Approximation algorithms for feasible cut and multicut problems

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Abstract

Let $G = (V, E)$ be an undirected graph with a capacity function $u : E \rightarrow \Re_+$ and let $S_1, S_2, \ldots, S_k$ be k commodities, where each S_i consists of a pair of nodes. A set X of nodes is called feasible if it contains no S_i, and a cut $(X, \overline{X})$ is called feasible if X is feasible. Several optimization problems on feasible cuts are shown to be **NP**-hard. A 2-approximation algorithm for the minimum-capacity feasible v^*-cut problem is presented. The multicut problem is to find a set of edges $F \subseteq E$ of minimum capacity such that no connected component of $G \setminus F$ contains a commodity S_i. It is shown that an $\alpha-$approximation algorithm for the minimum-ratio feasible cut problem gives a $2\alpha(1 + \ln T)$-approximation algorithm for the multicut problem, where T denotes the cardinality of $\bigcup_i S_i$. A new approximation guarantee of $O(t \log T)$ for the minimum capacity-to-demand ratio Steiner cut problem is presented; here each commodity S_i is a set of two or more nodes and t denotes the maximum cardinality of a commodity S_i.

Key words. feasible cut, multicut, flow, set covering problem, NP-hard problem, approximation algorithm

AMS(MOS) subject classifications. 90C27, 68Q25, 05C85, 90C35, 90B10

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1 Introduction

Polynomial-time approximation algorithms for solving various **NP**-hard problems on graphs involving cuts and multicuts have recently attracted a great deal of research. For example, Dahlhaus et al introduce and study the multiterminal cut problem [DJPSY 94], Leighton and Rao study the sparsest cut problem [LR 88], and Garg et al study the multicut problem [GVY 93]; see also [KRAR 90, KPRT 94]. There are both practical and theoretical reasons behind this increased research interest. Practical applications include minimizing communication costs in parallel computers, partitioning files among the nodes of a network, deleting the minimum number of edges to get a bipartite graph, VLSI design, etc., see [DJPSY 94, LR 88, GVY 93]. Theoretical motivations include obtaining the best approximation guarantees possible, designing and analyzing simple general-purpose algorithms such as the greedy heuristic, and applying techniques from combinatorial optimization such as linear programming relaxations and duality theory.

Let $G = (V, E)$ be an undirected graph, and let every edge $e \in E$ have a nonnegative capacity $u(e)$. In addition, there are k commodities $S_1, \ldots, S_k$ associated with G, where each commodity S_i is a set of nodes. In Section 2, each S_i is a pair of nodes, so we refer to each commodity as a *demand edge*. We call a cut $(X, \overline{X})$ *feasible* if no demand edge has both its end nodes in X; possibly, every demand edge has both end nodes in $\overline{X}$. The minimum-capacity feasible v^*-cut problem is to find a feasible cut $(X, \overline{X})$ such that X contains a given node v^* and the cut has minimum capacity $u(X, \overline{X})$. The minimum node cover problem is the special case of the minimum-capacity feasible v^*-cut problem in which every edge of $E(G)$ is incident to v^*, i.e., G is the "star" of node v^*; see Theorem 2.1. The minimum node cover problem is **NP**-hard, and despite extensive research, the best approximation guarantee known, assuming that $|V| \to \infty$, is 2. In Section 2, we give a 2-approximation algorithm for the minimum-capacity feasible v^*-cut problem, see Theorem 2.4. Our algorithm is based on solving the dual of a linear programming (LP) relaxation. The dual LP is a single-commodity flow problem, and for the special case when there is exactly one demand edge, this flow problem corresponds to the so-called sharing problem, see [Br 79]. Brown [Br 79] and Gallo et al [GGT 89] give combinatorial algorithms for the sharing problem. This gives combinatorial algorithms for solving our dual LP when there is one demand edge.

Another interesting problem on feasible cuts is to find a minimum-ratio feasible cut, see Section 2. Theorem 2.1 shows that this problem too is **NP**-hard. An approximation algorithm for it may be used as a subroutine in approximately solving the multicut problem, a key **NP**-hard problem that is being intensively studied, see [GVY 93, GVY 94, KPRT 94, BTV 95]. In particular, we show that an α-approximation algorithm for the minimum-ratio feasible cut problem gives an $O(\alpha \log |V|)$-approximation algorithm for the multicut problem, see Theorem 3.1. Our work here gives a simple, general scheme for analyzing greedy heuristics for multicut problems. Our scheme may be used to replace ad hoc arguments used previously, see e.g., [KRAR 90, Lemma 3.1] and [KPRT 94, Theorem 2.8].

We also study the Steiner multicut problem, and the minimum capacity-to-demand ratio Steiner cut problem. Both problems are **NP**-hard. Klein et al [KPRT 94] introduced both problems. In Section 4, we give a new approximation guarantee for the latter problem that is independent of the number of commodities, k. Our approximation guarantee is $O(t \log T)$ compared to the $O(\log kt \log T)$ guarantee of [KPRT 94], where T denotes the cardinality of $\bigcup_i S_i$, and t denotes the maximum cardinality of a commodity S_i. When t is fixed, then our guarantee is $O(\log T)$ versus $O(\log k \log T)$. Our analysis uses the method of Linial et al [LLR 95] and is simpler than the one in [KPRT 94].

1.1 Notation

We usually denote the graph under consideration by $G = (V, E)$, and use n for $|V(G)|$. For a node $v \in V$, $N(v)$ denotes the set of neighbors, $\{w : vw \in E\}$. For a set of nodes S, $\overline{S}$ denotes $V \setminus S$. We usually allow a nonnegative capacity $u : E \rightarrow \Re_+$ on the edges, and in some cases there is a nonnegative weight $w : V \rightarrow \Re_+$ on the nodes. For a set $S \subseteq V$ or $F \subseteq E$, $w(S)$ and $u(F)$ denote the sum of the weights of the nodes in S and the sum of the capacities of the edges in F, respectively. For $S \subseteq V$, $\delta(S)$ denotes the set of edges that have exactly one end in S. If $\emptyset \neq S \neq V$, then $\delta(S)$ is called a *cut* and is also denoted by $(S, \overline{S})$; and $u(\delta(S)) = u(S, \overline{S})$ is called the capacity of the cut.

