Invariant Vector Calculus

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Contents

The purpose of these notes

Often, those who take differential geometry at Waterloo are in Pure Mathematics, and so have not taken the fluid dynamics sequence in Applied Mathematics. This is unfortunate because fluid dynamics in three dimensions has, historically, motivated the study of differential geometry to a great extent. Because fluid dynamics has so many real world applications and ramifications there are many many people who spend their whole lives studying it. As a result there is a well developed set of notations for this field of mathematics, but unfortunately it is not the same one as we've developed for differential geometry. These notes are an attempt to bridge the notational gap between the disciplines of fluid dynamics and differential geometry, and so to give Pure Mathematics students a short introduction in how to read the Applied Mathematics notations, and to show the equivalence of both languages.

Amath 231 is Calculus 4 here at Waterloo, and unsurprisingly has only calculus 3 as a prerequisite. Therefore 231, and the follow up course AMATH 361 Continuum Mechanics do not mention many of the connections of their material to differential geometry. One of the goals of these notes is to illuminate these connections. These notes should then also give those who are only familiar with the theoretical side of differential geometry a better understanding of how to calculate standard integrals, and an appreciation of the physical interpretations behind many of the definitions in differential geometry.

1 A review of submanifold theory

Definition 1.0.0.1. *Let M^n and L^k be smooth manifolds. A smooth map $\iota : L^k \to M^n$ is called an injective immersion if*

1. *ι is injective*

2. *ι is an immersion, meaning $(\iota_*)_p : T_pL \to T_{\iota(p)}M$ is injective $\forall p \in L$*

Notice here that if ι is an injective immersion, we can use $(\iota_*)_p$ to identify each T_pL with a subspace of $T_{\iota(p)}M$. If $\iota : L \to M$ is an injective immersion then L is called an immersed submanifold of M. The name of this definition is suggestive of its physical meaning, because we can think of L as sitting inside M. The notation ι is also suggestive because perhaps the most naturally arising case of an injective immersion is the inclusion map from L to M when $L \subset M$ where L is also a smooth manifold.

If L is oriented, then condition 2 ensures that orientation is preserved or reversed, but there still is one, since ι is injective at every point of L and so has either positive determinant of its pushforward at every point, or negative determinant of its pushforward at every point. Which one it is will be irrelevant for us, because we will be using $\iota =$ inclusion, which is clearly an injective immersion and orientation preserving:

The coordinate representation of ι is

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$$

where there are $n-k$ zeros on the right. Taking the Jacobian of this map gives us the pushforward in local coordinates, an $n \times k$ matrix which looks like the $k \times k$ identity matrix sitting on top of a block of zeroes, which is injective.

Using the inclusion gives L the metric $\iota^*(g_M)$, which is the induced metric on L from M. By definition $\iota^*(g_M) = g_M\big|_L$, and so we also have that the restriction of any smooth map on M is also smooth on L, because its pullback by the inclusion is then the composition of smooth maps and so smooth.

The astute reader may object that in general, if X is a smooth vector field on M, then $X\big|_L$ may not be a smooth vector field on L, because if $p \in L$, $X_p \in T_pM \setminus T_pL$ is possible. This is a concern if we use a map other than the inclusion, but the definitions from vector calculus implicitly use the inclusion, so for these notes we don't have to worry about these concerns. For a proof that the inclusion map allows us to restrict see [smooth lee lemma 5.39]. Because the pullback by a smooth map of a form is a smooth form, we can always restrict forms to L.

Disclaimer:
Throughout this document we assume everything is as smooth as it needs to be so that we can apply our various definitions. Many of the vector calculus results and identities only require C^1 or C^2, but we want to view things as immersed manifolds, and will assume smoothness where it is convenient.

Since parameterizations are inverses of charts, by definition most of the immersed submanifolds studied below have a global parameterization, but technically not a global chart. The reason for this is that some of the submanifolds have a boundary. However we are in large part discussing integrals over submanifolds, and the boundary always has codimension 1, which is a zero set, and so does not effect the integral. We will therefore omit any further discussion of the actual required number of charts to cover the manifolds below, realizing that for our purposes the question is irrelevant.

2 AMATH 231

In this section we survey the course notes for AMATH 231 to show how all the definitions and theorems from the vector calculus portion of that course can be rewritten invariantly (that is "in differential geometry terms"). Unless otherwise stated, every vector calculus result from this section is taken directly from the AMATH 231 coursenotes [1], and every differential geometry result is taken from the differential geometry courses I've taken at Waterloo.

2.1 1 dimensional submanifolds

We will study 1 dimensional immersed submanifolds of $\mathbb{R}^n$ in the form of smooth curves $\gamma = \gamma(t)$ where $a \leq t \leq b$. Then γ is a smooth manifold with boundary of its own endpoints by the global smooth chart induced by its own parameterization, meaning γ is diffeomorphic to $[a, b]$ through the global chart $\phi = \gamma^{-1}$. The inclusion ι, which is an injective immersion as mentioned, makes γ an immersed submanifold of $\mathbb{R}^n$.

It should be noted here that technically γ is a manifold with boundary, and so if γ is not closed it cannot be covered with a single chart. However in what follows we will be concerned with integrals, and the boundary of a manifold always has one less dimension than the manifold (we say it has codimension 1), and so is a set of measure zero. Therefore the integral's value is not changed by the boundary, and so we will assume one chart is sufficient.

