SIAM J. Discrete Math.
Vol. 18, No. 3, pp. 638–646

BRIDGING SEPARATIONS IN MATROIDS*

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Abstract. Let (X_1, X_2) be an exact k-separation of a matroid N. If M is a matroid that contains N as a minor and the k-separation (X_1, X_2) does not extend to a k-separation in M, then we say that M *bridges* the k-separation (X_1, X_2) in N. One would hope that a minor minimal bridge for (X_1, X_2) would not be much larger than N. Unfortunately there are instances in which one can construct arbitrarily large minor-minimal bridges. We restrict our attention to the class of matroids representable over a fixed finite field and show that here minor-minimal bridges are bounded in size.

Key words. matroids, connectivity, blocking sequences

AMS subject classification. 05B35

DOI. 10.1137/S089548010139638X

1. Introduction. Seymour's Decomposition Theorem [4] states that any regular matroid can be obtained from graphic matroids, cographic matroids, and copies of R_{10} using 1-, 2-, and 3-sums. The main step in the proof of this remarkable theorem is to prove that

(1) *If M is a 3-connected regular matroid that is neither graphic nor cographic, then M contains a minor isomorphic to R_{10} or R_{12}.*

The matroids R_{10} and R_{12} are particular 3-connected regular matroids that are neither graphic nor cographic. It is easy to handle the regular matroids containing R_{10}.

(2) *If M is a 3-connected regular matroid that contains R_{10} as a minor, then $M = R_{10}$.*

Somewhat more complicated structures arise when considering R_{12}. Let N be a matroid with an exact k-separation (X_1, X_2), and let M be a matroid containing N as a minor. If there exists a k-separation (Y_1, Y_2) of M where $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then we say that the k-separation (X_1, X_2) of N is *induced* in M. If (X_1, X_2) is not induced in M, then we say that M *bridges* the k-separation (X_1, X_2) in N.

(3) *R_{12} has a 3-separation (X_1, X_2) such that $|X_1|, |X_2| = 6$. Moreover, if M is a regular matroid that contains R_{12} as a minor, then the 3-separation (X_1, X_2) of R_{12} is induced in M.*

The proof of (2), and of results like (2), is reduced to an elementary finite case check by Seymour's Splitter Theorem [4]. However, there is no satisfactory analogue of Seymour's Splitter Theorem that can be applied to prove results like (3). We are interested in minor-minimal matroids that bridge the k-separation (X_1, X_2) in N. Unfortunately, in some cases such matroids are arbitrarily large. Nevertheless, Seymour [4] and Geelen, Gerards, and Kapoor [1] have shown that such matroids are highly structured (see Theorem 3.4). The main result of this paper is that when

*Received by the editors October 12, 2001; accepted for publication (in revised form) August 10, 2004; published electronically April 8, 2005.

http://www.siam.org/journals/sidma/18-3/39638.html

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we restrict our attention to matroids over a fixed finite field, the situation improves significantly.

THEOREM 1.1. *For any finite field* $\mathbb{F}$ *and integer* k *there exists an integer* n *such that if* (X_1, X_2) *is an exact* k*-separation in an* $\mathbb{F}$*-representable matroid* N *and* M *is a minor-minimal* $\mathbb{F}$*-representable matroid that bridges the* k*-separation* (X_1, X_2) *in* N*, then* $|E(M)| \leq |E(N)| + n$*.*

We actually prove a stronger result, Theorem 7.1, that gives an explicit bound on n. This reduces the proof of (3) to a finite case check. Of course this is not practical as the number n is quite large. Nevertheless, we hope to use these results in subsequent papers to obtain excluded minor characterizations.

For 1- and 2-separations we obtain stronger results that are independent of representation. The first of these follows readily from results of Lemos and Oxley [3].

THEOREM 1.2. *If* (X_1, X_2) *is a separation in a matroid* N *and* M *is a minor-minimal matroid that bridges the separation* (X_1, X_2) *in* N*, then* $|E(M)| \leq |E(N)| + 2$*.*

THEOREM 1.3. *If* (X_1, X_2) *is an exact* 2*-separation in a matroid* N *and* M *is a minor-minimal matroid that bridges the* 2*-separation* (X_1, X_2) *in* N*, then* $|E(M)| \leq |E(N)| + 5$*.*

There is no analogue of Theorems 1.2 and 1.3 for 3-separations. Nevertheless, while there may be arbitrarily large minor-minimal bridging matroids, we can bound the branch-width of such matroids. (Branch-width is defined in section 8.)

THEOREM 1.4. *Let* (X_1, X_2) *be an exact* k*-separation in matroid* N *with branch-width* n*. If* M *is a minor-minimal matroid that bridges the* k*-separation* (X_1, X_2) *in* N*, then* M *has branch-width at most* $n + k$*.*

Let $(M_1, M_2, \ldots)$ be an infinite sequence of matroids each of which is representable over the same finite field and each with branch-width at most n. In [2] it is proved that there exists $i < j$ such that M_i is isomorphic to a minor of M_j. Combining this with Theorem 1.4, we can obtain a result similar to Theorem 1.1. However, there are two differences. First, in Theorem 1.1 we keep N as a minor while the other approach keeps a minor isomorphic to N. More importantly, we obtain an explicit upper-bound on the size of M, which cannot be done using the methods in [2].