2 Feasible cut problems

In this section, we focus on the special case where each commodity has precisely two nodes. We first show that three variants of the feasible cut problem are **NP**-hard. Then we give a 2-approximation algorithm for the minimum-capacity feasible v^*-cut problem. A $(2-\epsilon)$-approximation algorithm (ϵ an absolute constant) would be a major result, since it would give a $(2-\epsilon)$-approximation algorithm for the minimum node cover problem (Corollary 2.2). Finally, we study approximation algorithms for a variant that has a fractional objective function.

Let $G = (V, E)$ be an undirected graph with a capacity function $u : E \rightarrow \Re_+$ and a set of commodities. In this section, each commodity is a pair of nodes, and so we call each commodity a *demand edge*. We use k to denote the number of commodities, and Q to denote the set of end nodes of demand edges, i.e., $Q = \bigcup_{1=1}^{k} \{s_i, t_i\}$, where $s_1t_1, \ldots, s_kt_k$ are the demand edges. Every node in Q is called a *terminal* node. A set of nodes is called *feasible* if it contains no commodity, i.e., no demand edge has both end nodes in the set; possibly, no demand edge has an end node in the set. A *feasible cut* $(S, \overline{S})$ is one such that S is feasible. We study three related problems on feasible cuts:

(P1): (*minimum-ratio feasible cut problem*) minimize $\dfrac{u(\delta(S))}{|S \cap Q|}$ such that S is a feasible set of nodes, $\emptyset \neq S \neq V$.

(P2): Given nonnegative node weights $w : V \rightarrow \Re_+$, minimize $u(\delta(S)) - w(S)$ such that S is feasible, $\emptyset \neq S \neq V$.

(P3): (*minimum-capacity feasible v^*-cut problem*) Given a fixed node v^*, minimize $u(\delta(S))$ such that S is feasible and $v^* \in S \neq V$.

2.1 Hardness of feasible cut problems

Theorem 2.1 *Problems (P1), (P2) and (P3) are* **NP**-*hard.*

Proof
Problem (P1): We reduce the minimum node cover problem to (P1). Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be an instance of the minimum node cover problem. Clearly, $\widetilde{G}$ has a node cover of size $\leq |\widetilde{V}| - 1$. Assume that $\widetilde{G}$ has no isolated nodes. We construct an instance of (P1) from $\widetilde{G}$ as follows, see Figure 1:

- take two copies of $\widetilde{V}$, say V and V', and an extra node z;
- form a perfect matching between V and V' with each matching edge having unit capacity;
- fix a node v' in V' and add an edge between v' and every other node in V' with capacity $+\infty$; also, add the edge $v'z$ with unit capacity;
- for each edge of $\widetilde{G}$, we take the corresponding node pair of V to be a demand edge; also, we add the demand edges zw, for each $w \in V'$. Note that $Q = \{z\} \cup V \cup V'$.

We claim that if S is an optimal solution of (P1), then $V \backslash S$ gives a minimum node cover of the graph $\widetilde{G}$, and that conversely, if C is a minimum node cover of $\widetilde{G}$, then $(V \backslash C) \bigcup V'$ is an optimal solution of (P1). First, suppose that $S \bigcap V' = \emptyset$. Then $\frac{u(\delta(S))}{|S \cap Q|} = \frac{|S|}{|S \cap Q|} = 1$. Now, suppose that $S \bigcap V' \neq \emptyset$. Then $S \supseteq V'$, since there is a path of infinite capacity between any two nodes of V', and so $\dfrac{u(\delta(S))}{|S \cap Q|} = \dfrac{u(\delta(S))}{|S|} = \dfrac{1 + |V \backslash S|}{2|V| - |V \backslash S|}$. Note that this ratio is

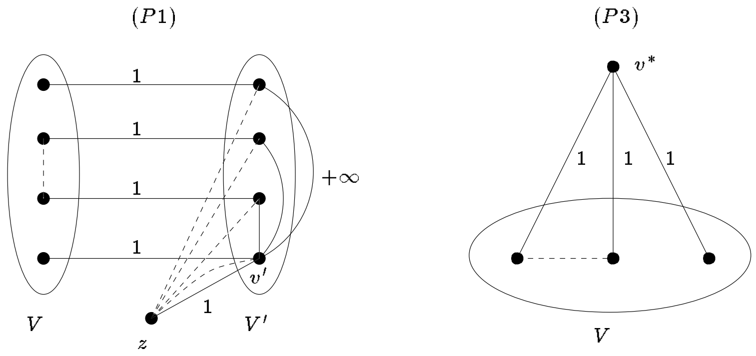

Figure 1: Reducing the minimum node cover problem to problems (P1) (left) and (P3) (right). Dashed lines indicate demand edges.

minimum when $|V \backslash S|$ is minimum, and if $|V \backslash S| \leq |V| - 1 = |\widetilde{V}| - 1$, then the ratio is less than 1. By the feasibility of S, every demand edge has at least one end in $\{z\} \cup (V \backslash S)$, so $V \backslash S$ corresponds to a node cover of $\widetilde{G}$. Hence the optimal solution S of (P1) must contain V' with $V \backslash S$ corresponding to a minimum node cover of $\widetilde{G}$. Our claim follows easily.

Problem (P2): We reduce the maximum independent set problem to (P2). Taking an instance $G = (V, E)$ of the maximum independent set problem, we assign each vertex in G with unit weight and each edge in G with zero capacity. Further we have a demand edge for each edge in G. Then for any feasible set S in the instance of (P2), the objective function $u(\delta(S)) - w(S)$ equals $-|S|$. Therefore, by the feasibility condition, S is a maximum independent set of G if and only if S is an optimal solution of the instance of (P2).

Problem (P3): We reduce the minimum node cover problem to (P3). Given an instance $G = (V, E)$ of the minimum node cover problem, we construct an instance $G' = (V', E')$ of (P3) as follows, see Figure 1: (i) $V' = V \bigcup \{v^*\}$, $E' = \{v^*v | v \in V\}$ and each edge has unit capacity. (ii) Each edge in G gives a demand edge in G'. Then for any feasible set S with $v^* \in S$ we have $u(\delta(S)) = |V \backslash S|$, and $V \backslash S$ is a node cover in G by the feasibility of S. Hence $(S, \overline{S})$ is a minimum-capacity feasible v^*-cut in G' if and only if $\overline{S}$ is a minimum node cover in G. □

From the above proof, we get two easy corollaries. The construction for Corollary 2.3 is a variant of the construction for problem (P1) in Theorem 2.1.

