

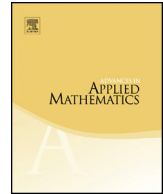


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A geometric version of the Andrásfai–Erdős–Sós Theorem[☆]



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ARTICLE INFO

Article history:

Received 27 January 2014
 Accepted 19 February 2014
 Available online xxxx

MSC:

05B25
 05B35

Keywords:

Matroid
 Projective geometry
 Blocking set

ABSTRACT

For each odd integer $k \geq 5$, we prove that, if M is a simple rank- r binary matroid with no odd circuit of length less than k and with $|M| > k2^{r-k+1}$, then M is isomorphic to a restriction of the rank- r binary affine geometry; this bound is tight for all $r \geq k - 1$. We use this to give a simpler proof of the following result of Govaerts and Storme: for each integer $n \geq 2$, if M is a simple rank- r binary matroid with no $\text{PG}(n-1, 2)$ -restriction and with $|M| > (1 - \frac{11}{2^{n+2}})2^r$, then M has critical number at most $n - 1$. That result is a geometric analogue of a theorem of Andrásfai, Erdős and Sós in extremal graph theory.

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1. Introduction

Our main result is:

Theorem 1.1. *For each odd integer $k \geq 5$ and each integer $r \geq k - 1$, if M is a simple rank- r binary matroid with no odd circuit of length less than k and with $|M| > \frac{k}{2^{k-1}}2^r$, then M is isomorphic to a restriction of the rank- r binary affine geometry.*

Examples showing that the bound is tight are given in Section 4. We will prove Theorem 1.1 in Section 3. In the remainder of this introduction we discuss the motivation.

[☆] This research was partially supported by a grant from the Office of Naval Research [N00014-10-1-0851].
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We will call a matroid N -free if it has no restriction isomorphic to N . Bose and Burton [3] proved the following theorem.

Theorem 1.2 (*Bose–Burton Theorem*). *For all integers r and n with $r \geq n \geq 2$, if M is a simple rank- r $\text{PG}(n-1, 2)$ -free binary matroid, then $|M| \leq (1 - \frac{1}{2^{n-1}})2^r$.*

Note that, if F is a rank- $(r-n+1)$ flat in $\text{PG}(r-1, 2)$, then the matroid $M = \text{PG}(r-1, 2) \setminus F$ attains equality in the Bose–Burton Theorem. We will denote $\text{PG}(r-1, 2) \setminus F$ by $\text{BB}(r, n-1)$. Note that $\text{BB}(r, 1)$ is isomorphic to the affine geometry $\text{AG}(r-1, 2)$. Bose and Burton proved that $\text{BB}(r, n-1)$ is the only matroid that attains equality in their theorem, but the following result is considerably stronger.

The *critical number* of a simple rank- r binary matroid M is equal to the minimum integer c such that M is isomorphic to a restriction of $\text{BB}(r, c)$. Equivalently, c is the minimum number of cocycles of M required to cover $E(M)$. (Here by a *cocycle* we mean a disjoint-union of cocircuits.) The following result was proved by Govaerts and Storme [4]; it is analogous to a theorem in extremal graph theory due to Andrásfai, Erdős and Sós [1]. We will review the related work in graph theory in the next section.

Theorem 1.3 (*Geometric Andrásfai–Erdős–Sós Theorem*). *Let $n \geq 2$ be an integer and let $\epsilon = \frac{3}{2^{n+2}}$. Then, for each integer $r \geq n+2$, if M is a simple rank- r $\text{PG}(n-1, 2)$ -free binary matroid with $|M| > (1 - \frac{1}{2^{n-1}} - \epsilon)2^r$, then M has critical number at most $n-1$.*

A simple binary matroid is called *affine* if it is isomorphic to a restriction of a binary affine geometry; that is, the ground set is itself a cocycle. Note that the $n=2$ instance of the Geometric Andrásfai–Erdős–Sós Theorem is the same as the $k=5$ instance of Theorem 1.1, and both equivalent to the following result.

Theorem 1.4. *For each integer $r \geq 4$, if M is a simple rank- r triangle-free binary matroid with $|M| > \frac{5}{16}2^r$, then M is affine.*

Govaerts and Storme prove the Geometric Andrásfai–Erdős–Sós Theorem by induction on n . The induction follows an existing method introduced by Beutelspacher [2], but the base case (Theorem 1.4) requires some work. Our proof of Theorem 1.4 is a little easier, though, Govaerts and Storme do prove a bit more; they characterize the non-affine simple rank- r triangle-free binary matroids with $\frac{5}{16}2^r$ elements.

2. Connections with graph theory

The following result is a weak version of Turán’s Theorem [6].