2. Tutte's Linking Theorem.

Let M be a matroid. For any subset A of $E(M)$ we let

$$\lambda_M(A) := r_M(A) + r_M(E(M) - A) - r_M(E(M));$$

λ_M is the *connectivity function* of M. For sets $A, B \subseteq E(M)$, we have

(i) $\lambda_M(A) = \lambda_M(E(M) - A)$,

(ii) $\lambda_M(A) \leq \lambda_M(A \cup \{e\}) + 1$ for each $e \in E(M)$, and

(iii) $\lambda_M(A) + \lambda_M(B) \geq \lambda_M(A \cup B) + \lambda_M(A \cap B)$.

If (X_1, X_2) is a partition of $E(M)$ such that $|X_1|, |X_2| \geq k$ and $\lambda_M(X_1) < k$, then we call (X_1, X_2) a *k-separation* of M. If, in addition, $\lambda_M(X_1) = k - 1$, then we call (X_1, X_2) an *exact k-separation* of M.

Let N be a minor of M and let (X_1, X_2) be an exact k-separation of N. We let $\kappa_M(X_1, X_2) = \min(\lambda_M(A) : X_1 \subseteq A \subseteq E(M) - X_2)$. Thus, M bridges (X_1, X_2) if and only if $\kappa_M(X_1, X_2) \geq k$. Note that if M' is a minor of M and $X_1, X_2 \subseteq E(M')$, then $\kappa_{M'}(X_1, X_2) \leq \kappa_M(X_1, X_2)$. The following theorem provides a good characterization for $\kappa(X_1, X_2)$; this theorem is in fact a generalization of Menger's theorem.

THEOREM 2.1 (Tutte's Linking Theorem [5]). *Let M be a matroid and let X_1, X_2 be disjoint subsets of $E(M)$. Then, there exists a minor M' of M such that $E(M') = X_1 \cup X_2$ and $\lambda_{M'}(X_1) = \kappa_M(X_1, X_2)$.*

We obtain the following easy corollary.

COROLLARY 2.2. *Let N be a matroid with an exact k-separation (X_1, X_2), and let M be a minor-minimal matroid that bridges the k-separation (X_1, X_2) in N. For any $e \in E(M) - E(N)$ either*

1. *$\kappa_{M\backslash e}(X_1, X_2) < k$ and N is not a minor of M/e, or*
2. *$\kappa_{M/e}(X_1, X_2) < k$ and N is not a minor of $M\backslash e$.*

By Corollary 2.2, there exists a unique partition (S, T) of $E(M) - E(N)$ such that $N = M\backslash S/T$. However, any minor can be obtained by contracting an independent set and deleting a coindependent set.

COROLLARY 2.3. *Let N be a matroid with an exact k-separation (X_1, X_2), and let M be a minor-minimal matroid that bridges the k-separation (X_1, X_2) in N. If $N = M\backslash S/T$, then S is coindependent and T is independent.*

We also require the following technical lemma.

LEMMA 2.4. *Let M be a matroid, let (Y_1, Y_2) be a partition of $E(M)$, and let $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. If $\kappa_M(X_1, Y_2) = \lambda_M(Y_2)$ and $\kappa_M(Y_1, X_2) = \lambda_M(Y_1)$, then $\kappa_M(X_1, X_2) = \lambda_M(Y_1)$.*

Proof. Let Y be a set such that $\lambda_M(Y) = \kappa_M(X_1, X_2)$ and $X_1 \subseteq Y \subseteq E(M) - X_2$. By submodularity, we have

$$\begin{aligned}
\kappa_M(X_1, X_2) &= \lambda_M(Y) \\
&\geq \lambda_M(Y \cap Y_1) + \lambda_M(Y \cup Y_1) - \lambda_M(Y_1) \\
&\geq \kappa_M(X_1, Y_2) + \kappa_M(Y_1, X_2) - \lambda_M(Y_1) \\
&= \lambda_M(Y_2) + \lambda_M(Y_1) - \lambda_M(Y_1) \\
&= \lambda_M(Y_2) \\
&\geq \kappa_M(X_1, X_2).
\end{aligned}$$

Thus, $\kappa_M(X_1, X_2) = \lambda_M(Y_1)$, as required. ∎

3. Blocking sequences.

In this section we review results from [1], but we use slightly different notation; similar results are given in [4]. Let N be a minor of a matroid M, and let $X = E(N)$. Then there exists a coindependent set S and an independent set T such that $N = M\backslash S/T$. Therefore, there exists a basis B of M such that $T \subseteq B \subseteq E(M) - S$. For any subset Y of $E(M)$, we define

$$M[Y, B] := M\backslash(E(M) - (Y \cup B))/(B - Y).$$

Thus $M[Y, B]$ is the minor of M on ground set Y obtained by contracting only elements of B and deleting only elements of $E(M) - B$. In particular, $N = M[X, B]$.