Corollary 2.2 *An α-approximation algorithm for problem (P3) (minimum-capacity feasible v^*-cut) gives an α-approximation algorithm for the minimum node cover problem.*

Corollary 2.3 *The maximum-capacity feasible cut problem (find a feasible cut $(S, \overline{S})$ of maximum capacity) is* **NP***-hard. An α-approximation algorithm gives an α-approximation algorithm for the maximum independent set problem.*

2.2 Algorithms for the minimum-capacity feasible v^*-cut problem

We focus on problem (P3): given a fixed node v^*, find a feasible cut $(S, \overline{S})$ of minimum capacity with $v^* \in S$. In some special cases, the optimal solution can be found by network flow techniques (see Proposition 2.5). For the general problem, we present a 2-approximation algorithm.

Problem (P3) can be formulated as an integer program; relaxing the integrality constraints gives a linear program. There is a nonnegative length variable l_e for every edge e. Each node v is assigned with a potential d_v such that the potential difference across each edge is no more than the length of that edge. Furthermore, for each demand edge, the sum of the potentials of its two end nodes is at least one. The following LP expresses this. We constrain the special node v^* to have zero potential.

$$(LP1) \begin{cases} Z_{LP} = \text{minimize } \sum_e u_e l_e \\ \text{subject to} \\ d_w \leq l_{vw} + d_v, \ \text{for every edge } vw \\ d_v \leq l_{vw} + d_w, \ \text{for every edge } vw \\ d_s + d_t \geq 1, \ \text{for every demand edge } st \\ d_{v^*} = 0 \\ l_e \geq 0, \ \text{for every edge } e. \end{cases}$$

Theorem 2.4 *Given an instance of (P3), there is a polynomial algorithm to find a feasible set S with $v^* \in S$ such that*

$$Z_{LP} \leq u(\delta(S^{opt})) \leq u(\delta(S)) \leq 2Z_{LP},$$

where S^{opt} denotes the optimal set.

We present two proofs of Theorem 2.4. The first proof was discovered earlier by R. Ravi [R 95], improving on our approximation guarantee of $(4 \ln 2)$ in a preliminary version of this paper. The second proof was discovered independently by us. Our proof applies the standard construction for proving the max-flow min-cut theorem to (LP1) and its dual.

Proof (R. Ravi) Let l, d be the optimal solution of (LP1). Let $dist_l(v, w)$ denote the length of a shortest v-w path with respect to l. Starting at v^*, we grow a shortest paths tree with respect to l, but stop just before a node at distance $\geq 1/2$ is added. More precisely, let

$S = \{w | dist_l(w, v^*) \leq \phi\}$, where we take $\phi = 1/2$ if there is no vertex w at distance $1/2$ from v^*, otherwise we take $\phi = 1/(2+\epsilon)$ (ϵ is a small positive number). The set S is feasible since every demand edge st has either $d_s \geq 1/2$ or $d_t \geq 1/2$, and so every demand edge has at least one end in $V \setminus S$. Let $v_0 = v^*, v_1, v_2, \ldots, v_p$, $p = |S| - 1$, be the order in which the nodes of S are added to the shortest paths tree. Denote $dist_l(v_i, v^*)$ by d_i, $i = 0, 1, 2, \ldots, p$ and let $d_{p+1} = \phi$. Let $S_i = \{v_0, v_1, \ldots, v_i\}$, for $i = 0, 1, 2, \ldots, p$, and let $y(S_i) = d_{i+1} - d_i$; so $y(S_p) = \phi - dist_l(v_p, v^*)$. Let $E' = \bigcup_{i=0}^{p} \delta(S_i)$. Then

$$\begin{aligned}
Z_{LP} &= \sum_{e \in E} u_e l_e &\geq& \sum_{e \in E'} u_e \sum_{\substack{e \in \delta(S_i) \\ i: 0 \leq i \leq p}} y(S_i) \\
&= \sum_{i=0}^{p} \Big(y(S_i) \sum_{e \in \delta(S_i)} u_e \Big) &\geq& \Big(\sum_{i=0}^{p} y(S_i) \Big) \min_{0 \leq i \leq p} u(\delta(S_i)) \\
&= (\phi) \min_{0 \leq i \leq p} u(\delta(S_i)).
\end{aligned}$$

Therefore, $\min_{0 \leq i \leq p} u(\delta(S_i)) \leq \frac{1}{\phi} Z_{LP}$. Hence, the capacity of one of the cuts $\delta(S_i), 0 \leq i \leq p$, is at most $(2+\epsilon)$ times the minimum capacity of a feasible v^*-cut. $\Box$

Proof (alternative proof of Theorem 2.4) To simplify the notation, and without loss of generality, we assume that no demand edge is incident to v^*. Recall that Q denotes $\bigcup_{i=1}^{k} \{s_i, t_i\}$, where $s_i t_i$, $i = 1, \ldots, k$, are the demand edges. To obtain the dual linear program of (LP1), first, rewrite (LP1) as follows. Remove the constraint $d_{v^*} = 0$, and for every demand edge st, replace the constraint $d_s + d_t \geq 1$ by the constraint $(d_s - d_{v^*}) + (d_t - d_{v^*}) \geq 1$. Now, the dual is:

$$\left\{ \begin{array}{llll}
Z_{LP} = \text{maximize } \sum_{i=1}^{k} r_i & & & \\
\text{subject to} & & & \\
\sum_{w \in N(v)} f(v,w) - \sum_{w \in N(v)} f(w,v) & = & 0, & \text{for each node } v \in V \backslash (\{v^*\} \cup Q) \\
\sum_{w \in N(v)} f(q,w) - \sum_{w \in N(v)} f(w,q) + \sum_{i: q \in \{s_i, t_i\}} r_i & = & 0, & \text{for each node } q \in Q \\
\sum_{w \in N(v)} f(v^*,w) - \sum_{w \in N(v)} f(w,v^*) - 2 \sum_{i=1}^{k} r_i & = & 0 & \\
f(v,w) + f(w,v) & \leq & u(vw), & \text{for each edge } vw \in E \\
f(v,w), f(w,v) & \geq & 0, & \text{for each edge } vw \in E \\
r_i & \geq & 0, & i = 1, 2, \ldots, k.
\end{array} \right.$$

A feasible solution to this LP is a flow f such that for each demand edge $s_i t_i$, the net inflow to both end nodes is the same, namely r_i, and such that v^* has net outflow $2 \sum_{i=1}^{k} r_i$. For every other node, the flow conservation condition holds, and every edge satisfies the capacity constraint. Let f and r form an optimal solution to this LP. We may assume that at most one direction of each edge vw has positive flow. Otherwise, if $f(v,w) \geq f(w,v) > 0$, we can substitute $f(v,w)$ by $f(v,w) - f(w,v)$ and substitute $f(w,v)$ by zero. It is easy to check that the new flow is still optimal.