Recall that if ω is a top form on a manifold M with a global chart ϕ, then

$$\int_M \omega = \int_{\phi(M)} (\phi^{-1})^*(\omega)$$

by definition. Taking $M = \gamma, \phi = \gamma^{-1}$ as in the preceding paragraph we get

$$\int_\gamma \omega = \int_{\phi(\gamma)} (\phi^{-1})^*(\omega) = \int_{\gamma^{-1}(\gamma([a,b]))} \gamma^*\omega = \int_a^b (\gamma^*\omega)_t$$

where $(\gamma^*\omega)_t$ is the pullback of ω by γ, paramaterized by t.

Using the inclusion map, we can identify $\gamma(t)$ with $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)) \in \mathbb{R}^n$. Recall also that if $t_0 \in [a, b]$ then we define $\gamma'(t_0) = (\gamma_*)_{t_0}(\frac{\partial}{\partial t}\big|_{t_0})$. So the tangent spaces $T_{\gamma(t)}\gamma$ are completely described as scalar multiples of $\gamma'(t)$. Suppose f is a scalar field, then $\gamma'(t_0)f$ is

$$(\gamma_*)_{t_0}(\frac{\partial}{\partial t}\big|_{t_0})f = \frac{\partial}{\partial t}\big|_{t_0}(f(\gamma(t))) = \frac{\partial}{\partial t}\big|_{t_0}(f(\gamma_1(t), \dots, \gamma_n(t))) = \frac{\partial f}{\partial x_i}(\gamma(t_0))\frac{\partial \gamma_i}{\partial t}(t_0) = (\gamma_i'(t_0)\frac{\partial}{\partial x^i}\big|_{\gamma(t_0)})f$$

so $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$ which agrees with the vector calculus definition.

2.1.1 Arc Length

Definition 2.1.1.1. *The arclength of the curve $\gamma = \gamma(t)$ in $\mathbb{R}^n, a \leq t \leq b$ is*

$$\int_\gamma ds = \int_a^b \|\gamma'(t)\|\, dt$$

As with most definitions in vector calculus, this definition makes physical sense. If $\gamma(t)$ is the path followed by a particle, then $\gamma'(t)$ is the velocity of that particle, and so $\|\gamma'(t)\|$ is the speed of that particle at a given time t. Then arclength is the distance travelled, and we can interpret distance travelled as the integral of the speed with respect to time, which is exactly our definition.

Now in $\mathbb{R}^n$ the standard metric is $\bar{g} = (dx^1)^2 + \cdots + (dx^n)^2 = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$ where really this last term on the right is what we should always write, but the middle sum is often written as a light abuse of notation. From section 2 we know that we can think of γ as an immersed submanifold of $\mathbb{R}^n$ with metric $\iota^*(\bar{g})$, where $\iota : \gamma \to \mathbb{R}^n$ is the inclusion, an injective immersion preserving the orientation of the positive t direction being the "outside." Now we have

$$g_\gamma := \iota^*(\bar{g}) = \bar{g}\big|_\gamma = (dx^1)\big|_\gamma^2 + \cdots + (dx^n)\big|_\gamma^2$$

Now if μ is the volume form of γ, to find $\int_\gamma \mu_\gamma$, we need to find $(\gamma^*\mu)_t$ by definition of the integral. The manifold γ is diffeomorphic to $[a, b]$ through the global parameterization $\gamma(t)$. Write $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ in components, so that $x^i(\gamma(t)) = \gamma_i(t)$ on $\gamma, \forall i = 1, \ldots, n$. Therefore

$$d\left(x^i(\gamma(t))\right) = d\left(\gamma^i(t)\right) = \gamma_i'(t)\,dt$$

for each i. Therefore we have

$$\begin{aligned}
(\gamma^* g_\gamma)_t &= \gamma^*[(dx^1)\big|_\gamma^2 + \cdots + (dx^n)\big|_\gamma^2]_t \\
&= (dx^1(\gamma(t)))^2 + \cdots + (dx^n(\gamma(t)))^2 \\
&= (\gamma_1'(t))^2\,(dt)^2 + \cdots + (\gamma_n'(t))^2\,(dt)^2 \\
&= [(\gamma_1'(t))^2 + \cdots + (\gamma_n'(t))^2]\,(dt)^2 \\
&= \|\gamma'(t)\|^2\,(dt)^2
\end{aligned}$$

Now we can express g_γ in global coordinates as

$$g_\gamma = (\gamma^{-1})^*\gamma^* g_\gamma = (\gamma^{-1})^*\left(\|\gamma'(t)\|^2\,(dt)^2\right)$$

and $\gamma^{-1} = \phi$ by definition, so

$$g_\gamma = \|\gamma'(t \circ \phi)\|^2\,(d(t \circ \phi))^2$$

But recall that the Riemannian Volume form in local coordinates $y^1, \cdots, y^k$ for an oriented manifold (M^k, g) is given by $\sqrt{\det(g)}dy^1 \wedge \cdots \wedge dy^k$. In this case our manifold is γ with the metric g_γ. We have $\sqrt{\det(g_\gamma)} = \sqrt{g_\gamma}$ since $k = 1$, so the volume form of g_γ is

$$\mu(g_\gamma) = \sqrt{\|\gamma'(t \circ \phi)\|^2}\,d(t \circ \phi) = \|\gamma'(t \circ \phi)\|\,d(t \circ \phi)$$