Theorem 2.1. *For all integers t and n with $n \geq t \geq 2$, if G is a simple n -vertex K_t -free graph, then $|E(M)| \leq \frac{t-2}{t-1} \binom{n}{2}$.*

One natural class of K_t -free graphs is the class of $(t - 1)$ -colourable graphs. The stronger version of Turán’s Theorem amounts to saying that the densest K_t -free graphs are all $(t - 1)$ -colourable. One might hope that, for some $\epsilon > 0$, all n -vertex K_t -free graphs with at least $(\frac{t-2}{t-1} - \epsilon)\binom{n}{2}$ edges are $(t - 1)$ -colourable. However, this is not true as one can take the direct sum of a triangle-free graph with chromatic number t and some sufficiently large and dense graph with chromatic number $t - 1$.

Andrásfai, Erdős and Sós [1] overcome this issue by considering minimum degree instead of the number of edges. Note that, if G is an n -vertex graph with minimum degree αn , then $|E(G)| > \alpha\binom{n}{2}$.

Theorem 2.2 (*Andrásfai–Erdős–Sós Theorem*). *Let $t \geq 3$ be an integer and let $\epsilon = \frac{1}{(t-1)(3t-4)}$. Then, for each integer $n \geq t$, if G is a simple n -vertex K_t -free graph with minimum degree $> (\frac{t-2}{t-1} - \epsilon)n$, then G is $(t - 1)$ -colourable.*

In some sense the geometric version is even nicer, since the Geometric Andrásfai–Erdős–Sós Theorem implies the Bose–Burton Theorem, but it is not immediately evident whether or not the Andrásfai–Erdős–Sós Theorem implies Turán’s Theorem.

The Andrásfai–Erdős–Sós Theorem is proved by induction on t ; the base case is:

Theorem 2.3. *For each integer $n \geq 5$, if G is a simple n -vertex triangle-free graph with minimum degree $> \frac{2}{5}n$, then G is bipartite.*

They prove the following strengthening.

Theorem 2.4. *For each odd integer $k \geq 5$ and each integer $n \geq k$, if G is a simple n -vertex graph with no odd-circuit of length less than k and with minimum degree $> \frac{2}{k}n$, then G is bipartite.*

The above results on graphs bear a striking resemblance to the results in the introduction, where the role of “chromatic number” in graphs replaces “critical number” in the geometric setting. It is well known that the critical number and the chromatic number are related. For example, if G is a simple graph of chromatic number χ and $M(G)$ has critical number $c \geq 1$, then

$$2^{c-1} < \chi \leq 2^c.$$

In particular, G is bipartite if and only if $M(G)$ is affine. Moreover, the characterization of bipartite graphs using odd circuits is in fact a specialization of well-known result about binary matroids; see Oxley [5, Proposition 9.4.1].

Theorem 2.5. *A simple binary matroid is affine if and only if it does not contain a circuit of odd size.*

3. The proofs

We start by proving [Theorem 1.1](#), which we reformulate here for convenience; the equivalence between these formulations requires [Theorem 2.5](#).

Theorem 3.1. *Let $k \geq 5$ be an odd integer, let M be a simple rank- r binary matroid with $r \geq k - 1$, and let C be a circuit of size k in M . If M does not have an odd-circuit of length $< k$, then $|M| \leq \frac{k}{2^{k-1}} 2^r$.*

Proof. We break the proof into three cases.

Case 1. $r = k - 1$.

Suppose, for a contradiction, that $|M| > k = |C|$. Let $e \in E(M) - E(C)$. Since M is simple and binary and since $r(C) = r(M)$ there is a partition (C_1, C_2) of C with $|C_1|, |C_2| \geq 2$ such that $C_1 \cup \{e\}$ and $C_2 \cup \{e\}$ are both circuits. However, since C is odd, one of $C_1 \cup \{e\}$ and $C_2 \cup \{e\}$ is odd. This contradicts that C is a smallest odd circuit in M .

Case 2. $r = k$.

Suppose, for a contradiction, that $|M| > |C| + k$. By [Case 1](#), $E(M) - C$ is a cocircuit of M .

Claim. *For each $u_1, u_2 \in E(M) - C$ there exist $v_1, v_2 \in C$ such that $\{u_1, u_2, v_1, v_2\}$ is a circuit.*

Since $M|(C \cup \{u_1, u_2\})$ is binary and has co-rank 2, its ground set partitions into three series classes $(\{u_1, u_2\}, C_1, C_2)$. Since C is odd, we may assume that $|C_1|$ is odd. Now $C_1 \cup \{u_1, u_2\}$ is an odd circuit. Since C is an odd circuit of minimum size, $|C_1| = |C| - 2$ and, hence, $|C_2| = 2$. Now $C_2 \cup \{u_1, u_2\}$ gives the required circuit.