We find a feasible v^*-cut $(S, \overline{S})$ of small capacity by constructing the node set S by first taking v^* in S. Then, repeatedly, for each edge vw with $v \in S$ and $w \notin S$, if either $f(v,w) < u(vw)$ or $f(w,v) > 0$, then we include w in S. We claim that S is feasible. Otherwise, S contains both end nodes of some demand edge $s_i t_i$. Then there exists an augmenting path from v^* to s_i and an augmenting path from v^* to t_i. Hence we can increase the net inflow r_i of t_i and s_i by a positive amount, contradicting the optimality of f. Furthermore, if $v \in S, w \notin S$ and $vw \in E$, then $f(v,w) = u(vw)$ and $f(w,v) = 0$. Therefore,

$$\begin{aligned} u(S,\overline{S}) &= \sum_{v \in S, w \in \overline{S}, vw \in E} f(v,w) - \sum_{w \in \overline{S}, v \in S, vw \in E} f(w,v) \\ &= \sum_{v \in S} \Big(\sum_{w \in N(v)} f(v,w) - \sum_{w \in N(v)} f(w,v) \Big), \text{ by Lemma 11.1 [BM 78]} \\ &\leq 2 \sum_{i=1}^{k} r_i, \end{aligned}$$

where the last inequality holds because the net outflow of v^* is $2\sum_{i=1}^k r_i$ and the net outflow of every node in $S \backslash \{v^*\}$ is non-positive. The cut $(S, \overline{S})$ is a 2-approximation for the minimum-capacity feasible v^*-cut since $u(S, \overline{S}) \leq 2\sum_{i=1}^k r_i = 2Z_{LP}$. □

The next result gives a polynomial algorithm for the minimum-capacity feasible v^*-cut for several special cases. For an instance of problem (P3), let H denote the set of demand edges; the graph (V, H) is called the *demand graph.*

Proposition 2.5 *If the number of maximal independent sets in the demand graph (V, H) is polynomial in n, and all the maximal independent sets can be found in polynomial time, then a minimum-capacity feasible v^*-cut can be found in polynomial time.*

Proof For each maximal independent set $I \subseteq V$ of the demand graph, we find a cut of minimum capacity separating $\{v^*\}$ from $V \setminus I = Q \setminus I$ by solving a single maximum flow problem (the source is v^* and the sinks are $Q \setminus I$). The best of these cuts is the required output. □

Corollary 2.6 *If the demand graph (V, H) is one of the following, then a minimum-capacity feasible v^*-cut can be found in polynomial time:*

(a) a graph with a bounded number of edges, i.e., $|H| = O(1)$,

(b) a clique,

(c) a complete bipartite graph,

(d) the complement of a triangulated graph,

(e) the complement of a bipartite graph, or

(f) the complement of a line graph.

See p. 302 of [GLS 88] for similar results on the maximum independent set problem. Unfortunately, problems (P1), (P2) and (P3) remain **NP**-hard for the special case when the demand graph is bipartite. This follows because each of these problems can be transformed into an "equivalent" problem such that no two demand edges have a node in common, i.e., the demand edges form a matching. To see this, repeatedly split a node incident with two or more demand edges into two new nodes joined by a new edge with a huge capacity.

Proposition 2.7 *Problems (P1), (P2) and (P3) remain* **NP***-hard under the restriction that the demand graph is bipartite.*

Consider solving (LP1) by a combinatorial algorithm. Since the dual of (LP1) is a single-commodity flow problem, the question arises whether the optimal flow can be found using standard techniques from network flows. If there is exactly one demand edge, then it can be seen that the optimal flow can be found by computing several s-t maximum flows. Brown [Br 79] gives an algorithm for the case of one demand edge that finds the optimal flow by computing two maximum s-t flows. In fact, Brown gives a combinatorial algorithm for a problem that is more general, the so-called sharing problem. Unfortunately, for the case of two or more demand edges, we do not know of any combinatorial algorithm. Even for the special case when every edge in G is incident to v^* (i.e., G is the "star" of node v^*) and no demand edge is incident to v^*, obtaining a combinatorial algorithm is nontrivial. In this special case, note that an optimal solution to (LP1) has $d_w = l_{v^*w}$ for each node $w \in V \backslash \{v^*\}$. Obtaining an optimal solution to (LP1) corresponds to solving an LP relaxation of the minimum-weight node cover problem; there is a combinatorial algorithm for this problem due to Nemhauser and Trotter [NT 75].

Proposition 2.8 *[Br 79] In the special case when there is one demand edge, an optimal solution to the dual of (LP1) can be found by computing two maximum s-t flows.*

2.3 Approximating minimum-ratio feasible cuts

We focus on finding a minimum-ratio feasible cut. It is well known that an optimization problem with a rational objective function $f(x)/g(x)$ can be solved in polynomial time if the associated problem with objective function $f(x) - \lambda g(x)$, where λ is a real-valued parameter, can be solved in polynomial time [GM 84, Appendix 5]. For the minimum-ratio feasible cut problem, the associated problem can be solved by solving at most n minimum-capacity feasible v^*-cut problems; see the first paragraph in the proof of Proposition 2.9. Consequently, we can efficiently solve the minimum-ratio feasible cut problem for the special cases in Corollary 2.6. In general, an approximation algorithm for an optimization problem with a rational objective function does *not* follow from an approximation algorithm for the associated problem. The main result of this subsection is an $O(1)$-approximation algorithm for the minimum-ratio feasible cut problem for the special case when the feasible shore B of a minimum-capacity feasible cut $(B, \overline{B})$ has $\geq (0.5 + \epsilon)|Q|$ of the terminals. For related work, see [HFKI 87]. Recall that Q denotes the set of end nodes of demand edges.

