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Disjoint cocircuits in matroids with large rank ☆

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Abstract

We prove that, for any positive integers n, k and q, there exists an integer R such that, if M is a matroid with no $M(K_n)$- or $U_{2,q+2}$-minor, then either M has a collection of k disjoint cocircuits or M has rank at most R. Applied to the class of cographic matroids, this result implies the edge-disjoint version of the Erdös–Pósa Theorem.

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1. Introduction

A rank-r matroid can have at most r pairwise disjoint cocircuits. There are, however, matroids with arbitrarily large rank but with no two disjoint cocircuits. For example, for any positive integer n, $M(K_n)$, the cycle matroid of the complete graph on n vertices, does not contain a pair of disjoint cocircuits. Also, for positive integers r and n, if $n \geqslant 2r-1$, then $U_{r,n}$, the rank-r uniform matroid with n elements, does not contain a pair of disjoint cocircuits. We prove the following theorem.

Theorem 1.1. *For any positive integers n, k and q, there exists an integer R such that, if M is a matroid with no $M(K_n)$- or $U_{2,q+2}$-minor, then either M has a collection of k disjoint cocircuits or M has rank at most R.*

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Note that, for a prime power q, $U_{2,q+2}$ is the smallest line that is not GF(q)-representable. The motivation for this theorem is to gain insight into minor-closed classes of GF(q)-representable matroids that omit a given clique. Let $\mathscr{H}(n,q)$ denote the class of GF(q)-representable matroids with no $M(K_n)$-minor. Note that the class of cographic matroids is contained in $\mathscr{H}(5,q)$; we expect that, for any n and q, the class $\mathscr{H}(n,q)$ behaves, qualitatively, much like the class of cographic matroids. Applied to the class of cographic matroids, Theorem 1.1 implies the edge-disjoint version of the Erdös–Pósa Theorem [1]. (Note that, for a matroid M, a basis of M^* is a minimum cardinality set of elements that intersects each circuit of M.)

Theorem 1.2 (Erdös–Pósa). *For any integer k there exists an integer R such that, if G is a graph, then either G has a collection of k edge-disjoint circuits or there exists a set of R edges that intersects all circuits of G.*

2. Preliminaries

We assume that the reader is familiar with standard definitions in matroid theory. We use the notation of Oxley [4], with the exception that we denote the simple matroid canonically associated with the matroid M by si(M).

For any positive integer q we define $\mathscr{U}(q)$ to be the class of matroids with no $U_{2,q+2}$-minor. It is well-known that a simple rank-r GF(q)-representable matroid has at most $\frac{q^r-1}{q-1}$ elements; Kung [3] showed that the same bound holds for matroids in $\mathscr{U}(q)$.

Lemma 2.1. *For any integer $q \geqslant 2$ and any simple rank-r matroid $M \in \mathscr{U}(q)$, we have $|E(M)| \leqslant \frac{q^r-1}{q-1}$.*

The simple matroids in $\mathscr{U}(1)$ have no circuits, so Theorem 1.1 is trivial when $q = 1$.

The *rank-deficiency* of a set A of elements of a matroid M is defined as $r(M) - r_M(A)$. The following proposition is elementary; we omit the proof.

Proposition 2.2. *Let M be a matroid and let X and Y be disjoint subsets of $E(M)$. Then, $r(M/X) - r_{M/X}(Y) \leqslant r(M) - r_M(Y)$. Moreover, equality holds if and only if $X \subseteq \mathrm{cl}_M(Y)$.*

We call a matroid M *round* if each cocircuit of M is spanning. Equivalently, M is round if and only if M does not contain a pair of disjoint cocircuits. Note that, for a simple graph G, $M(G)$ is round if and only if G is a clique. The property of roundness is, however, more common in matroids; for example, projective geometries and uniform matroids $U_{r,n}$ with $n \geqslant 2r-1$ are round.

The following properties of round matroids are straightforward.