Let $u \in E(M) - C$ and let $X = E(M) - (C \cup \{u\})$. By the claim, for each $e \in X$ there exists a two-element set $P_e \subseteq C$ such that $P_e \cup \{u, e\}$ is a circuit. Moreover, since M is binary, $P_e \neq P_f$ for distinct $e, f \in X$. Since $|X| \geq |C|$, there exist $e, f \in X$ such that P_e and P_f are disjoint. Since M is binary, the symmetric difference Z of C , P_e , and P_f can be partitioned into circuits. However Z is smaller than C and has odd size; this contradicts that C is a minimum sized odd-circuit.

Case 3. $r > k$.

By [Claim 1](#), $\text{cl}_M(C) = C$. By [Claim 2](#), each parallel class of M/C has size at most k . Moreover, M/C has rank $r - k + 1$ and hence it has at most $2^{r-k+1} - 1$ points. Therefore

$$|M| \leq k(2^{r-k+1} - 1) + k = \frac{k}{2^{k-1}} 2^r,$$

as required. \square

Now we prove the Geometric Andrásfai–Erdős–Sós Theorem from [Theorem 1.4](#). This proof is sketched in [\[4\]](#); Govaerts and Storme attribute the method to Beutelspacher [\[2\]](#). We reformulate the result here for convenience.

Theorem 3.2. *For all integers r and n with $r - 2 \geq n \geq 2$, if M is a simple rank- r PG($n - 1, 2$)-free binary matroid with $|M| > (1 - \frac{11}{2^{n+2}})2^r$, then M has critical number at most $n - 1$.*

Proof. Consider a counterexample (r, n, M) with n minimum. Thus M is a simple rank- r PG($n - 1, 2$)-free binary matroid with $|M| > (1 - \frac{11}{2^{n+2}})2^r$ and with critical number at least n . By [Theorem 1.4](#), $n \geq 3$.

Consider M as a restriction of PG($r - 1, 2$) and let B denote the set of points not in M . Thus $|B| < \frac{11}{2^{n+2}}2^r - 1$.

Claim 1. *There is a line l of PG($r - 1, 2$) containing exactly one point of M .*

If not, then B is a flat of PG($r - 1, 2$). Since M has critical number at least n , we have $r_M(B) \leq r(M) - n$. So there is a rank- n flat F of PG($r - 1, 2$) that is disjoint from B . But then $M|F$ is isomorphic to PG($n - 1, 2$). This contradiction proves the claim.

Claim 2. *There is a hyperplane H of PG($r - 1, 2$), such that $|B \cap H| \geq 2^{r-n+1} - 1$.*

Let l be a line containing exactly one point in M , let $p \in l \cap E(M)$, and let H_0 be a hyperplane of M that does not contain p . Let X be the set of all points $q \in H_0 \cap E(M)$ such that $\{p, q\}$ spans a triangle in M . There are at most $2^{r-1} - 2$ lines of PG($r - 1, 2$) that contain p and that contain at least one other point of M . Each of these lines contains at most one point of $M \setminus (X \cup \{p\})$, so

$$|M| \leq 2^{r-1} - 1 + |X|.$$

Thus $|X| > (1 - \frac{11}{2^{n+1}})2^{r-1}$. Since M is PG($n - 1, 2$)-free, $M|X$ is PG($n - 2, 2$)-free. Therefore, by the minimality of the counterexample, $M|X$ has critical number $\leq n - 2$. Let F_0 be a rank- $(r - n + 1)$ flat in H_0 that is disjoint from X and let F_1 be the flat spanned by $F_0 \cup \{p\}$. By definition, $|F_1 \cap B| \geq 2^{r-n+1} - 1$. We can extend F_1 to obtain the desired hyperplane; this proves the claim.

Let H be a hyperplane satisfying [Claim 2](#).

Claim 3. *There is a rank- $(n - 1)$ flat F of PG($r - 1, 2$) with $F \subseteq H \cap E(M)$.*

Suppose otherwise; thus $M|(E(M) \cap H)$ is $\text{PG}(n - 2, 2)$ -free. Since M has critical number $\geq n$, $M|(E(M) \cap H)$ has critical number $\geq n - 1$. Now, by the minimality of our counterexample,

$$|E(M) \cap H| \leq \left(1 - \frac{11}{2^{n+1}}\right)2^{r-1}.$$

Thus

$$|M| \leq |E(M) \cap H| + 2^{r-1} \leq \left(1 - \frac{11}{2^{n+2}}\right)2^r,$$

giving the required contradiction. This proves the claim.