Proposition 2.9 *Suppose that an α-approximation algorithm for the minimum-capacity feasible v^*-cut problem is available. Let ϵ be a number such that $1 > 1/\alpha \geq \epsilon > 0$. If the instance $G = (V, E)$, $u : E \rightarrow \Re_+$, $s_1t_1, \ldots, s_kt_k$ has a minimum-capacity feasible cut $(B, \overline{B})$ such that $|B \cap Q| \geq (1 + \epsilon - \frac{1}{\alpha})|Q|$, then there is a $(1/\epsilon)$-approximation algorithm for the minimum-ratio feasible cut problem.*

Proof Assume every cut has positive capacity and $Q \neq \emptyset$ (i.e., there is a demand edge). Let the feasible set S^* achieve the optimal ratio λ^*, i.e.,

$$\lambda^* \quad = \quad \frac{u(\delta(S^*))}{|S^* \cap Q|} \quad \leq \quad \frac{u(\delta(S))}{|S \cap Q|}, \quad \forall S \text{ feasible}, \ \emptyset \neq S.$$

Here is the method for solving the minimum-ratio feasible cut problem, given an algorithm for the minimum-capacity feasible v^*-cut problem: We "linearize" the minimum-ratio feasible cut problem to an instance of problem (P2)

$$\text{minimize } \{u(\delta(S)) - \lambda |S \cap Q| \ : \ S \text{ feasible}, \emptyset \neq S \cap Q\},$$

where λ is a nonnegative parameter. The optimal value of the linearized problem is zero iff we fix $\lambda = \lambda^*$; also, λ is less (greater) than λ^* iff the optimal value of the linearized problem is positive (negative). Note that $\lambda^* \geq \hat{u}/n$ and $\lambda^* \leq U$, where $\hat{u} = \min_e u_e$ and $U = \sum_e u_e$. To find λ^*, we execute a binary search over the interval $[\hat{u}/n, U]$. Each iteration of the binary search solves the linearized problem with the current value of λ. To solve the linearized problem, we transform it to a variant of the minimum-capacity feasible v^*-cut problem. We add a new node v^* and all the edges v^*w, $w \in Q$, and fix the capacity of each new edge at λ. The goal of our variant is to find a cut $(S', \overline{S'})$ of minimum capacity such that $S' = \{v^*\} \cup S$, S is feasible and $\emptyset \neq S \bigcap Q$, i.e., the feasible shore S' should contain v^* as well as a terminal node of the original problem. This can be achieved as follows: for each terminal node $w \in Q$, construct an instance of the minimum capacity feasible v^*-cut problem by "contracting" nodes v^* and w into a new node v^*. The best of the $|Q|$ feasible v^*-cuts gives the desired cut $(S', \overline{S'})$. In terms of the v^*-cut problem, the capacity of $(S', \overline{S'})$ is $u(\delta(S)) + \lambda|\overline{S} \cap Q|$. Subtracting $\lambda|Q|$, we get the optimal value of the linearized problem.

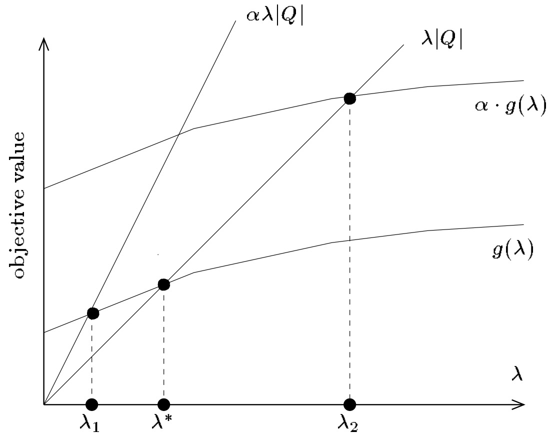

Figure 2: An illustration of the parametrized objective function $g(\lambda)$, and of $\lambda_1, \lambda^*, \lambda_2$.

The rest of the proof focuses on the solution of the linearized problem for a fixed value of the parameter λ, however, we keep the notation of the original minimum-ratio feasible cut problem, i.e., S, S^*, S_λ denote a feasible shore without the new node v^*. The objective function for the feasible v^*-cut problem as a function of λ is

$$g(\lambda) = \min_{S \text{ feasible},\ \emptyset \neq S \cap Q} \left\{ u(\delta(S)) + \lambda|\overline{S} \cap Q| \right\} = u(\delta(S_\lambda)) + \lambda|\overline{S_\lambda} \cap Q|,$$

where S_λ denotes the optimal set when the parameter is fixed at λ. The function $g(\lambda)$ is the "lower envelope" of the linear functions $u(\delta(S)) + \lambda|\overline{S} \cap Q|$ ($\forall S$ feasible, $\emptyset \neq S \bigcap Q$), each of which has positive slope, and so $g(\lambda)$ is a piece-wise linear, concave function with positive slope, see Figure 2. Note that $g(\lambda^*) = u(\delta(S^*)) + \lambda^*|\overline{S^*} \cap Q| = \lambda^*|Q|$. Let λ_1 and λ_2 denote the values of the parameter such that

$$\lambda_1|Q| = g(\lambda_1)/\alpha \quad \text{and} \qquad \lambda_2|Q| = \alpha \cdot g(\lambda_2);$$

recall that α is the approximation guarantee for the minimum-capacity feasible v^*-cut problem. Observe that if the parameter λ does not lie in the interval $[\lambda_1, \lambda_2]$, then the current iteration of the binary search (in the standard method for the minimum-ratio problem) gives the correct decision: if $\lambda < \lambda_1$, then the smallest possible objective value found by the approximation algorithm for the feasible v^*-cut problem is $g(\lambda)$, and since $1/\alpha$ of this value is $> \lambda|Q|$, we decide correctly that $\lambda^* > \lambda$; the other case, $\lambda > \lambda_2$, is similar. On the other hand, if λ is in the interval $[\lambda_1, \lambda_2]$, then we cannot make the correct decision. In this case, our approximation algorithm for the minimum-ratio problem terminates, and returns the current value of λ and the associated feasible set S, $\emptyset \neq S \bigcap Q$, as the approximately optimal solution.

