studying the graphs that are pivot-minor-minimal while containing a vertex-minor isomorphic to one of $W_5, W_7,$ or $BW_3$. We require the following two lemmas that are proved in Section 3. The proofs are direct but inelegant. These facts are transparent in the context of isotropic systems; see Bouchet [3]. However, the direct proofs are shorter than the requisite introduction to isotropic systems.
Lemma 1.4 (Bouchet [3, (9, 2)]). Let $H$ be a vertex-minor of a graph $G$, let $v \in V(G) - V(H)$, and let $w$ be a neighbor of $v$. Then $H$ is a vertex-minor of one of the graphs $G - v$, $(G * v) - v$, and $(G \times vw) - v$.

Note that the vertex $w$ in Lemma 1.4 is an arbitrary neighbor of $v$. Indeed, if $w_1$ and $w_2$ are neighbors of $v$, then $G \times vw_1 = (G \times vw_2) \times w_1 w_2$; see [8, Proposition 2.5]. (This fact is elementary and has been known for more than 20 years, but we could not find an earlier reference.) Therefore, $(G \times vw_1) - v$ is pivot-equivalent to $(G \times vw_2) - v$. We let $G/v$ denote the graph $(G \times vw) - v$ for some neighbor $w$ of $v$; if $v$ has no neighbors then we let $G/v$ denote $G - v$. Thus, $G/v$ is well defined up to pivot-equivalence and, hence, also up to local equivalence.

Let $H$ be a graph. A graph $G$ is called $H$-unique if $G$ contains $H$ as a vertex-minor and, for each vertex $v \in V(G)$, at most one of the graphs $G - v$, $(G * v) - v$, and $G/v$ has a vertex-minor isomorphic to $H$. Note that if $G$ is a graph that is pivot-minor-minimal with the property that it has a vertex-minor isomorphic to $H$, then $G$ is $H$-unique.

Lemma 1.5. Let $G$ be an $H$-unique graph and let $G'$ be a vertex-minor of $G$ that contains $H$ as a vertex-minor. Then $G'$ is $H$-unique.

As an immediate corollary to Lemma 1.5 we obtain the following result.

Lemma 1.6. Let $H$ be a graph and let $k > |V(H)|$. If there is no $H$-unique graph on $k$ vertices, then every $H$-unique graph has at most $k - 1$ vertices.

Using Lemma 1.6 and computer search we prove the following three results. The computation takes less than 3 minutes on a SUN Workstation; we use the package NAUTY for isomorphism-testing.

Lemma 1.7. Every $W_5$-unique graph is locally equivalent to a graph that is isomorphic to one of the 11 graphs depicted in Figure 4.

Lemma 1.8. If $G$ is $W_7$-unique then either $G$ is locally equivalent to $W_7$ or $G$ has a vertex-minor isomorphic to $W_5$.

Lemma 1.9. If $G$ is $BW_3$-unique then either $G$ is locally equivalent to $BW_3$ or $Q_3$, or $G$ has a vertex-minor isomorphic to $W_5$. (The graph $Q_3$ is depicted in Figure 3.)

Theorem 1.1 and the above lemmas imply that every pivot-minor-minimal non-circle-graph is locally equivalent to $W_7$, $BW_3$, $Q_3$, or to one of the 11 graphs depicted in Figure 4. The number below each of the graphs is the number of pair-wise non-isomorphic graphs that are locally equivalent to it; in total there are 4,239 such graphs. In addition, there are $9 + 22 + 4$ graphs locally equivalent to $BW_3$, $W_7$, and $Q_3$. To prove Theorem 1.2, it suffices to check which of these 4,274 graphs is a pivot-minor-minimal non-circle-graph. This is also done by computer and takes less than 3 minutes. This includes 2.5 minutes to generate the 4,274 graphs, 3 seconds to generate all circle graphs up to 9 vertices, and 2 seconds to test which of the 4,274 graphs is a pivot-minor-minimal non-circle-graph.