Let (X_1, X_2) be an exact k-separation in N. A sequence $v_1, \ldots, v_p \in E(M)$ is a *blocking sequence* for the k-separation (X_1, X_2) of N, with respect to B, if

1.(a) $\lambda_{M[X \cup \{v_1\}, B]}(X_1) \geq k$,
(b) $\lambda_{M[X \cup \{v_p\}, B]}(X_1 \cup \{v_p\}) \geq k$,
(c) for all $i \in \{1, \ldots, p-1\}$, we have $\lambda_{M[X \cup \{v_i, v_{i+1}\}, B]}(X_1 \cup \{v_i\}) \geq k$, and

2. no proper subsequence of $v_1, \ldots, v_p$ satisfies 1.

If there is a blocking sequence for (X_1, X_2), then M clearly bridges (X_1, X_2). The converse is also true and is proved in [1, Theorem 4.14].

THEOREM 3.1. *Let B be a basis of the matroid M, let $N = M[X, B]$, and let (X_1, X_2) be an exact k-separation of N. Then, M bridges the k-separation (X_1, X_2) in N if and only if there exists a blocking sequence for (X_1, X_2) with respect to B.*

The following propositions give additional properties of blocking sequences; the first follows easily from the definitions, while the second is proved in [1, Proposition 4.15].

PROPOSITION 3.2. *Let B be a basis of the matroid M, let $N = M[X, B]$, and let $v_1, \dots, v_p$ be a blocking sequence, with respect to B, for an exact k-separation (X_1, X_2) of N. Now, let $i, j \in \mathbb{Z}$, where $0 \le i < j - 1 \le p$; let $Y_1 \subseteq X_1 \cup \{v_1, \dots, v_i\}$, where $X_1 \cup \{v_i\} \subseteq Y_1$; and let $Y_2 \subseteq X_2 \cup \{v_j, \dots, v_p\}$, where $X_2 \cup \{v_j\} \subseteq Y_2$. Then, (Y_1, Y_2) is an exact k-separation in $M[Y_1 \cup Y_2, B]$, and $v_{i+1}, \dots, v_{j-1}$ is a blocking sequence for this exact k-separation with respect to B.*

PROPOSITION 3.3. *Let B be a basis of the matroid M, let $N = M[X, B]$, and let $v_1, \dots, v_p$ be a blocking sequence, with respect to B, for an exact k-separation (X_1, X_2) of N. Then, the sequence $v_1, \dots, v_p$ alternates between elements of B and $E(M) - B$.*

In summary, we obtain the following theorem.

THEOREM 3.4. *Let N be a matroid with an exact k-separation (X_1, X_2), and let M be a minor-minimal matroid that bridges the k-separation (X_1, X_2) of N. Then there exists a unique partition (S, T) of $E(M) - E(N)$ such that $N = M\backslash S/T$. Moreover, there exists an ordering $v_1, \dots, v_p$ of the elements in $E(M) - E(N)$ that alternates between elements of S and T such that, for each $i \in \{1, \dots, p\}$,*

(i) *if $v_i \in S$, then $(X_1 \cup \{v_1, \dots, v_{i-1}\}, X_2 \cup \{v_{i+1}, \dots, v_p\})$ is a k-separation in $M\backslash v_i$, and*

(ii) *if $v_i \in T$, then $(X_1 \cup \{v_1, \dots, v_{i-1}\}, X_2 \cup \{v_{i+1}, \dots, v_p\})$ is a k-separation in M/v_i.*

4. Guts and coguts elements.

We let $\mathrm{cl}_M(X)$ denote the closure of the set X in a matroid M. The *coclosure* of X, denoted $\mathrm{cl}^*_M(X)$, is the closure of X in M^*. If $e \notin X$, it is easy to show that $e \in \mathrm{cl}^*_M(X)$ if and only if $e \notin \mathrm{cl}_M(E(M) - (X \cup \{e\}))$. The following proposition is well known and straightforward.

PROPOSITION 4.1. *Let M be a matroid, let $X \subseteq E(M)$, and let $e \in E(M) - X$. Then*

(i) *$\lambda_{M/e}(X) < \lambda_M(X)$ if and only if $e \in \mathrm{cl}_M(X)$ and e is not a loop;*

(ii) *dually, $\lambda_{M\backslash e}(X) < \lambda_M(X)$ if and only if $e \in \mathrm{cl}^*_M(X)$ and e is not a coloop.*

Let (X_1, X_2) be an exact k-separation of M. An element e is in the *guts* of (X_1, X_2) if $e \in \mathrm{cl}_M(X_1 - \{e\})$ and $e \in \mathrm{cl}_M(X_2 - \{e\})$. Similarly, e is in the *coguts* of (X_1, X_2) if $e \in \mathrm{cl}^*_M(X_1 - \{e\})$ and $e \in \mathrm{cl}^*_M(X_2 - \{e\})$. Equivalently, e is in the coguts of (X_1, X_2) if $e \notin \mathrm{cl}_M(X_1 - \{e\})$ and $e \notin \mathrm{cl}_M(X_2 - \{e\})$.