But we want $\gamma^*(\mu(g_\gamma))$, and this is easy to calculate:

$$\gamma^*(\mu(g_\gamma)) = \gamma^*(\|\gamma'(t \circ \phi)\|\,d(t \circ \phi)) = \|\gamma'(t \circ \phi \circ \phi^{-1})\|\,d(t \circ \phi \circ \phi^{-1})) = \|\gamma'(t)\|dt$$

since $\gamma = \phi^{-1}$. Therefore we have shown that $(\gamma^*\mu)_t = \|\gamma'(t)\|dt$. But now we see that the arclength definition from vector calculus is just the integral over the manifold γ of the volume form μ of γ, so

$$\int_a^b \|\gamma'(t)\|\,dt = \int_a^b (\gamma^*\mu)_t = \int_\gamma \mu = vol(\gamma)$$

by definition of volume of a manifold. Intuitively, this is exactly what you would expect. So we have shown

$$\boxed{\int_\gamma ds = \int_\gamma \mu} \tag{1}$$

2.1.2 Line Integral of a Scalar Field

A scalar field is just a function defined on some subset of $\mathbb{R}^n$. This term is analogous to the definition of vector field, in that a vector field gives a vector at each point, and a scalar field gives a scalar at each point.

Definition 2.1.2.1. *Consider a curve $\gamma = \gamma(t), a \le t \le b$ in $\mathbb{R}^n$ which is at least C^1, and a scalar field f which is at least continuous on γ. Then the line integral of f along γ is defined to be*

$$\int_\gamma f\, ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\|\, dt$$

Physically, the line integral's meaning depends on f's physical meaning. If f is a mass density function, the line integral is the total mass of γ. If f is the height of fence whose base is on γ, then the line integral is the area of one side of the fence. There are many examples.

From the previous section we immediately recognize the volume form of γ in global coordinates $(\gamma^*\mu)_t = \|\gamma'(t)\|\, dt$. We can also write $f(\gamma(t)) = (\gamma^* f)_t$ Therefore

$$\int_\gamma f\, ds = \int_a^b f(\gamma(t))(\gamma^*\mu)_t = \int_a^b (\gamma^* f)_t(\gamma^*\mu)_t = \int_a^b (\gamma^* f \gamma^*\mu)_t = \int_a^b (\gamma^*[f\mu])_t = \int_\gamma f\mu$$

because $f\mu$ is a top form, so this last equality is just the definition of integration on the manifold γ, as we saw above. Note that if $f \equiv 1$ on γ then arclength is the result of taking the line integral of f. Therefore we have shown

$$\boxed{\int_\gamma f\, ds = \int_\gamma f\, \mu} \tag{2}$$

2.1.3 Line Integral of a Vector Field

Definition 2.1.3.1. *Consider a curve $\gamma = \gamma(t) = \mathbf{x}, a \le t \le b$ in $\mathbb{R}^n$ which is at least C^1, and a vector field $\mathbf{F}$ which is at least continuous on γ. Then the line integral of $\mathbf{F}$ along γ is defined to be*

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt$$

Physically, if we interpret $\mathbf{F}$ as a force field, then the line integral is the work done on a particle traversing γ in the direction of positive parameterization. It can also be thought of as a measure of the tendency for γ to align with $\mathbf{F}$ because by looking at the integrand, we can see that the value $\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = \|\mathbf{F}(\gamma(t))\| \|\gamma'(t)\| cos(\theta)$, where θ is the angle between the vectors, is greatest when $\theta = 0$. Therefore a positive result indicates a tendency to follow the field, and a negative result indicates a tendency to go against the field, but these are weighted sums remember.

Write $\mathbf{F} = (f^1, \cdots, f^n)$ where the f^i are continuous functions on $\mathbb{R}^n$. We also have $\gamma'(t) = (\gamma_1'(t), \cdots, \gamma_n'(t))$. Therefore

$$(\mathbf{F}(\gamma(t)) \cdot \gamma'(t))\, dt = f^i(\gamma(t))\gamma_i{}'(t)\, dt = [f^i(\gamma(t))\gamma_i{}'(t)]\, dt$$

Note to each vector $\mathbf{v}$ in $\mathbb{R}^n$ we can associate the map $\mathbf{w} \mapsto \mathbf{v} \cdot \mathbf{w}$ which is by definition a 1-form. Since $\mathbf{v} \cdot \mathbf{w} = \bar{g}(\mathbf{v}, \mathbf{w}) = (\mathbf{v})^\flat \mathbf{w}$ we see that this identification is just identifying $\mathbf{v}$ with $\mathbf{v}^\flat$. This was done all for a fixed vector $\mathbf{v}$, so for our vector field $\mathbf{F}$ we have $\mathbf{F}^\flat = \bar{g}_{ij} f^i dx^j = \sum_i f^i dx^i$ since $\bar{g}_{ij} = \delta_i^j$. This is why the identification between $\mathbf{F}$ and $\mathbf{F}^\flat$ is so often made in $\mathbb{R}^n$: you can use the same coefficient functions for both the vector field and its associated 1 form. However this is only valid if the metric is δ_i^j.