In the context of delta-matroids, Theorem 1.2 is an excluded-minor characterization for the class of even Eulerian delta-matroids. Using Lemmas 1.7, 1.8, and 1.9 one can
prove that all excluded-minors for the class of Eulerian delta-matroids have at most 10 elements. We discuss this further in Section 4.

We conclude the introduction by proving the following theorem that immediately implies Theorem 1.3.

**Theorem 1.10.** Let \( H \) be a graph with \( |V(H)| = k \). Then every \( H \)-unique graph has at most \( 2^k - 1 \) vertices.

**Proof.** Let \( G \) be an \( H \)-unique graph. Up to local equivalence we may assume that \( H \) is an induced subgraph of \( G \).

Consider any vertex \( v \in V(G) - V(H) \). Let \( G_v \) denote the subgraph of \( G \) induced by the vertex set \( V(H) \cup \{v\} \). By Lemma 1.5, \( G_v \) is \( H \)-unique. Note that \( G_v - v = H \) and, hence, \( (G_v * v) - v \neq H \). Therefore, \( v \) has at least two neighbors in \( V(H) \).

Now consider any two distinct vertices \( u, v \in V(G) - V(H) \). Let \( G_{uv} \) denote the subgraph of \( G \) induced by the vertex set \( V(H) \cup \{u, v\} \). By Lemma 1.5, \( G_{uv} \) is \( H \)-unique. Note that \( G_{uv} - u - v = H \). Suppose that \( u \) and \( v \) are adjacent, then \( G_{uv} * uv = G_{uv} \) and, hence, both \( G_{uv} - u \) and \( G_{uv} - v \) have \( H \) as a vertex-minor. If \( u \) and \( v \) are not adjacent, then \( G_{uv} * u * v = G_{uv} \) and, hence, both \( G_{uv} - u \) and \( (G_{uv} * u) - u \) have \( H \) as a vertex-minor. In either case we contradict the fact that \( G_{uv} \) is \( H \)-unique, and hence \( u \) and \( v \) have distinct neighbors among \( V(H) \).

In summary, each vertex in \( V(G) - V(H) \) has at least two neighbors in \( V(H) \) and no two vertices in \( V(G) - V(H) \) have the same neighbors in \( V(H) \). Therefore, \( |V(G)| \leq |V(H)| + 2^k - (k + 1) = 2^k - 1 \). \( \blacksquare \)

We remark that we can slightly improve the above bound to \( 2^k - 2k - 1 \) when the graph \( H \) has minimum degree at least 2 and \( H \) has no “twin” vertices. Two distinct vertices \( u, v \in V(H) \) are twins if \( N_H(u) - \{v\} = N_H(v) - \{u\} \); here \( N_H(v) \) denotes the set of all neighbors of \( v \).

Theorem 1.10 has an interesting consequence.

**Theorem 1.11.** Let \( G_1, G_2 \) be graphs. If \( G \) is a vertex-minor-minimal graph containing both \( G_1 \) and \( G_2 \) as vertex-minors, then

\[
|V(G)| \leq 2^{|V(G_1)|} + 2^{|V(G_2)|} - 2.
\]

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**Proof.** We claim that if a graph $G$ has a vertex-minor isomorphic to $H$, then there exists a set $X$ of at most $2^{|V(H)|} - 1$ vertices of $G$ such that, for each vertex $v \in V(G) - X$, at least two of the graphs $G - v$, $(G*v) - v$, and $G/v$ have vertex-minors isomorphic to $H$.

To prove the claim, we may assume that $H$ is an induced subgraph of $G$. Let $X$ be the vertices of $G$ such that at most one of the graphs $G - v$, $(G*v) - v$, and $G/v$ has vertex-minors isomorphic to $H$. Let $G'$ be the subgraph of $G$ induced on $V(H) \cup X$. Obviously $G'$ has a vertex-minor isomorphic to $H$ and for each vertex $v$ in $X$, at most one of the graphs $G' - v$, $(G'*v) - v$, and $G'/v$ has vertex-minors isomorphic to $H$. Therefore, $G'$ has a vertex-minor $G''$ such that $X \subseteq V(G'')$, and $G''$ is $H$-unique. By Theorem 1.10, $|V(G'')| \leq 2^{|V(H)|} - 1$ and therefore $|X| \leq 2^{|V(H)|} - 1$.