(i) If M is round and $e \in E(M)$, then M/e is round.
(ii) If N is a spanning minor of M and N is round, then M is round.
(iii) If M is round, then si(M) is round.

From these properties it easily follows that:

(iv) If $N = M \setminus D/C$ is a minor of M where D is coindependent and N is round, then $\mathrm{si}(M/C)$ is round.

Throughout most of this paper, when we take minors we typically only use contraction and simplification. There is one situation, however, in which we delete a cocircuit.

Lemma 2.3. *Let $q \geqslant 2$ be an integer, let $M \in \mathcal{U}(q)$, and let C be a minimum-sized cocircuit of M. Then, for any cocircuit C' of $M \setminus C$, we have $|C'| \geqslant |C|/q$.*

Proof. Set $F = E(M) - (C \cup C')$. Then F is a flat of M, with rank-deficiency 2, contained in $E(M) - C$. Now, $\mathrm{si}(M/F)$ is a line with at most $q+1$ points. Thus, there are at most $q+1$ hyperplanes containing F, one of which is $E(M) - C$. Let the others be $H_1, H_2, \ldots, H_{q'}$. Then $q' \leqslant q$, and $\{H_1 - F, H_2 - F, \ldots, H_{q'} - F\}$ is a partition of C. So, since C is a cocircuit of minimum size, we have

$$\sum_{i=1}^{q'} (|C| + |C'| - |H_i - F|) = \sum_{i=1}^{q'} |E(M) - H_i| \geqslant q'|C|.$$

That is, $q'|C| + q'|C'| - |C| \geqslant q'|C|$, so that $|C'| \geqslant |C|/q$. □

3. Round minors

In this section we prove a weaker version of Theorem 1.1.

Lemma 3.1. *There exists an integer-valued function $f_1(k,n,q)$ such that, for any integers $k \geqslant 1$, $n \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a matroid with rank at least $f_1(k,n,q)$, then either M has k disjoint cocircuits or M has a round minor with rank at least n.*

Let $\Gamma(M)$ denote the maximum rank-deficiency among all cocircuits of M. Thus, $\Gamma(M) = 0$ if and only if M is round. We will prove Lemma 3.1 as a corollary of the following lemma.

Lemma 3.2. *There exists an integer-valued function $h_1(k,n,q)$ such that, for any integers $k \geqslant 1$, $n \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a matroid with rank at least $h_1(k,n,q)$, then either*

(i) *$\Gamma(M) \geqslant k$, or*
(ii) *M has a round minor with rank at least n.*

In its turn, Lemma 3.2 follows easily from the next lemma.

Lemma 3.3. *There exists an integer-valued function $h_2(n,t,q)$ such that, for any integers $n\geqslant 1$, $t\geqslant 1$, and $q\geqslant 2$, if $M\in\mathscr{U}(q)$ is a matroid with rank at least $h_2(n,t,q)$ and $\Gamma(M)=t$, then M has a minor N with $r(N)\geqslant n$ and $\Gamma(N)<\Gamma(M)$.*

Proof. Let $h_2(n,t,q)=tq^{n^2}+n$. Now, let $M\in\mathscr{U}(q)$ be a matroid with $r(M)\geqslant h_2(n,t,q)$ and $\Gamma(M)=t>0$. We first find a minor N of M with $\Gamma(N)<\Gamma(M)$; we then show that the minor has rank at least n.