Now since $(\gamma^* \mathbf{F}^\flat)_t = \sum_i f^i(\gamma(t)) d\left(x^i(\gamma(t))\right) = [f^i(\gamma(t))\gamma_i{}'(t)]\, dt$ because $d\left(x^i(\gamma(t))\right) = \gamma_i'(t)\, dt$ we know that

$$(\mathbf{F}(\gamma(t)) \cdot \gamma'(t))\, dt = (\gamma^* \mathbf{F}^\flat)_t$$

and so

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b (\gamma^* \mathbf{F}^\flat)_t = \int_\gamma \mathbf{F}^\flat\big|_\gamma$$

This shows us that our definitions agree, giving us

$$\boxed{\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma \mathbf{F}^\flat\big|_\gamma} \tag{3}$$

2.1.4 First and Second Fundamental Theorems for Line Integrals

Fix $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{R}^n$. Recall that if γ is a curve joining $\mathbf{x}_0, \mathbf{x}_1$ then the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$ is said to be path independent if the value of this integral is independent of the curve γ joining $\mathbf{x}_0, \mathbf{x}_1$. An important point of interest in vector calculus is to determine when $\mathbf{F}$ has the property that its line integral between two points is path independent, largely because it makes calculation easier:

First Fundamental Theorem for Line Integrals 2.1.4.1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be open and connected. Let $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$ be a continuous vector field whose line integral is path independent in $\mathcal{U}$. Fix $\mathbf{x}_0 \in \mathcal{U}$. Define $\phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x}$, then $\nabla\phi = \mathbf{F}$ on $\mathcal{U}$*

Proof. (Sketch) We wish to show that $\frac{\partial \phi}{\partial x^i} = f^i$ where f^i is the ith component of $\mathbf{F}$. Since the integral ϕ is path independent, we are free to choose a path that traverses from $\mathbf{x}_0$ to $\mathbf{x}$ in such a way that only one of $\mathbf{x}$ is changing at a time. Then by the definition of the line integral and the Fundamental Theorem of Calculus we can write ϕ as a sum of integrals each of which are dependent on a single x^i, and the partial derivative of that integral with respect to that x^i is exactly f^i, and zero otherwise. □

Lets consider this theorem invariantly. It says that if $\phi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{x}$, then $\nabla\phi = \mathbf{F}$. Let $\gamma(\mathbf{x})$ be some curve joining $\mathbf{x}_0$ to $\mathbf{x}$, then $\gamma(\mathbf{x})$ is an immersed submanifold of $\mathbb{R}^n$ with the inclusion, and we can write $\phi(\mathbf{x}) = \int_{\gamma(\mathbf{x})} \mathbf{F}^\flat$. By definition $\nabla\phi = (d\phi)^\sharp$, so $\nabla\phi = \mathbf{F} \Leftrightarrow (d\phi)^\sharp = \mathbf{F} \Leftrightarrow df = \mathbf{F}^\flat$.

Now the theorem says that if $\phi(\mathbf{x}) = \int_{\gamma(\mathbf{x})} \mathbf{F}^\flat$ then $d\phi = \mathbf{F}^\flat$.

So really this theorem gives a way to construct a primitive of a differential form on a subset of $\mathbb{R}^n$. This theorem is about cohomology! Lets try the converse theorem.

Second Fundamental Theorem for Line Integrals 2.1.4.1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be open and connected. Let $\mathbf{F} : \mathcal{U} \to \mathbb{R}^n$ be a continuous vector field, and let $\mathbf{x}_1, \mathbf{x}_2$ be two fixed points in $\mathcal{U}$. If $\mathbf{F} = \nabla\phi$, where $\phi : \mathcal{U} \to \mathbb{R}^n$ is a C^1 scalar field, and γ is any curve in $\mathcal{U}$ joining $\mathbf{x}_1$ to $\mathbf{x}_2$, then*

$$\int_\gamma \mathbf{F} \cdot\, d\mathbf{x} = \phi(\mathbf{x}_2) - \phi(\mathbf{x}_1)$$

Proof. Chain rule and fundamental theorem of calculus □

Invariantly, if $\mathbf{F} = \nabla\phi$, then $\mathbf{F}^\flat = d\phi$, so using the fact that $\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma \mathbf{F}^\flat$ we proved above this theorem says:

$$\int_\gamma d\phi = \phi(\gamma(b)) - \phi(\gamma(a))$$

Which is a result that's usually proven right after integration of forms along curves is defined in a differential geometry class.

Putting these two results together we have that if $\mathbf{F}$ is a continuous vector field, then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{x} \text{ is path independent in } \mathbf{F}\text{'s domain} \Leftrightarrow \mathbf{F} = \nabla\phi \text{ for some scalar field } \phi \text{ on } \mathbf{F}\text{'s domain.}$$

We've also shown that this is equivalent to:

$$\int_\gamma \mathbf{F}^\flat \text{ is path independent in } \mathbf{F}^\flat\text{'s domain} \Leftrightarrow \mathbf{F}^\flat \text{ is exact on its domain}$$

If γ is a closed curve in $\mathbf{F}^\flat$'s domain, then we can pick fix two points t_1, t_2 on gamma and break γ into two smooth curves, γ_1 from t_1 to t_2, and γ_2 from t_2 to t_1. $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} = \int_{\gamma_1} - \int_{-\gamma_2}$ because reversing the orientation of γ_2 gives minus the integral. If we have path independence then since γ_1 and $-\gamma_2$ are both curves from t_1 to t_2, $\int_{\gamma_1} = \int_{-\gamma_2}$. Therefore $\int_\gamma = 0$