By the claim, for $i \in \{1, 2\}$, there exists a set $X_i$ of at most $2^{|V(G_i)|} - 1$ vertices such that for each vertex $v$ in $V(G) - X_i$, at least two of the graphs $G - v$, $(G*v) - v$, and $G/v$ have vertex-minors isomorphic to $G_i$. If $|V(G)| > 2^{|V(G_1)|} + 2^{|V(G_2)|} - 2$, then there exists a vertex $v \notin X_1 \cup X_2$. So at least one of the graphs $G - v$, $(G*v) - v$, and $G/v$ has both $G_1$ and $G_2$ as vertex-minors.

An analogous statement for graph minors was conjectured by Lovász and Milgram (see Ungar [13]) and the only known proofs are highly non-trivial and depend upon the graph minors structure theorem of Robertson and Seymour [10].

2. **DEFINITIONS**

We assume that readers are familiar with elementary definitions in matroid theory including cycle matroids, binary matroids, regular matroids, duality, and minors; see Oxley [9]. However, all references to matroids are peripheral to the main results in the paper.

All graphs in this paper are finite. The following definitions are mostly well known.

**Circle graphs.** A chord of a circle is a straight line segment whose two ends lie on the circle. Let $V$ be a finite set of chords of a circle; the intersection graph of $V$ is the graph $G=(V,E)$ where $uv \in E$ if and only if the chords $u$ and $v$ intersect. A circle graph is the intersection graph of a set of chords of a circle.

**PU-orientable graphs.** A principally unimodular matrix is a square matrix over the reals such that each non-singular principal submatrix has determinant $\pm 1$. Let $G=(V,E)$ be an orientation of a graph. The signed adjacency matrix of $G$ is the $V \times V$ matrix $(a_{uv})$ where $a_{uv} = 1$ when $uv \in E$, $a_{uv} = -1$ when $vu \in E$, and $a_{uv} = 0$ otherwise. A graph $G$ is PU-orientable if it admits an orientation whose signed adjacency matrix is principally unimodular.

**Local complementation and vertex-minors.** Let $v$ be a vertex of a graph $G$. The graph $G*v$ is the graph obtained from $G$ by applying local complementation at $v$; that is, to replace the subgraph induced on the neighbors of $v$ in $G$ with its complement graph.

If $G'$ can be obtained by a sequence of local complementations from $G$, then we say that $G$ and $G'$ are locally equivalent. A vertex-minor of $G$ is an induced subgraph of any graph that is locally equivalent to $G$. (An induced subgraph is one that is obtained by vertex deletion.)
Pivot-minors. Let \( uv \) be an edge of a graph \( G \). Let \( G \times uv = G * u * v * u \); this operation is referred to as pivoting. It is straightforward to verify that \( G * u * v * u = G * v * u * v \) and, hence, that pivoting is well defined. If \( G' \) can be obtained by a sequence of pivots from \( G \), the we say that \( G \) and \( G' \) are pivot equivalent. A pivot-minor of \( G \) is an induced subgraph of any graph that is pivot equivalent to \( G \).

Fundamental graphs. Let \( B \) be a basis of a matroid \( M \). The fundamental graph of \( M \) with respect to \( B \) is the graph with vertex set \( E(M) \) and edges \( uv \) where \( u \in B \), \( v \in E(M) - B \), and \( (B - \{u\}) \cup \{v\} \) is a basis of \( M \). Note that the fundamental graph is bipartite. A fundamental graph of a graph \( G \) is a fundamental graph of the cycle matroid of \( G \).

3. VERTEX-MINORS

In this section we prove Lemmas 1.4 and 1.5. As noted in the introduction, these results are easy in the context of isotropic systems [3], but the direct proofs given here avoid a lengthy introduction to isotropic systems. We start by proving the following key lemma.