The following proposition is also well known.

PROPOSITION 4.2. *Let M be a matroid, let (X_1, X_2) be a partition of $E(M)$, and let $e \in X_2$. Then*

(i) *$\lambda_M(X_1) < \lambda_M(X_1 \cup \{e\})$ if and only if $e \in \mathrm{cl}_M(X_2 - \{e\})$ and $e \notin \mathrm{cl}_M(X_1)$, and*

(ii) *$\lambda_M(X_1) = \lambda_M(X_1 \cup \{e\})$ if and only if e is either in the guts or in the coguts of (X_1, X_2).*

The following technical lemma is crucial.

LEMMA 4.3. *Let (X_1, X_2) be an exact k-separation of a matroid N, and let M be a minor-minimal matroid bridging the k-separation (X_1, X_2) of N. Moreover, let B be a basis of a matroid M such that $N = M[X_1 \cup X_2, B]$, let $v_1, \dots, v_p$ be a blocking sequence for (X_1, X_2) with respect to B, and let $M' = M[X_1 \cup X_2 \cup \{v_2, \dots, v_{p-1}\}, B]$.*

If $p \geq 2k+2$, then there exists $i \in \{2,3,\ldots,p-1\}$ such that $\kappa_{M'\backslash v_i}(X_1,X_2) = k-1$ and $\kappa_{M'/v_i}(X_1,X_2) = k-1$.

Proof. Given disjoint subsets A_1, A_2 of $E(M)$, we let $\sqcap_M(A_1,A_2) = r_M(A_1) + r_M(A_2) - r_M(A_1 \cup A_2)$. Thus, if (A_1,A_2) is a partition of $E(M)$, then $\sqcap_M(A_1,A_2) = \lambda_M(A_1)$. Moreover, it is straightforward to see that $\sqcap_M(A_1,A_2) \leq \kappa_M(A_1,A_2)$. We prove the stronger result that if $p \geq 2(k - \sqcap_M(X_1,X_2)) + 2$, then there exists $i \in \{2,\ldots,p-1\}$ such that $\kappa_{M'\backslash v_i}(X_1,X_2) = k-1$ and $\kappa_{M'/v_i}(X_1,X_2) = k-1$. By Proposition 3.2 and Lemma 2.4, we may assume that $p = 2(k - \sqcap_M(X_1,X_2)) + 2$. Note that $\sqcap_M(X_1,X_2) \leq k-1$, so $p \geq 4$.

By duality, we may assume that $v_1 \in B$; thus, by Proposition 3.3, $v_2 \notin B$ and $v_p \notin B$. Since N is a minor of $M'\backslash v_2$, we have $\kappa_{M'\backslash v_2}(X_1,X_2) = k-1$. Suppose that $\kappa_{M'/v_2}(X_1,X_2) < k-1$. Then there exists a $(k-1)$-separation (Y_1,Y_2) of M'/v_2 such that $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. Note that $\lambda_M(Y_1 \cup \{v_2\}) \leq \lambda_{M/v_2}(Y_1) + 1$ and $\kappa_M(X_1,X_2) = k$, so $(Y_1 \cup \{v_2\}, Y_2)$ is a k-separation of M'. Therefore, by the definition of a blocking sequence, $v_3 \in Y_1$. Similarly, we see that $v_4,\ldots,v_{p-1} \in Y_1$. Thus, $Y_1 = X_1 \cup \{v_3,\ldots,v_{p-1}\}$ and $Y_2 = X_2$.

By Proposition 4.2, $v_2 \in \mathrm{cl}_{M'}(X_2)$. Therefore, $v_2 \in \mathrm{cl}_M(X_2 \cup \{v_1\})$. Now, by Proposition 3.2, $(X_1 \cup \{v_1\}, X_2 \cup \{v_3,\ldots,v_p\})$ is a k-separation in $M\backslash v_2$. Thus, by Proposition 4.1, $v_2 \notin \mathrm{cl}_M(X_1 \cup \{v_1\})$. Similarly, $(X_1, X_2 \cup \{v_2,v_3,\ldots,v_p\})$ is a k-separation in M/v_1. Thus, by Proposition 4.1, $v_1 \in \mathrm{cl}_M(X_1)$. Let $X_1' = X_1 \cup \{v_2\}$, $X = X_1 \cup X_2$, and $X' = X \cup \{v_2\}$. Since $v_2 \notin \mathrm{cl}_M(X_1 \cup \{v_1\})$, we have $r_M(X_1') = r_M(X_1) + 1$ and, since $v_1 \in \mathrm{cl}_M(X_1)$ and $v_2 \in \mathrm{cl}_M(X_2 \cup \{v_1\})$, we have $r_M(X') = r_M(X)$. Hence, $\sqcap_M(X_1',X_2) > \sqcap_M(X_1,X_2)$. Moreover, by Proposition 3.2, $v_3,\ldots,v_p$ is a blocking sequence for the k-separation (X_1',X_2) in $M[X',B]$. Now let $M'' = M[X' \cup \{v_4,\ldots,v_{p-1}\}, B]$. Inductively, we find $i \in \{4,5,\ldots,p-1\}$ such that $\kappa_{M''\backslash v_i}(X_1',X_2) = k-1$ and $\kappa_{M''/v_i}(X_1',X_2) = k-1$. Now, the result follows by Lemma 2.4. ∎