Let F be such a flat. There are 2^{r-n} flats of rank n in $\text{PG}(r - 1, 2)$ that contain F but are not contained in H . Since M is $\text{PG}(n - 1, 2)$ -free, each of these flats contains a point in B . Thus $|B - H| \geq 2^{r-n}$. Therefore

$$|B| \geq 2^{r-n} + 2^{r-n+1} - 1 = \frac{12}{2^{n+2}}2^r - 1.$$

This contradiction completes the proof. \square

4. Extremal examples

Our constructions are based on the following result.

Lemma 4.1. *Let M be a simple rank- r matroid, let $v \in E(M)$ such that each line containing v has 3 points, and let N be the restriction of M to a hyperplane not containing v . Then*

- (i) $|M| = 2|N| + 1$.
- (ii) $M \setminus v$ and N have same critical number.
- (iii) For each odd integer $k \geq 3$, if $M \setminus v$ has an odd circuit of length $\leq k$, then N has an odd circuit of length $\leq k$.
- (iv) For each integer $n \geq 2$, if $M \setminus v$ has a $\text{PG}(n - 1, 2)$ -restriction, then N has a $\text{PG}(n - 1, 2)$ -restriction.
- (v) For each integer $n \geq 2$, if N has a $\text{PG}(n - 1, 2)$ -restriction, then M has a $\text{PG}(n, 2)$ -restriction.

Before we prove [Lemma 4.1](#), we introduce some definitions. Note that M is defined, up to isomorphism, from N . We say that M is a *conical lift* of N and that $M \setminus e$ is a *doubling* of N .

Proof of Lemma 4.1. Note that (i) is trivial. Moreover, since $\text{PG}(n, 2)$ is a conical lift of $\text{PG}(n - 1, 2)$, (v) is also trivial.

Consider M as a restriction of $\text{PG}(r-1, 2)$ and let H be the hyperplane of $\text{PG}(r-1, 2)$ containing N . Let \bar{N} be the restriction of $\text{PG}(r-1, 2)$ to $H - E(N)$ and let \bar{M} be the restriction of $\text{PG}(r-1, 2)$ to $E(\text{PG}(r-1, 2) - E(M)) \cup \{v\}$. Note that \bar{M} is a conical lift of \bar{N} . Hence (ii) follows from (v).

Now consider (iv). Suppose that N_1 is a restriction of $M \setminus v$ that is isomorphic to $\text{PG}(n-1, 2)$. Now $(M/v)|E(N_1)$ is also isomorphic to $\text{PG}(n-1, 2)$. Since N is isomorphic to the simplification of M/v , N has a restriction isomorphic to $\text{PG}(n-1, 2)$, as required.

Finally, consider (iii). Let C be an odd circuit in $M \setminus v$. We may assume that C spans v since otherwise the proof goes as the proof of (iv). Then there is an odd subset C' of C such that $C' \cup \{v\}$ is a circuit in M . Thus C' is an odd circuit in M'/v . Since N is isomorphic to the simplification of M/v , N has an odd circuit of length $|C'| \leq k$. \square

The following result shows that [Theorem 1.1](#) is tight.

Theorem 4.2. *For each odd integer $k \geq 5$ and each integer $r \geq k - 1$, there exists a non-affine rank- r simple $(\frac{k}{2^{k-1}}2^r)$ -element binary matroid with no odd circuit of length less than k .*

Proof. When $r = k - 1$, we take the circuit of length k . Then we construct examples in higher rank by repeatedly doubling. \square

The next result shows that the Geometric Andrásfai–Erdős–Sós Theorem is tight; these examples were given in [\[4\]](#).

Theorem 4.3. *For all integers n and r with $r - 2 \geq n \geq 2$, there is a simple rank- r $\text{PG}(n-1, 2)$ -free binary matroid with critical number n and with $(1 - \frac{11}{2^{n+2}})2^r$ elements.*

Proof. For $n = 2$, the examples come from [Theorem 4.2](#). Suppose that $n \geq 3$ and that there exists a simple rank- $(r-1)$ $\text{PG}(n-2, 2)$ -free binary matroid N with $|N| = (1 - \frac{11}{2^{n+1}})2^{r-1}$, and with critical number $n - 1$. Let H be a hyperplane in $\text{PG}(r-1, 2)$ and construct a restriction M of $\text{PG}(r-1, 2)$ by taking a copy of N in H along with all points outside H . Thus M is $\text{PG}(n-1, 2)$ -free, has critical number n , and has $2^{r-1} + (1 - \frac{11}{2^{n+1}})2^{r-1} = (1 - \frac{11}{2^{n+2}})2^r$ points. \square

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