Let $\{C_1,C_2,\dots,C_k\}$ denote the set of cocircuits of M whose rank-deficiency is exactly t. For $i\in\{1,2,\dots,k\}$, set $D_i=E(M)-\mathrm{cl}(C_i)$, $X_i=C_i\cup D_i$, and $G_i=E(M)-X_i$. Let $X=\{x_1,x_2,\dots,x_p\}$ be a minimal cover of $(X_1,X_2,\dots,X_k)$; that is, X is minimal with respect to the property that each X_i contains at least one member of X. Consider the minor $N=M/X$ of M. Note that, for $C\subseteq E(N)$, C is a cocircuit of N if and only if it is a cocircuit of M. Then, by Proposition 2.2, $\Gamma(N)\leqslant\Gamma(M)$. Suppose that $\Gamma(N)=\Gamma(M)$, and let C be a cocircuit of N with rank-deficiency t. Thus, $C\in\{C_1,C_2,\dots,C_k\}$, say $C=C_i$. Since X is a cover of $(X_1,\dots,X_k)$ there exists $x\in X$ such that $x\in X_i$. Clearly $x\notin C_i$, so $x\in D_i$. But then $x\notin\mathrm{cl}(C_i)$, so, by Proposition 2.2, the rank-deficiency of C_i in N is strictly less than its rank-deficiency in M. This shows that $\Gamma(N)<\Gamma(M)$.

It remains to show that N has sufficiently large rank. If $p\leqslant tq^{n^2}$ then $r(N)\geqslant r(M)-p\geqslant n$. Thus, we may assume that $p\geqslant tq^{n^2}$.

By the minimality of X, for each $i\in\{1,2,\dots,p\}$ there exists $j\in\{1,2,\dots,k\}$ such that $X\cap X_j=\{x_i\}$. By possibly reordering $(X_1,\dots,X_k)$, we may assume that, for $i\in\{1,\dots,p\}$, $X\cap X_i=\{x_i\}$. Now, it may be the case that the $D_i=D_j$ for distinct $i,j\in\{1,\dots,p\}$. Suppose that $D_1=\cdots=D_a$, where $2\leqslant a\leqslant p$. Since M is not round, D_1 contains a cocircuit. Thus, the rank-deficiency of D_1 is at most t. Now, $C_1,\dots,C_a$ are cocircuits disjoint from D_1 and $x_1,\dots,x_a$ is a system of distinct representatives of $C_1,\dots,C_a$. So the rank-deficiency of D_1 is at least a; hence $a\leqslant t$. That is, among $(D_1,\dots,D_p)$ no set is repeated more than t times. By possibly reordering, we may assume that $D_1,\dots,D_b$ are distinct and that $b\geqslant p/t\geqslant q^{n^2}$. For $i,j\in\{1,\dots,b\}$, it is easy to show that $C_i\neq D_j$.

For $i\in\{1,\dots,b\}$, the element $x_i\in X$ either belongs to C_i or to D_i. If $x_i\in C_i$, set $F_i=(E(M)-D_i)$. Since $(X-x_i)\subseteq G_i$, we have $X\subseteq F_i$. On the other hand, if $x_i\in D_i$, set $F_i=(E(M)-C_i)$; again F_i contains X. Now, by the discussion above, flats $(F_1,\dots,F_b)$ are distinct. So $(F_1-X,F_2-X,\dots,F_b-X)$ are distinct flats of N. Let $m=r(N)$. The number of distinct flats of a rank-m matroid in $\mathscr{U}(q)$ is at most q^{m^2}. Therefore, $q^{m^2}\geqslant b\geqslant q^{n^2}$, and, hence, $r(N)=m\geqslant n$; as required. □

Lemma 3.2 follows easily by successively applying Lemma 3.3.

Proof of Lemma 3.1. Let $f_1(1,n,q)=1$ and, for $k\in\{2,3,\dots\}$, we recursively define $f_1(k,n,q)=h_1(f_1(k-1,n,q),n,q)$. We prove the result by induction on k. The case when $k=1$ is trivial, as a matroid with non-zero rank has at least one cocircuit. Suppose that $k>1$ and that the result holds for smaller values of k. Let $M\in\mathscr{U}(q)$ be a matroid with rank at least $f_1(k,n,q)$. We may assume that M does not contain a round minor with rank at least n. Then, by Lemma 3.2, M has a cocircuit C_1 with rank-deficiency at least $f_1(k-1,n,q)$. Thus, M/C_1 has rank at least $f_1(k-1,n,q)$. Moreover, M/C_1 has no round minor with rank at least n. Then, by the induction hypothesis, M/C_1 contains $k-1$ disjoint cocircuits $(C_2,\dots,C_k)$. But then $(C_1,\dots,C_k)$ are disjoint cocircuits of M, as required. □

4. Building density

By the *density* of a matroid M we mean $|E(M)|/r(M)$. The next task is to show that, given a round matroid with sufficiently large rank, we can find a round minor that is dense.