5. Bridging 1- and 2-separations.

In this section we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $X = E(N)$, let B be a basis of M such that $N = M[X,B]$, and let $v_1,\ldots,v_p$ be a blocking sequence for the separation (X_1,X_2) with respect to B. Suppose that $p \geq 3$, and let $M' = M[X \cup \{v_2\}]$. By the definition of a blocking sequence, $(X_1 \cup \{v_2\}, X_2)$ and $(X_1, X_2 \cup \{v_2\})$ are both separations of M'. Hence, N is a minor of both $M'\backslash v_2$ and M'/v_2. But then N is a minor of both $M\backslash v_2$ and M/v_2. So by Corollary 2.2, we obtain a contradiction. ∎

To prove Theorem 1.3 we require the following key lemma, whose proof we leave as an exercise.

LEMMA 5.1. *Let N be a minor of a matroid M, let (X_1,X_2) be an exact 2-separation of N, and suppose that $\lambda_M(X_1) = \lambda_M(X_2) = 1$. If N' is a minor of M such that $E(N') = X_1 \cup X_2$ and $\lambda_{N'}(X_1) = 1$, then $N' = N$.*

Proof of Theorem 1.3. Let $X = E(N)$, let B be a basis of M such that $N = M[X,B]$, and let $v_1,\ldots,v_p$ be a blocking sequence for the separation (X_1,X_2) with respect to B. Suppose that $p \geq 6$. By Proposition 3.2, we may assume that $p = 6$. Let $M' = M[X \cup \{v_2,v_3,v_4,v_5\}, B]$. By Lemma 4.3, there exists $i \in \{2,3,4,5\}$ such that $\kappa_{M'\backslash v_i}(X_1,X_2) = 1$ and $\kappa_{M'/v_i}(X_1,X_2) = 1$. Then, by Tutte's Linking Theorem and Lemma 5.1, N is a minor of both $M'\backslash v_i$ and M'/v_i. But then N is a minor of both $M\backslash v_i$ and M/v_i, contradicting Corollary 2.2. ∎

6. Bridging larger separations.

In this section we give examples showing that there is no analogue of Theorems 1.2 and 1.3 for 3-separations. The same examples also show that there is no analogue of Theorem 1.1 for infinite fields. In particular, we prove the following proposition.

PROPOSITION 6.1. *For any infinite field* $\mathbb{F}$ *and integer* n*, there exist* $\mathbb{F}$*-representable matroids* N *and* M *such that* N *has an exact* 3*-separation* (X, Y)*,* M *is a minor-minimal matroid bridging this separation in* N*, and* $|E(M)| \geq |E(N)| + n$*.*

Let $p \geq n/2$ be an integer and let

$$A = \begin{array}{cc} & \begin{array}{ccccccccc} y_1 & y_2 & x_3 & x_4 & v_1 & v_2 & \cdots & v_{p-1} & v_p \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ y_3 \\ y_4 \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_p \end{array} & \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \alpha_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha_2 & \alpha_3 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & \alpha_{2p-2} & \alpha_{2p-1} \end{pmatrix} \end{array}.$$

Let $U = \{u_1, \ldots, u_p\}$ and $V = \{v_1, \ldots, v_p\}$, and let $D = A[U \cup \{y_3\}, V \cup \{x_3\}]$ (that is, D is the submatrix of A with rows indexed by $U \cup \{y_3\}$ and columns indexed by $V \cup \{x_3\}$). Now, choose $\alpha_1, \ldots, \alpha_{2p-1} \in \mathbb{F}$ so that each subdeterminant of D is nonzero unless it is identically zero as a polynomial in $\alpha_1, \ldots, \alpha_{2p-1}$.

Now, let M be the matroid represented over $\mathbb{F}$ by $[I|A]$, let $B = U \cup \{x_1, x_2, y_3, y_4\}$, let $X = \{x_1, x_2, x_3, x_4\}$, let $Y = \{y_1, y_2, y_3, y_4\}$, and let $N = M[B, X \cup Y]$. Note that $|E(M)| \geq |E(N)| + n$. Also, it is routine to check that (X, Y) is an exact 3-separation in N, and that $u_1, v_1, \ldots, u_p, v_p$ is a blocking sequence for (X, Y) with respect to B. Thus, M bridges the 3-separation (X, Y) in N; it remains to prove that M is minor-minimal with this property.

Claim. M is a minor-minimal matroid that bridges the 3-separation (X, Y) of N.