**Lemma 3.1.** Let \( G = (V, E) \) be a graph and let \( v, w \in V \).

1. If \( v \neq w \) and \( vw \notin E \), then \( (G * w) - v, (G * w * v) - v, \) and \( (G * w) / v \) are locally equivalent to \( G - v, (G * v) - v, \) and \( G / v \), respectively.
2. If \( v \neq w \) and \( uv \in E \), then \( (G * w) - v, (G * w * v) - v, \) and \( (G * w) / v \) are locally equivalent to \( G - v, G / v, \) and \( (G * v) - v \), respectively.
3. If \( v = w \), then \( (G * w) - v, ((G * u) * v) - v, \) and \( (G * w) / v \) are locally equivalent to \( (G * v) - v, G - v, \) and \( G / v \), respectively.

**Proof.** We first consider the case that \( v \neq w \). It is obvious that \( (G * w) - v = (G - v) * w \) and hence that \( (G * w) - v \) is locally equivalent to \( G - v \).

Suppose that \( vw \in E \). Note that \( (G * w * v) - v = (G * w * v * w) - v = ((G * v) * w - v) * w = (G / w) * v \) and hence \( (G * w * v) - v \) is locally equivalent to \( G / v \). Similarly, \( (G * w) / v = ((G * v) * w) - v = ((G * v) - v) * w \) and hence \( (G * w) / v \) is locally equivalent to \( (G * v) - v \).

Now suppose that \( vw \notin E \). Note that \( (G * w * v) - v = (G * v * w) - v = ((G * v) - v) * w \) and hence \( (G * w * v) - v \) is locally equivalent to \( G / v \). Let \( u \) be a neighbor of \( v \). If \( uw \notin E \), then \( ((G * w) * v) - v = (G / u) * w \) and hence \( (G * w) / v \) is locally equivalent to \( G / v \). Hence, we may assume that \( uw \in E \). Now \( (G * w) / v = (G * u * v * u) - v \) and \( (G * w * u * v * u) - v \) is locally equivalent to \( (G * w * u * v * u) - v = (G * w * u * w * w * v * w) - v = (G * w * u * w * v) - v = (G * w * u * w * v) - v = (G * u * v) - v = (G * v) - v = (G * w) - v \) and hence \( (G * w) - v \) is locally equivalent to \( G / v \).

Now suppose that \( v = w \). Then \( (G * w) - v = (G * v) - v \) and \( (G * w * v) - v = G - v \). Moreover, if \( uv \in E \), then \( (G * w) / v = ((G * v) * u) - v = (G * u * v * u * v) - v = (G * u * v * u) - v = (G * u * v) - v = (G * v) - v = (G * w) - v \) and hence \( (G * w) - v \) is locally equivalent to \( G / v \).

We now prove Lemma 1.4 which we restate here for convenience. This lemma appeared in [3, (9.2)].
Lemma 3.2. Let $H$ be a vertex-minor of a graph $G$ and let $v \in V(G) - V(H)$. Then $H$ is a vertex-minor of one of the graphs $G - v$, $(G \ast v) - v$, and $G/v$.

Proof. If $H$ is a vertex-minor of $G$, then there is a graph $G'$ that is locally equivalent to $G$ such that $H$ is an induced subgraph of $G$. Now $G' - v$ contains $H$ as a vertex-minor. Since $G$ is locally equivalent to $G'$ the result follows by Lemma 3.1.

Finally we now prove Lemma 1.5 which again we restate for convenience.

Lemma 3.3. Let $G$ be an $H$-unique graph and let $G'$ be a vertex-minor of $G$ that contains $H$ as a vertex-minor. Then $G'$ is $H$-unique.

Proof. By Lemma 3.1 every graph that is locally equivalent to $G$ is $H$-unique. Then, inductively, it suffices to consider the case that $G' = G - v$ for some vertex $v$. If $G - v$ is not $H$-unique, then there is a vertex $w \neq v$ such that at least two of $(G - v) - w$, $(G - v) \ast w) - w$, and $(G - v)/w$ contain $H$ as a vertex-minor. But then at least two of $G - w$, $(G \ast w) - w$, and $G/w$ contain $H$ as a vertex-minor, contradicting the fact that $G$ is $H$-unique.