Conversely if $\int_\gamma \mathbf{F}^\flat = 0$ for every closed curve γ in $\mathbf{F}^\flat$'s domain, then given any two points $\mathbf{x}_1, \mathbf{x}_2$ in $\mathbf{F}^\flat$'s domain, take any two smooth curves γ_1 and γ_2 in $\mathbf{F}^\flat$'s domain which join $\mathbf{x}_1$ to $\mathbf{x}_2$ and which do not intersect. Let $\gamma = \gamma_1 \cup (-\gamma_2)$ is a simple closed curve, and so $0 = \int_\gamma = \int_{\gamma_1} + \int_{-\gamma_2} = \int_{\gamma_1} - \int_{\gamma_2} \Rightarrow \int_{\gamma_1} = \int_{\gamma_2}$. Note if the two chosen curves do intersect, the intersection points again form closed curves, and the argument can be repeated. We have therefore shown that:

$$\int_\gamma \mathbf{F}^\flat \text{ is path independent in } \mathbf{F}^\flat\text{'s domain}$$

$$\Updownarrow$$

$$\mathbf{F}^\flat \text{ is exact on its domain}$$

$$\Updownarrow$$

$$\int_\gamma \mathbf{F}^\flat = 0 \text{ for every closed curve } \gamma \text{ in } \mathbf{F}^\flat\text{'s domain}$$

2.2 2 dimensional submanifolds

We now begin our study of surfaces in $\mathbb{R}^n$. We will denote our surfaces by Σ and assume we can write $\Sigma = \mathbf{p}(u,v) = (p_1(u,v), p_2(u,v), p_3(u,v))$ where $\mathbf{p} : \mathcal{D}_{uv} \xrightarrow{\sim} \Sigma$ is a diffeomorphism and $\mathcal{D}_{uv} \subset \mathbb{R}^2$. So letting $\phi = \mathbf{p}^{-1}, \phi$ is a global chart for Σ. Therefore Σ is a 2 manifold with boundary, and the inclusion is an orientation preserving injective immersion making Σ an immersed submanifold of $\mathbb{R}^n$. As with the one dimensional case Σ is a manifold with boundary means one chart may not be sufficient, but for the purposes of integration one chart is sufficient.

Note the similarity of this construction with that of the one dimensional case. In particular we have that, in general, parameterizations are inverses of charts on the manifold, because charts are thought of as mapping from the manifold, and parameterizations are thought of as mapping to the manifold. By assuming a smooth parameterization we are also assuming that our surface can be oriented. We will assume, as is standard, that closed surfaces have an outward pointing normal, and that the boundary of surfaces with boundary have the orientation such that if one were walking along the boundary the side with the normal pointing up would be on your left. This means if we're looking at the side where the normal vector points up, the boundary is oriented counter clockwise.

Taking our chart ϕ and the manifold Σ, by definition we have, for a 2 form ω

$$\int_\Sigma \omega = \int_{\phi(\Sigma)} (\phi^{-1})^*(\omega) = \int_{\mathbf{p}^{-1}(\mathbf{p}(\mathcal{D}_{uv}))} \mathbf{p}^*\omega = \iint\limits_{\mathcal{D}_{uv}} (\mathbf{p}^*\omega)_{(u,v)}$$

We remember this fact for use below, and by analogy with our study of curves, we being with the area of a surface.

2.2.1 Surface Area

The definition of Surface Area does not require $\mathbf{p}$ to be smooth, only C^1.

Definition 2.2.1.1. *If $\Sigma = \mathbf{p}(u,v)$, $(u,v) \in \mathcal{D}_{uv}$ where $\mathbf{p}$ is C^1, then the surface area of Σ is defined as*

$$S(\Sigma) = \iint\limits_{\mathcal{D}_{uv}} \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du \, dv$$

Note that when we write $du\,dv$ in calculus we actually mean $du \wedge dv$.
Lets find the volume form for Σ, because we'll need it for the rest of the 2 dimensional case. First we find the induced metric on Σ, call it g_Σ, and pull it back to $\mathcal{D}_{uv}$. We know that

$$g_\Sigma = \bar{g}\big|_\Sigma = (dx^1)\big|_\Sigma^2 + (dx^2)\big|_\Sigma^2 + (dx^3)\big|_\Sigma^2$$

To find $(\mathbf{p}^* g_\Sigma)_{(u,v)}$, we use the fact that

$$(x, y, z) = (p_1(u,v), p_2(u,v), p_3(u,v))$$

on Σ, so $d(x^i(\mathbf{p}(u,v))) = \frac{\partial p_i}{\partial u} du + \frac{\partial p_i}{\partial v} dv$ for each i. Therefore we can pullback g_Σ:

$$(\mathbf{p}^* g_\Sigma)_{(u,v)} = \sum_i \left(\frac{\partial p_i}{\partial u}\, du + \frac{\partial p_i}{\partial v}\, dv\right)^2$$

but now

$$\begin{aligned}(\frac{\partial p_i}{\partial u}\, du + \frac{\partial p_i}{\partial v}\, dv)^2 &= (\frac{\partial p_i}{\partial u}\, du + \frac{\partial p_i}{\partial v}\, dv) \otimes (\frac{\partial p_i}{\partial u}\, du + \frac{\partial p_i}{\partial v}\, dv) \\ &= \frac{\partial p_i}{\partial u}^2 (du)^2 + 2\frac{\partial p_i}{\partial u}\frac{\partial p_i}{\partial v}\, du \otimes dv + \frac{\partial p_i}{\partial v}^2 (dv)^2\end{aligned}$$

so summing over i we have

$$(\mathbf{p}^* g_\Sigma)_{(u,v)} = (\sum_i \frac{\partial p_i}{\partial u}^2)(du)^2 + 2(\sum_i \frac{\partial p_i}{\partial u}\frac{\partial p_i}{\partial v})\, du \otimes dv + (\sum_i \frac{\partial p_i}{\partial v}^2)(dv)^2$$