Proof. Suppose, for a contradiction, that there is a proper minor M' of M that bridges the 3-separation (X, Y) of N. Since M' is a minor of M, there exists a basis B' of M such that $M' = M[B', E(M')]$. Since N is a minor of M', we may assume without loss of generality that $E(N) \cap B' = \{x_1, x_2, y_3, y_4\}$. Since $u_1, v_1, \ldots, u_p, v_p$ is a blocking sequence with respect to B and M' is a *proper* minor of M, we see that $B \neq B'$. Now, since B' is a basis of M, $A[B - B', B' - B]$ is nonsingular. Note that $B - B' \subseteq U$ and $B' - B \subseteq V$. By our choice of $\alpha_1, \ldots, \alpha_{2p-1}$, we see that $A[(B - B') \cup \{y_3\}, (B' - B) \cup \{x_3\}]$ is nonsingular. Hence, $(B' - \{y_3\}) \cup \{x_3\}$ is a basis of M. So, $\{x_1, x_2, x_3, y_4\}$ is a basis of $M[B', E(N)] = N$. However, from A we can see that $\{x_1, x_2, x_3, y_4\}$ is not a basis of $M[B, E(N)] = N$. This contradiction completes the proof. $\square$

7. Representation over finite fields. The difficulty when we go from 2-separations to 3-separations is that the analogue of Lemma 5.1 fails. Let N be a minor of a matroid M, let (X_1, X_2) be an exact 3-separation of N, and suppose that $\lambda_M(X_1) = \lambda_M(X_2) = 2$. If N' is a minor of M such that $E(N') = X_1 \cup X_2$ and $\lambda_{N'}(X_1) = 2$, then it need not be the case that $N' = N$. Lemma 5.1 essentially says that there is a unique way to compose two matroids at given points, but it is well known that there is no bound on the number of ways to compose two matroids along given lines. However, over a finite field, lines have bounded length, and hence there is a bound on the number of ways to compose two representations along given lines. Similarly, there is also a bound on the number of ways to compose two representations on subspaces of any fixed dimension.

Throughout the remainder of this section we let $\mathbb{F}$ be a fixed finite field with q elements and we let $V(r,\mathbb{F})$ denote a vector space over $\mathbb{F}$ with rank r. Thus, the number of points in $V(r,\mathbb{F})$ is q^r. We let $n(r,q)$ denote the number of ordered bases of $V(r,\mathbb{F})$. It is straightforward to see that

$$n(r,q) = (q^r-1)(q^r-q)\cdots(q^r-q^{r-1}).$$

Let V_1 and V_2 be two rank-k subspaces of $V(r,\mathbb{F})$. Then the number of invertible linear transformations from V_1 to V_2 is $n(k,q)$. The following theorem is a strengthening of Theorem 1.1. (Sharper bounds can be obtained by using projective equivalence.)

THEOREM 7.1. *Let $\mathbb{F}$ be a finite field with q elements. If (X_1,X_2) is an exact k-separation in an $\mathbb{F}$-representable matroid N and M is a minor-minimal $\mathbb{F}$-representable matroid that bridges the k-separation (X_1,X_2) in N, then $|E(M)| \le |E(N)| + (2k+1)n(k-1,q)$.*

To make the proof of Theorem 7.1 rigorous, we need to be particular about the way we define representations. A *configuration over* $\mathbb{F}$ is a set of labelled elements in the vector space $V(r,\mathbb{F})$, for some integer r, where all labels are distinct, but a vector may receive more than one label. Two configurations are *isomorphic* if one can be obtained from the other by relabelling. Formally, a configuration is a pair $(E,\mathbb{V})$, where E is a finite set and $\mathbb{V} = V(r,\mathbb{F})$, with a function ψ from E to the set of vectors in $\mathbb{V}$. Let $\mathcal{L}:\mathbb{V}\to\mathbb{V}$ be a linear transformation. We let $\mathcal{L}(E,\mathbb{V})$ denote the configuration obtained by applying $\mathcal{L}$ to $\mathbb{V}$ and relabelling accordingly. That is, in $\mathcal{L}(E,\mathbb{V})$ an element $e\in E$ labels the vector $\mathcal{L}(\psi(e))$. If $\mathcal{L}$ is invertible, then we call $(E,\mathbb{V})$ and $\mathcal{L}(E,\mathbb{V})$ *equivalent*.

We associate a matroid M with a configuration $(E,\mathbb{V})$ in the natural way. That is, E is the ground set of M and for a set X of elements, $r_M(X)$ is the rank of X in $\mathbb{V}$. Thus, a matroid M is *representable* over a field $\mathbb{F}$ if it is induced by a configuration over $\mathbb{F}$ in this way. Note that equivalent configurations represent the same matroid.

The notion of minors extends naturally to configurations. Let $(E,\mathbb{V})$ be a configuration, let D and C be disjoint subsets of E, and let $\mathcal{L}:\mathbb{V}\to\mathbb{V}$ be a linear transformation whose kernel is the subspace spanned by C. We let $(E,\mathbb{V})\backslash D/C$ denote the configuration $\mathcal{L}(E-(D\cup C),\mathbb{V})$; we call any such configuration a *minor* of $(E,\mathbb{V})$. Obviously, if $(E,\mathbb{V})$ is a representation of M, then $(E,\mathbb{V})\backslash D/C$ is a representation of $M\backslash D/C$. Note that $\mathcal{L}$ is not uniquely defined, so $(E,\mathbb{V})\backslash D/C$ is not uniquely determined by D and C; but all such configurations are equivalent. When it is necessary to distinguish the particular linear transformation used, we say that $\mathcal{L}$ *projects* $(E,\mathbb{V})$ onto $(E,\mathbb{V})\backslash D/C$.