We claim that under the hypothesis of the proposition $\lambda_1 \geq \epsilon\lambda^*$ and $\lambda_2 \leq \lambda^*/\epsilon$. The proposition follows from this claim. Now, we prove the claim. For the feasible shore B of the minimum-capacity feasible cut in the hypothesis, i.e., $B = S_\lambda$ for $\lambda = 0$, let β denote $|B \bigcap Q|/|Q|$, and note that for all $\lambda \geq 0$, $|S_\lambda \bigcap Q|/|Q| \geq \beta$.

Since $\alpha\lambda_1|Q| = g(\lambda_1) = u(\delta(S_{\lambda_1})) + \lambda_1|\overline{S_{\lambda_1}} \cap Q|$,

$$(\alpha - 1)\lambda_1 \frac{|Q|}{|S_{\lambda_1} \cap Q|} = \frac{u(\delta(S_{\lambda_1}))}{|S_{\lambda_1} \cap Q|} - \lambda_1 \geq \lambda^* - \lambda_1.$$

Hence, $\lambda_1 \geq \dfrac{1}{1 + (\alpha - 1)/\beta}\ \lambda^* \geq \epsilon\lambda^*$, since by the hypothesis $\beta \geq \epsilon + 1 - \dfrac{1}{\alpha} \geq \beta\epsilon + \dfrac{\alpha - 1}{\alpha}$ (since $\beta \leq 1$) $\geq \beta\epsilon + (\alpha - 1)\epsilon$ (since $\epsilon \leq \dfrac{1}{\alpha}$), hence $\dfrac{1}{1 + (\alpha - 1)/\beta} \geq \epsilon$.

For λ_2 we have,

$$\begin{aligned}
\frac{\lambda_2|Q|}{\alpha} &= g(\lambda_2) \\
&\leq u(\delta(S^*)) + \lambda_2|\overline{S^*} \cap Q|, \text{ by definition of } g(\lambda_2) \\
&= \lambda^*|Q| + (\lambda_2 - \lambda^*)|\overline{S^*} \cap Q|.
\end{aligned}$$

Hence,

$$\lambda_2 \leq \frac{\alpha|S^* \cap Q|}{|Q| - \alpha|\overline{S^*} \cap Q|}\ \lambda^* \leq \frac{\alpha|Q|}{|Q| - \alpha|\overline{S^*} \cap Q|}\ \lambda^* \leq \frac{1}{\beta + (1/\alpha) - 1}\ \lambda^* \leq \frac{\lambda^*}{\epsilon},$$

where the last inequality follows from the hypothesis: $\beta \geq 1 + \epsilon - \frac{1}{\alpha}$. $\square$

3 An approximate greedy method for multicut problems

We present a greedy method for approximately solving **NP**-hard Steiner/simple multicut problems. In the Steiner multicut problem introduced by Klein et al [KPRT 94], the input consists of $G = (V, E), u : E \rightarrow \Re_+$, and k commodities $S_1, S_2, \ldots, S_k$, where each S_i is a set of two or more nodes ($|S_i| \geq 2,\ i = 1, \ldots, k$). The problem is to find a set of edges $F \subseteq E$ of minimum capacity such that no connected component of $G \backslash F$ contains a commodity S_i, i.e., F should separate each commodity $S_i (1 \leq i \leq k)$ in the sense that there exist two nodes v, w in S_i such that every v-w path uses an edge of F. A node is called a *terminal* if it belongs to some commodity S_i. We use T to denote the number of terminals, $T = |\bigcup_i S_i|$, and t to denote the maximum cardinality of a commodity, $t = \max_i |S_i|$. Klein et al [KPRT 94], gave an $O(\log T \log k \log kt)$-approximation algorithm for the Steiner multicut problem. In the *simple* multicut problem, each commodity has precisely two nodes. This problem was studied by Klein et al and Garg et al [KRAR 90], [GVY 93]; they proved approximation guarantees of $O(\log^3 T)$ and $4 \ln(T + 1)$, respectively.

First, we formulate a variant of the multicut problem as a set covering problem with an exponential (in n) number of sets. This follows Bertsimas and Vohra [BV 94]; however, Bertsimas and Vohra never gave a polynomial-time approximation algorithm for this formulation. We directly apply an approximate greedy heuristic to the set covering problem: each iteration of this heuristic solves a minimum-ratio feasible cut subproblem. The subproblem turns out to be **NP**-hard too. However, if we can find an α-approximation for the subproblem, then our iterative method finds an $O(\alpha \log n)$-approximation to the multicut problem.

Recall that a set $S \subseteq V$ is *feasible* if it contains no commodity S_i, $1 \leq i \leq k$. Let $\mathcal{F}$ denote the family of all feasible sets. The goal is to "cover" all the terminals using feasible sets. For each feasible set S chosen in the covering, all edges in $\delta(S)$ are deleted from G, i.e., the multicut corresponding to this formulation is the union of $\delta(S)$ over all S in the covering. The following integer program (SC) for the set covering formulation has a 0–1 variable x_S for each feasible set S. The optimal value of (SC) is at least half and at most twice the capacity of an optimal multicut.

$$\begin{array}{llrcll}
(SC) & \text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} u(\delta(S)) x_S & & & \\
& \text{subject to} & \displaystyle\sum_{S \in \mathcal{F} : v \in S} x_S & \geq & 1, & \forall v \in Q = \bigcup_i S_i \\
& & x_S & \in & \{0, 1\}, & \forall S \in \mathcal{F}.
\end{array}$$

Recall that the greedy heuristic for solving the set covering problem repeatedly chooses a set S minimizing the ratio c_S / N'_S, where c_S is the coefficient of S in the objective function, and N'_S is the number of nonzero coefficients in S's column in the current constraints matrix [Ch 79]. The greedy heuristic is guaranteed to produce a set covering with objective value $\leq (1 + \ln N_{max}) z$, where N_{max} is the maximum over all columns of the number of nonzero coefficients and z is the optimal value (actually, z is the optimal value of the LP relaxation).