Now we can write

$$\begin{aligned}g_\Sigma &= (\mathbf{p}^{-1})^* \mathbf{p}^* g_\Sigma \\ &= \phi^* \mathbf{p}^* g_\Sigma \\ &= \phi^* \left((\sum_i \frac{\partial p_i}{\partial u}^2)(du)^2 + 2(\sum_i \frac{\partial p_i}{\partial u}\frac{\partial p_i}{\partial v})\, du \otimes dv + (\sum_i \frac{\partial p_i}{\partial v}^2)(dv)^2 \right) \\ &= (\sum_i [\frac{\partial p_i}{\partial u} \circ \phi]^2)(d(u\circ\phi))^2 + 2(\sum_i \frac{\partial p_i}{\partial u}\frac{\partial p_i}{\partial v} \circ \phi)\, d(u\circ\phi) \otimes d(v\circ\phi) + (\sum_i [\frac{\partial p_i}{\partial v} \circ \phi]^2)(d(v\circ\phi))^2\end{aligned}$$

and now to find the volume form we need to take the determinant. Suppressing the "$\circ\phi$" that's in all the terms, taking the determinant gives:

$$\begin{aligned}\det(g_\Sigma) &= (\sum_i \frac{\partial p_i}{\partial u}^2)(\sum_i \frac{\partial p_i}{\partial v}^2) - (\sum_i \frac{\partial p_i}{\partial u}\frac{\partial p_i}{\partial v})^2 \\ &= (\frac{\partial p_1}{\partial u}\frac{\partial p_2}{\partial v} - \frac{\partial p_2}{\partial u}\frac{\partial p_1}{\partial v})^2 + (\frac{\partial p_1}{\partial u}\frac{\partial p_3}{\partial v} - \frac{\partial p_3}{\partial u}\frac{\partial p_1}{\partial v})^2 + (\frac{\partial p_2}{\partial u}\frac{\partial p_3}{\partial v} - \frac{\partial p_3}{\partial u}\frac{\partial p_2}{\partial v})^2\end{aligned}$$

where this last equality follows by expanding everything out, collecting terms, and recognizing the squared terms. We also recognize that this last line is in fact $(\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}) \cdot (\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}) = \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\|^2$, so that writing in the "$\circ\phi$" that's been in every term all along we have

$$\det(g_\Sigma) = \left(\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \circ \phi \right)^2$$

Now we can define $\mu(g_\Sigma)$, the volume form for Σ with the metric g_Σ by

$$\mu(g_\Sigma) = \sqrt{\det(g_\Sigma)}\, d(u\circ\phi) \wedge d(v\circ\phi) = \left(\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \circ \phi \right) d(u\circ\phi) \wedge d(v\circ\phi)$$

But now we have

$$\begin{aligned}\mathbf{p}^*\mu(g_\Sigma) &= (\phi^{-1})^*\left(\left(\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \circ \phi\right) d(u \circ \phi) \wedge d(v \circ \phi)\right)\\ &= \left(\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \circ \phi \circ \phi^{-1}\right) d(u \circ \phi \circ \phi^{-1}) \wedge d(v \circ \phi \circ \phi^{-1})\\ &= \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du \wedge dv\end{aligned}$$

Now if we suppress the dependence on the metric, and express the dependence on u, v, we have shown that

$$(\mathbf{p}^*\mu)_{(u,v)} = \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du \wedge dv$$

And now we can express the surface area of Σ as

$$S(\Sigma) = \iint\limits_{\mathcal{D}_{uv}} \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du\, dv = \iint\limits_{\mathcal{D}_{uv}} (\mathbf{p}^*\mu)_{(u,v)} = \int\limits_{\Sigma} \mu = vol(\Sigma)$$

So again, this is exactly what we would expect.

2.2.2 Surface Integral of a Scalar Field

Definition 2.2.2.1. *If $\Sigma = \mathbf{p}(u, v)$, $(u, v) \in \mathcal{D}_{uv}$ where $\mathbf{p}$ is C^1, and if f is a continuous scalar field on Σ, then the surface integral of f over Σ is defined as*

$$\iint\limits_{\Sigma} f \, dS = \iint\limits_{\mathcal{D}_{uv}} f(\mathbf{p}(u, v))\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du\, dv$$

Physically, f will often represent the surface density of a physical quantity, and so the surface integral of f over Σ gives the total mass of that physical quantity on the surface Σ.

We see that little has changed from the case of surface area here. We know from the previous section that $(\mathbf{p}^*\mu)_{(u,v)} = \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du \wedge dv$, and we have

$$f(\mathbf{p}(u, v)) = (\mathbf{p}^* f)_{(u,v)}$$

so we have

$$\iint\limits_{\Sigma} f \, dS = \iint\limits_{\mathcal{D}_{uv}} f(\mathbf{p}(u, v))\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| \, du\, dv = \iint\limits_{\mathcal{D}_{uv}} (\mathbf{p}^* f)_{(u,v)}(\mathbf{p}^*\mu)_{(u,v)} = \iint\limits_{\mathcal{D}_{uv}} (\mathbf{p}^*[f\mu])_{(u,v)} = \int\limits_{\Sigma} f\mu$$