Lemma 4.1. *There exists an integer-valued function $f_2(\lambda,q)$ such that, for any integers $\lambda \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a round matroid with rank at least $f_2(\lambda,q)$, then M has a simple round minor N with $|E(N)| > \lambda r(N)$.*

To facilitate induction, we prove a stronger version of Lemma 4.1.

Lemma 4.2. *There exists an integer-valued function $h_3(\lambda,k,q)$ such that, for any integers $\lambda \geqslant 1$, $k \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a round matroid with rank at least $h_3(\lambda,k,q)$, then M has a simple round minor N with $|E(N)| > \lambda(r(N)-\lambda)$ and $r(N) \geqslant k$.*

Proof. Let $h_3(1,k,q)=k$, and, for $\lambda>1$, we recursively define

$$h_3(\lambda,k,q)=f_1(q(\lambda-1),h_3(\lambda-1,k,q),q)+1.$$

The proof is by induction on λ. The result is trivial when $\lambda=1$. Suppose that $\lambda>1$ and that the result holds for smaller values of λ.

Let $M \in \mathcal{U}(q)$ be a round matroid with rank at least $h_3(\lambda,k,q)$, and let C be a minimum-size cocircuit of M. By Lemma 3.1, either

(a) $M \setminus C$ has $q(\lambda-1)$ disjoint cocircuits, or
(b) $M \setminus C$ has a round minor N_1 with $r(N_1) \geqslant h_3(\lambda-1,k,q)$.

First consider case (a); that is, $M \setminus C$ has disjoint cocircuits $C_1,\dots,C_t$, where $t=q(\lambda-1)$. By Lemma 2.3, $|C_i| \geqslant r(M)/q$. Therefore,

$$|E(M)| \geqslant |C|+|C_1|+\cdots+|C_t| \geqslant r(M)+q(\lambda-1)r(M)/q=\lambda r(M).$$

In this case the result is satisfied by choosing $N=M$.

Now consider case (b); that is, $M \setminus C$ has a round minor N_1 with $r(N_1) \geqslant h_3(\lambda-1,k,q)$. By the induction hypothesis, N_1 has a simple round minor N_2 such that $|E(N_2)| > (\lambda-1)(r(N_2)-(\lambda-1))$ and $r(N_2) \geqslant k$. Now, N_2 is a minor of $M \setminus C$, so there exists an independent set I and coindependent set J of $M \setminus C$ such that $N_2 =$

$(M \setminus C) \setminus J/I$. Now, define $N = \mathrm{si}(M/I)$. Since M is round, N is round. Let $C' = C \cap E(N)$. Note that, C' is a spanning cocircuit of N, thus, $r(N) = r(N_2) + 1 > k$. Now,

$$\begin{aligned}|E(N)| &\geqslant |C'| + |E(N_2)| \\ &> r(N) + (\lambda - 1)(r(N_2) - (\lambda - 1)) \\ &= r(N) + (\lambda - 1)(r(N) - \lambda) \\ &\geqslant \lambda(r(N) - \lambda)\end{aligned}$$

as required. □

Proof of Lemma 4.1. Let $f_2(\lambda, q) = h_3(\lambda + 1, (\lambda + 1)^2, q)$. Now, let $M \in \mathcal{U}(q)$ be a round matroid with rank at least $f_2(\lambda, q)$. By Lemma 4.2, M has a simple round minor N with $r(N) \geqslant (\lambda + 1)^2$ and

$$\begin{aligned}|E(N)| &> (\lambda + 1)(r(N) - (\lambda + 1)) \\ &= \lambda r(N) + (r(N) - (\lambda + 1)^2) \\ &\geqslant \lambda r(N)\end{aligned}$$