Proof of Theorem 7.1. Let $X = E(N)$, let $E = E(M)$, let B be a basis of M such that $N = M[X,B]$, and let $v_1,\ldots,v_p$ be a blocking sequence for the separation (X_1,X_2) with respect to B. Let $n = n(k-1,q)$, and suppose that $p > n(2k+1)$. By Proposition 3.2, we may assume that $p = n(2k+1)+1$. For $i\in\{0,\ldots,n\}$ let $u_i = v_{i(2k+1)+1}$ and, for $i\neq 0$, let $W_i = \{v_{(i-1)(2k+1)+2},\ldots,v_{i(2k+1)}\}$. Thus $(\{u_0\},W_1,\{u_1\},\ldots,W_n,\{u_n\})$ is a partition of $\{v_1,\ldots,v_p\}$. Now let $E' = E - \{u_0,\ldots,u_n\}$ and let $M' = M[E',B]$. For each $i\in\{0,\ldots,n\}$, let $L_i = X_1\cup(W_1\cup\cdots\cup W_i)$ and $R_i = X_2\cup(W_{i+1}\cup\cdots\cup W_n)$. Thus, (L_i,R_i) is a k-separation in M' for each $i\in\{0,\ldots,n\}$. By Proposition 3.2 and Lemma 4.3, there exists $x_i\in W_i$ such that $\kappa_{M'\backslash x_i}(L_{i-1},R_i) = k-1$ and $\kappa_{M'/x_i}(L_{i-1},R_i) = k-1$ for each $i\in\{1,\ldots,n\}$.

Consider a configuration $(E', \mathbb{V})$ representing M' over $\mathbb{F}$. For any $A \subseteq \mathbb{V}$ let $\langle A \rangle$ denote the span of A. Then, for $i \in \{0, \dots, n\}$, let V_i denote $\langle L_i \rangle \cap \langle R_i \rangle$; thus V_i is a subspace of rank $k-1$. Now let $D_i = W_i - B$ and $C_i = W_i \cap B$. Note that N is a minor of $M' \backslash D_i / C_i$, so $\lambda_{M' \backslash D_i / C_i}(R_i) = k - 1$. Note that, since $\lambda_{M'}(R_i) = k-1$, $\langle C_i \rangle$ intersects $\langle R_i \rangle$ trivially. Choose a linear transformation $\mathcal{L}_i$ such that the kernel of $\mathcal{L}_i$ is $\langle C_i \rangle$ and $\mathcal{L}_i$ acts as the identity on $\langle R_i \rangle$. Thus $\mathcal{L}_i$ projects $(E', \mathbb{V})$ onto $(E', \mathbb{V}) \backslash D_i / C_i$. Let π_i be the restriction of $\mathcal{L}_i$ to V_{i-1}. Note that π_i is an invertible linear transformation from V_{i-1} to V_i. Now let $\mathcal{L} = \mathcal{L}_n \mathcal{L}_{n-1} \cdots \mathcal{L}_0$. Thus, $\mathcal{L}(X, V)$ is a configuration representing N.

Recall that $\kappa_{M' \backslash x_i}(L_{i-1}, R_i) = k-1$ and $\kappa_{M'/x_i}(L_{i-1}, R_i) = k-1$. Therefore, there exists a partition (D'_i, C'_i) such that $\lambda_{M' \backslash D'_i / C'_i}(R_i) = k-1$ and x_i is in exactly one of D_i and D'_i. Choose a linear transformation $\mathcal{L}'_i$ such that the kernel of $\mathcal{L}'_i$ is $\langle C'_i \rangle$ and $\mathcal{L}'_i$ acts as the identity on R_i. Thus $\mathcal{L}'_i$ projects $(E', \mathbb{V})$ onto $(E', \mathbb{V}) \backslash D'_i / C'_i$. Let π'_i be the restriction of $\mathcal{L}'_i$ to V_{i-1}. Note that π'_i is an invertible linear transformation from V_{i-1} to V_i. Now, for $i \in \{0, \dots, n\}$ we let $\sigma_i = (\pi'_n \cdots \pi'_{i+1})(\pi_i \cdots \pi_1)$. So σ_i is an invertible linear transformation from V_0 to V_{n-1}. The number of such distinct transformations is $n = n(k-1, q)$. Therefore, there exists $i > j$ such that $\sigma_i = \sigma_j$. Since each of these linear transformations is invertible, we see that $\pi'_i \pi'_{i-1} \cdots \pi'_{j+1} = \pi_i \pi_{i-1} \cdots \pi_{j+1}$; therefore

$$\pi_n \cdots \pi_{i+1} \pi'_i \cdots \pi'_{j+1} \pi_j \cdots \pi_1 = \pi_n \pi_{n-1} \cdots \pi_1.$$