Therefore we have shown

$$\boxed{\iint\limits_{\Sigma} f \, dS = \int\limits_{\Sigma} f\mu} \tag{4}$$

2.2.3 Surface Integral of a Vector Field

Definition 2.2.3.1. *Let* $\Sigma = \mathbf{p}(u,v)$, $(u,v) \in \mathcal{D}_{uv}$ *be an oriented surface with unit normal* $\mathbf{n}$. *If* $\mathbf{p}$ *is* C^1, *and if* $\mathbf{F}$ *is a continuous vector field on* Σ, *then the surface integral of* $\mathbf{f}$ *over* Σ *is defined as*

$$\iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{\mathcal{D}_{uv}} \mathbf{F}(\mathbf{p}(u,v)) \cdot (\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v})\, du\, dv$$

Physically, if $\mathbf{F}$ is a force field of some kind, then the surface integral gives the net flux through Σ caused by $\mathbf{F}$, where flux is "stuff per unit time"

$\mathbf{F}(\mathbf{p}(u,v)) \cdot (\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v})\, du \wedge dv$ is

$$\mathbf{F}(p_1,p_2,p_3) \cdot (\frac{\partial p_2}{\partial u}\frac{\partial p_3}{\partial v} - \frac{\partial p_3}{\partial u}\frac{\partial p_2}{\partial v}, \frac{\partial p_1}{\partial u}\frac{\partial p_3}{\partial v} - \frac{\partial p_3}{\partial u}\frac{\partial p_1}{\partial v}, \frac{\partial p_1}{\partial u}\frac{\partial p_2}{\partial v} - \frac{\partial p_2}{\partial u}\frac{\partial p_1}{\partial v})\, du \wedge dv$$

Lets compute $*(\mathbf{F}^\flat)$:

$$*(\mathbf{F}^\flat) = *(f_i\, dx^i) = f_i * (dx^i) = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy$$

Lets pull back this form by $\mathbf{p}(u,v)$. Recall from the previous section that

$$d(x^i(\mathbf{p}(u,v))) = \frac{\partial p_i}{\partial u}\, du + \frac{\partial p_i}{\partial v}\, dv$$

for each i. Lets pull back $f_1 dy \wedge dz$ using this formula

$$\begin{aligned}(\mathbf{p}^*(f_1 dy \wedge dz))_{(u,v)} &= f_1(\mathbf{p}(u,v)) d(y(\mathbf{p}(u,v))) \wedge d(z(\mathbf{p}(u,v))) \\ &= f_1(\mathbf{p}(u,v))(\frac{\partial p_2}{\partial u}\, du + \frac{\partial p_2}{\partial v}\, dv) \wedge (\frac{\partial p_3}{\partial u}\, du + \frac{\partial p_3}{\partial v}\, dv) \\ &= f_1(\mathbf{p}(u,v))(\frac{\partial p_2}{\partial u}\frac{\partial p_3}{\partial v} - \frac{\partial p_3}{\partial u}\frac{\partial p_2}{\partial v}) du \wedge dv\end{aligned}$$

which is the first term in $\mathbf{F}(\mathbf{p}(u,v)) \cdot (\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v})\, du \wedge dv$. Pulling back the other two terms in $*(\mathbf{F}^\flat)$ gives the second and third terms respectively. Putting it all together by linearity we have that

$$\mathbf{F}(\mathbf{p}(u,v)) \cdot (\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v})\, du \wedge dv = (\mathbf{p}^*[*(\mathbf{F}^\flat)])_{(u,v)}$$

Therefore

$$\iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \mathbf{F}(\mathbf{p}(u,v)) \cdot (\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v})\, du \wedge dv = \iint\limits_{\mathcal{D}_{uv}} = (\mathbf{p}^*[*(\mathbf{F}^\flat)])_{(u,v)} = \int\limits_{\Sigma} *(\mathbf{F}^\flat)$$

So now we have shown that

$$\boxed{\iint\limits_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \int\limits_{\Sigma} *(\mathbf{F}^\flat)} \tag{5}$$

2.3 Grad, Div, Curl, and the Laplacian

Before we discuss the major theorems from AMATH 231, we must show the equivalence of some concepts from vector calculus with definitions from differential geometry.

2.3.1 The Gradient of a Scalar Field

We define the vector differential operator ∇ as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

and define the action of ∇ on a scalar field f as $\nabla(f) = \nabla f$ the gradient of f.

Recall that in differential geometry the gradient of f is defined to be the vector field $(df)^\sharp$. In local coordinates, then,

$$\nabla f = (df)^\sharp = (\frac{\partial f}{\partial x^i}\, dx^i)^\sharp = (g^{ij}\frac{\partial f}{\partial x^i})\frac{\partial}{\partial x^j}$$

However we're in the case $(\mathbb{R}^3, \bar{g})$, so we have global coordinates, so we can write

$$\nabla f = \delta_i^j \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

so the two definitions agree.

Physically, the gradient of a scalar field is a vector field that points in the direction of greatest increase of that function.

2.3.2 The Divergence of a Vector Field

We define the divergence of a vector field by using the symbol $\nabla\cdot$ to be the operator that acts on a vector field $\mathbf{F}$ as

$$\nabla \cdot \mathbf{F} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (f_1, f_2, f_3) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

where this last equality is by analogy with the operation of the inner product on $\mathbb{R}^3$.