as required. □

Let e be an element of a simple matroid M. Define $\delta_M(e) = |E(M)| - |E(\mathrm{si}(M/e))|$. A rank-2 flat with at least 3 elements is called a *long line*. Obviously $\delta_M(e) \geqslant 1$, since we lose e in the contraction. We also lose elements on long lines through e. If L is a long line containing e, then the elements $L - \{e\}$ are represented by a single element of $\mathrm{si}(M/e)$. If $M \in \mathcal{U}(q)$, then $3 \leqslant |L| \leqslant q + 1$. We let $\ell_M(e)$ denote the number of long lines through e. Then,

$$(\delta_M(e) - 1)/q \leqslant \ell_M(e) \leqslant (\delta_M(e) - 1)/2.$$

Lemma 4.3. *If M is a simple matroid such that $|E(M)| > \lambda r(M)$, then there exists a subset X of $E(M)$ such that $\delta_{\mathrm{si}(M/X)}(e) > \lambda$ for each $e \in E(\mathrm{si}(M/X))$.*

Proof. Choose $X \subseteq E(M)$ maximal such that $|E(\mathrm{si}(M/X))| > \lambda r(\mathrm{si}(M/X))$ and let $N = \mathrm{si}(M/X)$. By the maximality of X, we have $|E(\mathrm{si}(N/e))| \leqslant \lambda r(\mathrm{si}(N/e))$ for any $e \in E(N)$. Now,

$$\begin{aligned}\delta_N(e) &= |E(N)| - |E(\mathrm{si}(N/e))| \\ &> \lambda r(N) - \lambda r(\mathrm{si}(N/e)) \\ &= \lambda\end{aligned}$$

as required. □

5. Nests

In order to extract a specific clique minor from a sufficiently large round matroid, we go through an intermediate class of matroids called nests. We use the following lemma to recognize cliques; the result is well known but we include the proof for completeness.

Lemma 5.1. *Let M be a matroid with ground set $B \cup H$ where $B = \{b_1, \dots, b_n\}$ is a basis of M, $H = \{h_{ij}: 1 \leqslant i < j \leqslant n\}$ is a hyperplane of M disjoint from B, and $\{b_i, h_{ij}, b_j\}$ is a triangle of M for each $i < j$. Then M is isomorphic to $M(K_{n+1})$.*

Proof. Construct a complete graph G with vertex set $V = \{v_0, \dots, v_n\}$ and edges labeled by $B \cup H$ where $b_i \in B$ labels the edge incident with v_0 and v_i and $h_{ij} \in H$ labels the edge incident with v_i and v_j. We claim that $M = M(G)$; they clearly have the same rank. Consider a spanning tree T of G. If there exists an edge $h_{ij} \in T \cap H$ such that v_i has degree-one in T then $(T - \{h_{ij}\}) \cup \{b_i\}$ is a spanning tree of G and $r_M((T - \{h_{ij}\}) \cup \{b_i\}) = r_M(T)$. By repeatedly applying such changes, we see that $r_M(T) = r_M(B)$. Thus, T is a basis of M. Now, consider a circuit C of G, and let X be the set of edges in B that are incident with a vertex of $C - v_0$ in G. Note that $C \subseteq \mathrm{cl}_M(X)$. If $B \cap C \neq \emptyset$ then $|X| < |C|$, so C is dependent in M. On the other hand, if $C \subseteq H$ then, since $|C| = |X|$ and $C \subseteq H \cap \mathrm{cl}_M(X)$, we see that C is dependent in M. Hence, $M = M(G)$ as required. □

A *nest* is a matroid that contains a basis $B = \{b_1, \dots, b_n\}$ such that, for any integers i,j where $1 \leqslant i < j \leqslant n$, the pair (b_i, b_j) spans a long line in $\mathrm{si}(M/\{b_1, \dots, b_{i-1}\})$; the elements of B are called *joints*. The main result of this section is that large nests contain big cliques; to prove this we use an elegant method introduced by Kung [2].