For the multicut problem, the greedy heuristic applied to (SC) gives an approximation guarantee of $2(1+\ln T) = O(\log T)$; recall that T denotes $|\bigcup_{i=1}^{k} S_i|$. Each iteration of the greedy heuristic applied to (SC) has to solve the **NP**-hard minimum-ratio feasible cut problem, since we have to find a feasible set S minimizing $u(\delta(S))/|S \cap Q|$. Our next result shows that if we can find an α-approximation to the minimum-ratio feasible cut problem, then by iterating this we can find a $2\alpha(1+\ln T)$-approximation to the multicut problem via (SC).

Theorem 3.1 *Consider an approximate greedy heuristic for the set covering problem that in each iteration finds a set $\widetilde{S}$ such that*

$$\frac{c_{\widetilde{S}}}{N'_{\widetilde{S}}} \leq \alpha \cdot \min_S \frac{c_S}{N'_S}$$

where c_S is the coefficient of S in the objective function, and N'_S denotes the number of nonzero coefficients in S's column in the current constraints matrix (after deleting rows of points already covered). Then the final set covering found by the heuristic has objective value at most $\alpha(1+\ln N_{max})$ times the optimal value.

Proof The proof is similar to Chvatal's analysis of the greedy heuristic [Ch 79]. See also Lemma 3.2.1 in [RV 93].

Consider the LP relaxation of the set covering problem and its dual

$$(P)\begin{cases} \min \sum_S c_S x_S \\ \text{subject to} \\ \sum_{S:i\in S} x_S \geq 1, \text{ for each } i \\ x \geq 0 \end{cases} \qquad (D)\begin{cases} \max \sum_i y_i \\ \text{subject to} \\ \sum_{i\in S} y_i \leq c_S, \text{ for each } S \\ y \geq 0. \end{cases}$$

We show that the heuristic constructs a feasible dual solution y such that its objective value is $\geq \dfrac{Z_H}{\alpha(1+\ln N_{max})}$, where Z_H is the objective value of the set covering found by the heuristic. The theorem follows because for every feasible dual solution y, and an optimal solution x of (P),

$$\sum_i y_i \leq \sum_S c_S x_S \leq Z^*,$$

where Z^* is the optimal value of the set covering problem.

Let $\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3, \ldots$ be the sequence in which the heuristic chooses sets, and let $\widetilde{S}_{f(i)}$ be the first set chosen by the heuristic that contains element i. For each element i, let $w_i = \dfrac{c(\widetilde{S}_{f(i)})}{|\widetilde{S}_{f(i)} \backslash (\widetilde{S}_1 \cup \ldots \cup \widetilde{S}_{f(i)-1})|}$. Clearly, $\sum_i w_i = Z_H$.

Now we claim that for each set S, $\sum_{i\in S} w_i \leq c_S \cdot \alpha(1+\ln|S|)$. We prove the claim as follows. Order the elements in S in the reverse order in which they were first covered:

$i_1, i_2, \ldots, i_{|S|}$. Consider i_l. When i_l is first covered, say by $\widetilde{S}_p$, S has at least l uncovered elements, so $|S \backslash (\widetilde{S}_1 \bigcup \ldots \bigcup \widetilde{S}_{p-1})| \geq l$. Hence, by our assumption,

$$\frac{c_{\widetilde{S}_p}}{|\widetilde{S}_p \backslash (\widetilde{S}_1 \bigcup \ldots \bigcup \widetilde{S}_{p-1})|} \leq \alpha \cdot \frac{c_S}{|S \backslash (\widetilde{S}_1 \bigcup \ldots \bigcup \widetilde{S}_{p-1})|} \leq \frac{\alpha c_S}{l},$$

so $w_{i_l} \leq \dfrac{\alpha c_S}{l}$. The claim follows since $\sum_{i_l \in S} w_{i_l} \leq \alpha c_S \sum_{i_l \in S} \dfrac{1}{l} \leq \alpha c_S (1 + \ln |S|)$. To get the dual solution y, let $y_i = \frac{w_i}{\alpha(1+\ln |N_{max}|)}$. □

The above theorem applies to another set covering formulation of a variant of the multicut problem. Instead of covering by feasible sets, we allow arbitrary node sets. We have a 0–1 variable x_S for every set $S \subseteq V$. The objective function is similar to the one in (SC), but the constraints are different. For each commodity $S_i \subseteq V$, $1 \leq i \leq k$, we require that at least one of the sets S chosen in the covering "separates" S_i, i.e., S includes some node of S_i and does not include some other node of S_i. The optimal value of (SC$'$) is at least half and at most twice the capacity of an optimal multicut.

$$\begin{array}{llrcll}
(SC') & \text{minimize} & \sum_{S \subseteq V} u(\delta(S)) x_S & & & \\
& \text{subject to} & \sum_{S \subseteq V : \emptyset \neq S \cap S_i \neq S_i} x_S & \geq & 1, & \forall i = 1, \ldots, k \\
& & x_S & \in & \{0,1\}, & \forall S \subseteq V.
\end{array}$$

Now, the greedy heuristic iteratively finds a set S minimizing $\frac{u(S,\overline{S})}{dem(S,\overline{S})}$, where $dem(S, \overline{S})$ denotes the number of commodities separated by $(S, \overline{S})$. (This agrees with our notation in Sections 4 since each commodity here has unit demand.) The problem of finding a cut minimizing the capacity-to-demand ratio is **NP**-hard [MS 90], however, extensive research has been devoted to designing approximation algorithms. For the case of $|S_i| = 2$, $1 \leq i \leq k$, there is an $O(\log T) = O(\log k)$ approximation algorithm due to Linial et al [LLR 95], see also [KRAR 90, PT 95]. For the general case, approximation guarantees of $O(\log kt \log T)$ and $O(t \log T)$ can be achieved in polynomial time; these results are due to Klein et al [KPRT 94], and Section 4 of this paper, respectively. We obtain approximation guarantees of $O(\log^2 k)$ and $O(\min(t, \log kt)\ \log T \log k)$ for the simple and Steiner multicut problems, respectively, by directly applying Theorem 3.1. However, this does not improve on the approximation guarantees of [GVY 93] and [KPRT 94], respectively.