Let

$$\mathcal{L}' = \mathcal{L}_n \cdots \mathcal{L}_{i+1} \mathcal{L}'_i \cdots \mathcal{L}'_{j+1} \mathcal{L}_j \cdots \mathcal{L}_1.$$

Now, $\mathcal{L}'(X, \mathbb{V})$ is equivalent to $\mathcal{L}(X, \mathbb{V})$, which is a representation of N. It follows that $M' \backslash x_i$ and M'/x_i both contain N as a minor. But then $M \backslash x_i$ and M/x_i both contain N as a minor, contradicting the minimality of M. $\square$

8. Branch-width. A tree is *cubic* if all vertices have degree 1 or 3; the vertices with degree 1 are the *leaves*. A *branch-decomposition* of a matroid M on a finite ground set E is a cubic tree such that E labels a set of the leaves of T. (No leaf gets more than one label, but there may be unlabelled leaves.) The set *displayed* by a given subtree of T is the set of elements of E that label leaves of that subtree. A set of elements A of E is *displayed* by an edge e of T if it is displayed by one of the two components of $T \backslash e$; the *width* $\lambda(e)$ of the edge e of T is $\lambda_M(A) + 1$. The *width* of a branch-decomposition is the maximum of the widths of its edges, and the *branch-width* of M is the minimum among the widths of its branch-decompositions.

Let T be a branch-decomposition of M and let e be an edge of width k in T. There are two subsets A and B of E that are displayed by e. These two sets partition E, and, if $|A|, |B| \geq k$, then (A, B) is a k-separation of M; we say that such k-separations are *displayed* by T.

LEMMA 8.1. *Let (X_1, X_2) be an exact k-separation in a matroid N and let M be a minor-minimal matroid that bridges the k-separation (X_1, X_2) in N. If N has a branch-decomposition of width n that displays (X_1, X_2), then M has branch-width at most $n + 1$.*

Proof. Let $X = E(N)$, let B be a basis of M such that $N = M[X, B]$, and let $v_1, \dots, v_p$ be a blocking sequence for the separation (X_1, X_2) with respect to B. By duality we may assume that $v_1 \notin B$. Let T be a width-n branch-decomposition of N that displays (X_1, X_2) and let $e = ab$ be the edge of T that displays X_1 and

X_2. Now let T_a and T_b be the components of $T - e$ containing a and b, respectively. We construct a tree-decomposition T' of M as follows. Connect T_a to T_b with a path $P = (a, x_1, \dots, x_p, b)$, and for each $i \in \{i, \dots, p\}$ add a leaf, labelled v_i, adjacent to x_i. Note that e has width $k \le n$ in T and, by Proposition 3.2, each edge of P has width $k+1$ in T'. The edges of T' incident with any of $v_1, \dots, v_p$ all have width 2. Consider any other edge f of T'. By symmetry we may assume that f is an edge of T_a. Let A and B be the sets displayed by f in T', where $A \subseteq X_1$. Note that A is displayed by f in T, so $\lambda_N(A) \le n$. By Proposition 3.2, $\lambda_N(X_1) = \lambda_{M\backslash v_1}(X_1)$. Therefore, since $A \subseteq X_1$, $\lambda_N(A) = \lambda_{M\backslash v_1}(A)$. Thus $\lambda_M(A) \le \lambda_{M\backslash v_1}(A) + 1 = \lambda_N(A) + 1 \le n + 1$. Therefore, T' has width at most $n + 1$, as required. ∎

Theorem 1.4 follows immediately from Lemma 8.1 and the next lemma.

LEMMA 8.2. *Let N be a matroid with branch-width n and let (X_1, X_2) be a k-separation of N. Then, there exists a branch-decomposition of N that displays (X_1, X_2) and that has width at most $n + k - 1$.*

Proof. Let T be a width-n branch-decomposition of N. We may assume that T has some unlabelled leaf r. Let s be the neighbor of r in T. Construct two copies T_1 and T_2 of $T - r$ such that for each vertex v of T the corresponding copies are labelled v_1 and v_2. Now construct a cubic tree T' by connecting T_1 and T_2 with the edge $s_1 s_2$. We now make T' into a new branch-decomposition of N as follows. For each $i \in \{1, 2\}$ and $e \in X_i$, if e labels the leaf x in T, then we label the leaf x_i with e in T'. Therefore, X_1 and X_2 are displayed by $s_1 s_2$ in T'.

Consider an edge f of $T - r$. Let A be the set that is displayed by the component of $T - f$ that does not contain s. Thus $A \cap X_1$ and $A \cap X_2$ are displayed by the copies of f in T'. Now, $\lambda_N(A \cap X_i) \le \lambda_N(A) + \lambda_N(X_i) \le n + (k - 1)$ for each $i \in \{1, 2\}$. Therefore, T' has width at most $n + k - 1$, as required. ∎

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