Lemma 5.2. *There exists an integer-valued function $h_4(n,q)$ such that, for any integers $n \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a nest with rank at least $h_4(n,q)$, then M contains an $M(K_n)$-minor.*

Proof. Let $h_4(n,q) = q^{n-2}$. Let $M \in \mathcal{U}(q)$ be a simple nest with rank $t \geqslant h_4(n,q)$, let $B = \{b_1, \dots, b_t\}$ be the set of joints of M, and, for each pair of integers i,j where $1 \leqslant i < j \leqslant t$, let e_{ij} be an element of M such that $\{b_i, b_j, e_{ij}\}$ is a triangle of $M/\{b_1, \dots, b_{i-1}\}$.

5.2.1. *For each $k \in \{1, \dots, t\}$, $e_{1k}, \dots, e_{k-1,k} \notin \mathrm{cl}_M(\{b_1, \dots, b_{k-1}\})$, and the set $\{e_{1k}, \dots, e_{k-1,k}\} \cup \{b_k\}$ is independent in M.*

Let $i \in \{1, \dots, k-1\}$. By the definition of e_{ik} we see that $e_{ik} \notin \mathrm{cl}_M(\{b_1, \dots, b_i\})$ but that $e_{ik} \in \mathrm{cl}_M(\{b_1, \dots, b_i\} \cup \{b_k\})$. Then, since B is a basis, we see that $e_{ik} \notin \mathrm{cl}_M(\{b_1, \dots, b_{k-1}\})$, as claimed. For the second part, we prove by induction on $i \in \{1, \ldots, k-1\}$ that $\mathrm{cl}_M(\{e_{1k}, \ldots, e_{ik}\} \cup \{b_k\}) = \mathrm{cl}_M(\{b_1, \ldots, b_i\} \cup \{b_k\})$. The case that $i = 1$ is trivial; suppose that $i > 1$ and that $\mathrm{cl}_M(\{e_{1k}, \ldots, e_{i-1,k}\} \cup \{b_k\}) = \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}\} \cup \{b_k\})$. By the definition of e_{ik} we readily see that $e_{ik} \notin \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}\} \cup \{b_k\})$ but $e_{ik} \in \mathrm{cl}_M(\{b_1, \ldots, b_i\} \cup \{b_k\})$. Thus, $\mathrm{cl}_M(\{b_1, \ldots, b_i\} \cup \{b_k\}) = \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}\} \cup \{e_{ik}, b_k\})$. However, $\mathrm{cl}_M(\{e_{1k}, \ldots, e_{i-1,k}\} \cup \{b_k\}) = \mathrm{cl}_M(\{b_1, \ldots, b_{i-1}\} \cup \{b_k\})$, so $\mathrm{cl}_M(\{e_{1k}, \ldots, e_{ik}\} \cup \{b_k\}) = \mathrm{cl}_M(\{b_1, \ldots, b_i\} \cup \{b_k\})$; as required. This proves 5.2.1.

Note that, for each $k \in \{1, \ldots, t\}$, the restriction of M to $\mathrm{cl}_M(\{b_1, \ldots, b_k\})$ is a nest. Let $X = \{b_1, \ldots, b_{n-2}\}$; our next objective is to make the flat spanned by X dense. We define a maximal sequence of matroids $(N_t, N_{t-1}, \ldots, N_k)$ such that $N_t = M$ and, for each $i \in \{k+1, \ldots, t\}$, $N_{i-1} = \mathrm{si}(N_i/a)$ for some $a \in E(N_i) - \mathrm{cl}(\{b_1, \ldots, b_{i-1}\})$ such that there exists $b \in \mathrm{cl}_M(X \cup \{a\}) - \mathrm{cl}(\{b_1, \ldots, b_{i-1}\})$ with $\mathrm{cl}_{N_i}(a, b) \cap \mathrm{cl}_{N_i}(X) = \emptyset$. (That is, to obtain N_{i-1} from N_i we look for a point $a \notin \mathrm{cl}(\{b_1, \ldots, b_{i-1}\})$ to contract that throws a new point into the flat spanned by X.) Note that,

$$n - 2 \leqslant |\mathrm{cl}_{N_t}(X)| < |\mathrm{cl}_{N_{t-1}}(X)| < \cdots < |\mathrm{cl}_{N_k}(X)| \leqslant q^{n-2} - 1.$$