4 A new approximation guarantee for minimum-ratio Steiner cuts

In the minimum capacity-to-demand ratio Steiner cut problem introduced by Klein et al [KPRT 94], the input consists of $G = (V, E), u : E \rightarrow \Re_+$, and k commodities $S_1, S_2, \ldots, S_k$ where each S_i is a set of nodes (possibly, $|S_i| > 2$). There is an additional input, namely, a nonnegative real-valued demand dem_i for each commodity S_i, $1 \leq i \leq k$. We say that a cut $(X, \overline{X})$ separates commodity S_i if $\emptyset \neq X \cap S_i \neq S_i$. The demand $dem(X, \overline{X})$ across a cut $(X, \overline{X})$ is the sum of the demands of the separated commodities. The minimum capacity-to-demand ratio Steiner cut problem is to find a cut that minimizes the ratio of the capacity of the cut and the demand across the cut, $u(X, \overline{X})/[\sum_{(i:\emptyset \neq S_i \cap X \neq S_i)} dem_i]$.

Theorem 4.1 *Given an instance of the minimum capacity-to-demand ratio Steiner cut problem, there is a (deterministic) polynomial algorithm to find a cut $(X, \overline{X})$ such that*

$$z^* \leq \min_{\emptyset \neq Y \neq V} \left\{ \frac{u(Y, \overline{Y})}{dem(Y, \overline{Y})} \right\} \leq \frac{u(X, \overline{X})}{dem(X, \overline{X})} \leq O(t \log T) z^*.$$

Here, z^* denotes the optimal value of the LP relaxation (LP2) (see below) of the problem, t denotes $\max_i |S_i|$, and T denotes $|\bigcup_i S_i|$.

We will use the following result due to Linial et al, see [LLR 95, Corollary 3.4]. For a graph G and length function $l : E \rightarrow \Re_+$, let $dist_l(v, w)$ denote the length of a shortest v-w path with respect to l.

Proposition 4.2 *Given a graph G, a length l_e on each edge e, and a set of nodes Q, there is a deterministic polynomial algorithm that constructs an l_1-metric $\rho : V \times V \rightarrow \Re_+$ such that*

1. *for every pair of nodes $\{v, w\}$ in Q,* $\dfrac{dist_l(v, w)}{O(\log |Q|)} \leq \rho(v, w) \leq dist_l(v, w)$*;*
2. *for every pair of nodes $\{v, w\}$ in V, $\rho(v, w) \leq dist_l(v, w)$.*

The next fact is well known [AD 91].

Fact 4.3 *Every l_1-metric on node set V can be written as a nonnegative linear combination of incidence vectors of cuts of the complete graph on V.*

Proof (Theorem 4.1) Let $l : E \rightarrow \Re_+$ be an optimal solution for the following LP relaxation of our problem. This LP relaxation is due to Klein et al [KPRT 94].

$$(LP2)\begin{cases} z^* = \text{minimize } \sum_e u_e l_e \\ \text{subject to} \\ \sum_{i=1}^k dem_i \cdot l(S_i) = 1 \\ l_e \geq 0, \ \forall e \in E, \end{cases}$$

where $l(S_i)$ denotes the minimum length of a spanning tree of the distance network $D_G(S_i)$ [HRW 92]. In more detail, given G, $l : E \to \Re_+$ and $S_i \subseteq V$, $D_G(S_i)$ consists of the complete graph on the node set S_i and edge lengths $dist_l$, i.e., the length of an edge vw, $v, w \in S_i$, equals the length $dist_l(v, w)$ of a shortest v-w path in G with respect to l. We use F_i to denote the set of edges of a minimum spanning tree of $D_G(S_i)$, so $l(S_i) = \sum_{vw \in F_i} dist_l(v, w)$.

Then,

$$\begin{aligned} z^* = \sum_e u_e l_e &= \frac{\sum_e u_e l_e}{\sum_{i=1}^k dem_i \cdot l(S_i)} \\ &\geq \frac{\sum_{vw\in E} u_{vw}\, dist_l(v,w)}{\sum_{i=1}^k dem_i (\sum_{vw\in F_i} dist_l(v,w))} \\ &\geq \frac{1}{O(\log T)} \frac{\sum_{vw\in E} u_{vw} \rho(v,w)}{\sum_{i=1}^k dem_i (\sum_{vw\in F_i} \rho(v,w))}, \end{aligned}$$

where ρ is an l_1-metric satisfying the two properties in Proposition 4.2, and we take the set Q in the proposition to be $\bigcup_{i=1}^k S_i$. By Fact 4.3, there exists a cut $(X, \overline{X})$ such that this quantity is

$$\begin{aligned} &\geq \frac{1}{O(\log T)} \frac{\sum_{vw\in(X,\overline{X})} u_{vw}}{\sum_{i=1}^k dem_i \cdot |(X,\overline{X}) \cap F_i|} \\ &\geq \frac{1}{O(\log T)} \frac{u(X,\overline{X})}{\sum_{(i:\emptyset\neq S_i\cap X\neq S_i)} dem_i \cdot (|S_i| - 1)} \\ &\geq \frac{1}{(t-1)O(\log T)} \cdot \frac{u(X,\overline{X})}{dem(X,\overline{X})}. \end{aligned}$$

Given the l_1 metric ρ in the form of an embedding of (V, ρ) into a real space with l_1 norm, the cut $(X, \overline{X})$ can be found in polynomial time, see [LLR 95, Theorem 4.1]. □

5 Conclusions

We conclude with two open problems. Is the problem of finding a feasible cut of minimum capacity **NP**-hard? Note that this is a variant of problem (P3) where we drop the constraint on the node v^*. Is there an $O(1)$ approximation algorithm for the minimum-ratio feasible cut problem? Our algorithm in Section 2.3 needs a "strong" assumption to give an $O(1)$ approximation guarantee, namely, there exists a minimum-capacity feasible cut $(B, \overline{B})$ such that B contains a large fraction of the terminal nodes. Note that such a cut $(B, \overline{B})$, if present, is an $O(1)$ approximation to the minimum-ratio feasible cut. The problem of finding such a cut $(B, \overline{B})$ is a bicriteria optimization problem, but unfortunately, we do not know how to solve it in polynomial time.

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