4. EULERIAN DELTA-MATROIDS

In this section we prove the following theorem.

Theorem 4.1. The excluded minors for the class of Eulerian delta-matroids have at most 10 elements.

The class of Eulerian delta-matroids is contained in the class of binary delta-matroids. Bouchet and Duchamp [6] determined the excluded minors for the class of binary delta-matroids; the largest of these has four elements. Then to prove Theorem 4.1, it suffices to consider binary delta-matroids. We give a terse introduction to binary delta-matroids and to Eulerian delta matroids, for more detail the reader is referred to Bouchet [1, 4].

Delta-matroids and minors. For sets $X$ and $Y$, we let $X \Delta Y$ denote the symmetric difference of $X$ and $Y$. A delta-matroid is a pair $M = (V, \mathcal{F})$ of a finite set $V$ and a non-empty set $\mathcal{F}$ of subsets of $V$, satisfying the symmetric exchange axiom: if $A, B \in \mathcal{F}$ and $x \in A\Delta B$, then there is $y \in A\Delta B$ such that $A\Delta\{x, y\} \in \mathcal{F}$. The sets in $\mathcal{F}$ are called feasible sets of $M$. For $X \subseteq V$, we abuse notation by letting $M\Delta X$ denote the set-system $(V, \mathcal{F}')$ where $\mathcal{F}' = \{F \Delta X : F \in \mathcal{F}\}$. It is straightforward to verify that $M\Delta X$ is a delta-matroid. Now let $M \setminus X$ denote the set-system $(V - X, \mathcal{F}'')$ where $\mathcal{F}'' = \{F \subseteq V - X : F \in \mathcal{F}\}$. If $M \setminus X$ has at least one feasible set, then $M \setminus X$ is a matroid. For any sets $X, Y \subseteq V$, if $(M\Delta X) \setminus Y$ has a feasible set, then we call it a minor of $M$. Two delta-matroids $M_1$, $M_2$ are equivalent if $M_1 = M_2\Delta X$ for some set $X$. A delta-matroid is even if its feasible sets either all have even cardinality or all have odd cardinality.

Binary delta-matroids. Let $A$ be a symmetric $V \times V$ matrix over the binary field GF(2). For $X \subseteq V$, we let $A[X]$ denote the principal submatrix of $A$ induced by $X$. A subset $X$ of $V$ is called feasible if $A[X]$ is non-singular. By convention, $A[\emptyset]$ is non-singular. A finite collection $\mathcal{C}$ of even binary delta-matroids is odd if it contains a binary delta-matroid that is odd. A finite collection $\mathcal{C}$ of even binary delta-matroids is even if it is not odd. According to Bouchet, a binary delta-matroid is even if and only if it contains at least one odd binary delta-matroid. Let $\mathcal{C}$ be the collection of all odd binary delta-matroids. Then $\mathcal{C}$ is a matroid. According to Bouchet, the class of even binary delta-matroids is the smallest matroid that contains the collection $\mathcal{C}$ of odd binary delta-matroids.
non-singular. We let \( \mathcal{F}_A \) denote the set of all feasible sets and let \( \text{DM}(A) = (V, \mathcal{F}_A) \). Bouchet [4] proved that \( \text{DM}(A) \) is indeed a delta-matroid. A delta-matroid is binary if it is equivalent to \( \text{DM}(A) \) for some symmetric matrix \( A \) over GF(2). We remark that \( \text{DM}(A) \) is even if and only if the diagonal of \( A \) is zero.

**Eulerian delta-matroids.** Let \( G = (V, E) \) be a graph and let \( X \subseteq V \). Let \( A(G, X) \) denote the symmetric \( V \times V \) matrix obtained from the adjacency matrix of \( G \) by changing the diagonal entries indexed by \( X \) from 0 to 1. Thus, any symmetric binary matrix can be written as \( A(G, X) \) for the appropriate choice of \( G \) and \( X \). The binary delta-matroid \( \text{DM}(A(G, X)) \Delta Y \) is Eulerian if and only if \( G \) is a circle graph. This is the most convenient definition for the purpose of proving Theorem 4.1, but Eulerian delta-matroids arise more naturally in relation to euler tours in a connected 4-regular graph; see Bouchet [1].