So $n - 2 + t - k \leqslant q^{n-2} - 1$. Hence, as $t \geqslant q^{n-2}$, we have $k \geqslant n - 1$.

Let N denote the restriction of N_k to $X \cup \{b_k\}$, let H denote the hyperplane of N spanned by X, and let $B' = \{e_{1k}, \ldots, e_{n-2,k}, b_k\}$. By 5.2.1, B' is disjoint from H and B' is a basis of N. Moreover, by the maximality of the sequence $(N_t, \ldots, N_k)$, for each pair (a, b) of distinct elements in B' there exists an element $c \in H$ such that $\{a, b, c\}$ is a triangle. So, by Lemma 5.1, N contains an $M(K_n)$-minor. $\square$

6. Building a nest

In this section we prove that round matroids with large rank contain large nests.

Lemma 6.1. *There exists an integer-valued function $f_3(n, q)$ such that, for any integers $n \geqslant 1$ and $q \geqslant 2$, if $M \in \mathcal{U}(q)$ is a round matroid with rank at least $f_3(n, q)$, then M contains a nest of rank n as a minor.*

We require the following technical lemma.

Lemma 6.2. *There exists an integer-valued function $h_5(k, q)$ such that, for any integers* $k \geqslant 1$ and $q \geqslant 2$, *if $M \in \mathcal{U}(q)$ is a round matroid with rank at least $h_5(k, q)$ and B is a basis of M, then there exists a simple round minor N of M, a $(k+1)$-element set $B' \subseteq B \cap E(N)$, and an element $e \in B'$ such that, for each element $x \in B' - \{e\}$, the pair $\{e, x\}$ spans a long line in N.*

Proof. Let $\lambda = q(k-1) + \frac{(q-1)(k-1)}{4} q^{k+3}$, let $h_5(k, q) = q^{f_2(\lambda, q)}$, let $M \in \mathcal{U}(q)$ be a round matroid with rank at least $h_5(k, q)$, and let B be a basis of M.

Consider any minor N of M. When constructing $\mathrm{si}(N)$ we keep a single representative of each parallel class of N; in this proof, we choose $\mathrm{si}(N)$ to contain as many elements of B as possible.

We say that a set $X \subseteq E(M)$ *dominates* M if each element in $E(M) - X$ is on a long line containing at least 2 elements of X. We claim that:

6.2.1. *There exists a simple round minor N_1 of M such that $B \subseteq E(N_1)$ and B dominates N_1. (Note that B need not be a basis in N_1.)*

Indeed, let N_1 be a minimal minor of M such that N_1 is simple and round and $B \subseteq E(N_1)$. Now, consider any element $f \in E(N_1) - B$. Certainly, $\mathrm{si}(N_1/f)$ is simple and round. Then, by the minimality of N_1, f is on a long line that contains at least 2 elements of B. That is, B dominates N_1; this proves 6.2.1.