Bouchet and Duchamp [6] proved that the class of binary delta-matroids is minor-closed. The class of Eulerian delta-matroids is also minor-closed, because the class of circle graphs is closed under local complementation.

If \( v \in X \), then it is straightforward to prove that

\[
\text{DM}(A(G, X)) \Delta \{v\} = \text{DM}(A(G \ast v, X \Delta N_G(v))).
\]

Similarly, if \( uv \in E \) and \( u, v \notin X \), then

\[
\text{DM}(A(G, X)) \Delta \{u, v\} = \text{DM}(A(G \times uv, X)).
\]

The operations \( A(G, X) \to A(G \ast x, X \Delta N_G(v)) \), for \( v \in X \), and \( A(G, X) \to A(G \times uv, X) \), for \( uv \in E \) and \( u, v \notin X \), are referred to as elementary pivots. If \( \text{DM}(A(G_1, X_1)) = \text{DM}(A(G_2, X_2)) \Delta Y \), then we can obtain \( A(G_2, X_2) \) from \( A(G_1, X_1) \) via a sequence of elementary pivots, implied by the uniqueness of binary representation for binary delta-matroids; see Bouchet and Duchamp [6, Property 3.1].

**Lemma 4.2.** Let \( G = (V, E) \) be a graph, let \( X \subseteq V \), and let \( v \in V \). If \( \text{DM}(A(G, X)) \) is an excluded minor for the class of Eulerian delta-matroids, then at least two of the graphs \( G - v, (G \ast v) - v \), and \( G \ast v \) are circle graphs.

**Proof.** Suppose that \( v \in X \). Then both \( \text{DM}(A(G, X)) \setminus \{v\} \) and \( (\text{DM}(A(G, X)) \setminus \{v\}) \) are Eulerian. Thus, \( G - v \) and \( (G \ast v) - v \) are both circle graphs, as required. Now suppose that \( v \notin X \). Since \( G - v \) is a circle graph but \( G \) is not, \( N_G(v) \neq \emptyset \); let \( w \in N_G(v) \). Now suppose that \( w \notin X \). Then \( \text{DM}(A(G, X)) \setminus \{v\} \) and \( (\text{DM}(A(G, X)) \setminus \{v, w\}) \) are both Eulerian. Thus, \( G - v \) and \( G \ast v \) are both circle graphs, as required. Finally suppose that \( w \in X \). Now \( \text{DM}(A(G, X)) \Delta \{w\} = \text{DM}(A(G \ast w, X \Delta N_G(w))) \) is an excluded minor for the class of Eulerian delta-matroids and \( v \in X \Delta N_G(w) \). Then, by the first case in the proof, \( (G \ast w) - v \) and \( ((G \ast w) \ast v) - v \) are both circle graphs. So, by Lemma 3.1, \( G - v \) and \( G \ast v \) are both circle graphs.

Lemma 4.2 and Theorem 1.1 imply that, if \( \text{DM}(A(G, X)) \) is an excluded minor for the class of Eulerian delta-matroids, then \( G \) is \( W_5 \)-, \( W_7 \)-, or \( BW_3 \)-unique. Then Theorem 4.1 follows immediately from Lemmas 1.7, 1.8 and 1.9.
FIGURE 5. Pairwise non-equivalent binary excluded minors $\text{DM}(\mathcal{A}(G, X))$ for the class of Eulerian delta-matroids (vertices in $X$ are denoted by squares).
By computer search, we found 166 non-equivalent binary excluded minors for the class of Eulerian delta-matroids. We list them in Figure 5. Combined with the 5 excluded minors for the class of binary delta-matroids (see Bouchet and Duchamp [6]), we conclude that there are exactly 171 excluded minors for the class of Eulerian delta-matroids. This computation takes 14 minutes if the list of all \( W_5 \)-unique graphs is given.

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REFERENCES


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