Now, $|B| = r(M)$ and $B \subseteq E(N_1)$, so $r(N_1) \geqslant \log_q(r(M)) \geqslant f_2(\lambda, q)$. Note that, by our convention on simplification, for any set $X \subseteq E(N_1)$, $B \cap E(\mathrm{si}(N_1/X))$ dominates $\mathrm{si}(N_1/X)$. By Lemma 4.1, there exists a simple minor N_2 of N_1 such that $|E(N_2)| > \lambda r(N_2)$. We may assume that $N_2 = \mathrm{si}(N_1/X)$ for some $X_1 \subseteq E(N_1)$. Thus, N_2 is round and $B \cap E(N_2)$ dominates N_2. Now, by Lemma 4.3, there exists $X_2 \subseteq E(N_2)$ such that $\delta_{\mathrm{si}(N_2/X_2)}(e) > \lambda$ for all $e \in E(\mathrm{si}(N_2/X_2))$. Let $N_3 = \mathrm{si}(N_2/X_2)$; note that N_3 is round and $B \cap E(N_3)$ dominates N_3. Now, each element e of N_3 is on at least λ/q long lines. Let $B_3 = B \cap E(N_3)$ and let $W_3 = E(N_3) - B_3$. We may assume that, for each $e \in B_3$, there are at most $k-1$ long lines of N_3 through e that contain another point of B_3 (since, otherwise, the result is clearly true). Since B_3 dominates N_3, we have:

6.2.2. $|W_3| \leqslant \frac{(k-1)(q-1)}{2}|B_3|$.

Let L denote the set of long lines in N_3 that contain at most one element of B_3. Thus, $|L| \geqslant (\lambda/q - (k-1))|B_3|$. Therefore, there exists an element $w \in W_3$ that is on at least $\frac{2|L|}{|W_3|} \geqslant \frac{4(\lambda/q-(k-1))}{(k-1)(q-1)} \geqslant q^{k+2}$ lines in L. Let X denote the set of all elements of B_3 that are on lines of L containing w. Now $|X| \geqslant q^{k+2}$ so $r_{N_3}(X) \geqslant k+2$. Then, there exists a $(k+1)$-element subset B' of X such that $B' \cup \{w\}$ is independent. Let $e \in B'$, let w' be an element of N_3 such that $\{e, w, w'\}$ is a triangle, and let $N = \mathrm{si}(N_3/w')$. Now, it is straightforward to check that e, B', and N have the desired properties. $\square$

Proof of Lemma 6.1. We let $f_3(1,q) = 1$ and, for $n \geqslant 2$, we recursively define $f_3(n,q) = h_5(q^{f_3(n-1,q)+1}, q)$. We will prove the stronger result that, for any integers $n \geqslant 1$ and $q \geqslant 2$, if $M \in \mathscr{U}(q)$ is a round matroid with rank at least $f_3(n,q)$ and B is a basis of M, then M contains a rank-n minor that is a nest whose joints are contained in B.

The proof is by induction on n. The case that $n=1$ is trivial; suppose that $k>1$ and that the result holds when $n = k-1$. Now, consider the case that $n=k$. Let $M \in \mathscr{U}(q)$ be a round matroid with rank at least $f_3(n,q)$ and let B is a basis of M. By

Lemma 6.2, there exists a simple round minor N_1 of M, a set $B' \subseteq B \cap E(N_1)$ with cardinality $(q^{f_3(n-1,q)+1} + 1)$, and an element $e \in B'$ such that, for each element $x \in B' - \{e\}$, the pair $\{e, x\}$ spans a long line in N_1. Note that $r_{N_1}(B') \geqslant f_3(n-1, q) + 1$, so there exists an $f_3(n-1, q)$-element set $B_1 \subseteq B' - \{e\}$ such that $B_1 \cup \{e\}$ is independent. By contraction and simplification, we can construct a simple round minor N_2 of N_1 such that $B_1 \cup \{e\}$ is a basis of N_2. Now let $N_3 = \mathrm{si}(N_2/e)$. Note that, N_3 is round, B_1 is a basis of N_3, and $r(N_3) \geqslant f_3(n-1, q)$. Then, by the induction hypothesis, N_3 contains a rank-$(n-1)$ minor N_4 that is a nest whose joints are contained in B_1. We may assume that $N_4 = \mathrm{si}(M/(X \cup \{e\}))$ for some set $X \subseteq E(M)$. Observe that $\mathrm{si}(M/X)$ is a rank-n nest whose joints are contained in B. $\square$

Theorem 1.1 is an immediate consequence of Lemmas 3.1, 5.2, and 6.1.

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