Recall that in differential geometry if (M, μ) is a manifold with volume form, we define the divergence $div : \Gamma(TM) \to C^\infty(M)$ by $div(\mathbf{F})\mu = d(\mathbf{F} \lrcorner\, \mu)$ and that the formula for divergence of a vector field $\mathbf{F}$ in local coordinates is

$$\text{div}(\mathbf{F}) = \frac{\partial f^k}{\partial x^k} + f^k \frac{\partial}{\partial x^k}(log\sqrt{\det(g)})$$

But in $(\mathbb{R}^3, \bar{g})$, we have global coordinates and $\bar{g} = \delta_i^j$, so $log\sqrt{\det(\bar{g})} = log\sqrt{1} = 0$, and therefore

$$\nabla \cdot \mathbf{F} = \frac{\partial f^1}{\partial x} + \frac{\partial f^2}{\partial y} + \frac{\partial f^3}{\partial z} = \text{div}(\mathbf{F})$$

so again the two formulas agree, but we can do better:

Lemma 2.3.2.1. *If* $\mathbf{F} = (f_1, f_2, f_3)$ *is a vector field in* $\mathbb{R}^3$*, then*

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = *d(*\mathbf{F}^\flat)$$

Proof.

$$*\mathbf{F}^\flat = *(f_i dx^i) = f_i * (dx^i) = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy$$

by linearity of $*$. So now

$$d(*\mathbf{F}^\flat) = (\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z})\, dx \wedge dy \wedge dz$$

but now $*dx \wedge dy \wedge dz = 1$, so we have our result. □

Physically, the divergence measures the tendency of a vector field to diverge from a given point.

2.3.3 The Curl of a Vector Field

We also define $\nabla \times \mathbf{F}$ to be the vector field

$$\nabla \times \mathbf{F} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (f_1, f_2, f_3) = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})$$

again by analogy with the cross product.

We want to write this invariantly, but the cross product is not usually defined in a differential geometry course, and this is just an analogy anyway. Again we can do better:

Lemma 2.3.3.1. *If* $\mathbf{F} = (f_1, f_2, f_3)$ *is a vector field in* $\mathbb{R}^3$*, then*

$$curl(\mathbf{F}) = \nabla \times \mathbf{F} = [*(d(\mathbf{F}^\flat))]^\sharp$$

Proof.

$$d(\mathbf{F}^\flat) = d(f_i dx^i) = (\frac{\partial f_i}{\partial x}\, dx + \frac{\partial f_i}{\partial y}\, dy + \frac{\partial f_i}{\partial z}\, dz) \wedge dx^i$$

$$= \frac{\partial f_1}{\partial y}\, dy \wedge dx + \frac{\partial f_1}{\partial z}\, dz \wedge dx + \frac{\partial f_2}{\partial x}\, dx \wedge dy + \frac{\partial f_2}{\partial z}\, dz \wedge dy + \frac{\partial f_3}{\partial x}\, dx \wedge dz + \frac{\partial f_3}{\partial y}\, dy \wedge dz$$

$$= (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})\, dx \wedge dy + (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})\, dy \wedge dz + (\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z})\, dx \wedge dz$$

now we can write

$$*d(\mathbf{F}^\flat) = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})\, dx + (\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x})\, dy + (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})\, dz$$

and now taking the sharp gives the result. □

Notice here we could define another operator

$$\begin{aligned} \text{curl} : \Omega^1(\mathbb{R}^3) &\to \Omega^1(\mathbb{R}^3) \\ \alpha = \alpha_i dx^i &\mapsto *d\alpha = *\left(\frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i\right) \end{aligned}$$

Where we're simply using the calculation we did above in the proof, rewriting the second line in the proof in summation notation.

In this definition we've seen the first case of the use of the cross product so far. Everything else we've encountered has meant something in differential geometry terms. The cross product is only defined in $\mathbb{R}^3$ and $\mathbb{R}^7$ as we shall see. We need a lemma for the 3 dimensional case:

Lemma 2.3.3.1. *If X and Y are two vector fields in $\mathbb{R}^3$, then $X \times Y = (*(X^\flat \wedge Y^\flat))^\sharp$*

Proof. Let $X = \sum_i X_i \frac{\partial}{\partial x^i}$ and $Y = \sum_j Y_j \frac{\partial}{\partial x^j}$. As we've seen before, in $\mathbb{R}^n$ this means that $X^\flat = X_i\, dx^i, Y^\flat = Y_j\, dx^j$. We're in $\mathbb{R}^3$, so we have, by definition,

$$X \times Y = (X_2 Y_3 - X_3 Y_2\, , X_3 Y_1 \, - \, X_1 Y_3,\, X_1 Y_2 - X_2 Y_1)$$

We also have

$$X^\flat \wedge Y^\flat = X_i\, dx^i \wedge Y_j\, dx^j = X_i\, Y_j\, dx^i \wedge dx^j$$

Now take $*$, $*(X_i\, Y_j\, dx^i \wedge dx^j) = X_i\, Y_j\, *(dx^i \wedge dx^j)$ by linearity of $*$. Recall that $*^2 = (-1)^{k(3-k)} Id$ on a k form, so on 1 and 2 forms in $\mathbb{R}^3, *^2 = Id$ If $i = j, dx^i \wedge dx^j = 0$ and if $i \neq j$ we have, using $*^2 = Id$:

- $*(dx \wedge dy) = dz$
- $*(dz \wedge dx) = dy$
- $*(dy \wedge dz) = dx$

Therefore

$$\begin{aligned} *(X_i\, Y_j\, dx^i \wedge dx^j) &= X_1\, Y_2\, dz - X_1\, Y_3\, dy + X_2\, Y_3\, dx - X_2\, Y_1\, dz + X_3\, Y_1\, dy - X_3\, Y_2\, dx \\ &= (X_2\, Y_3 \, - X_3\, Y_2\,)dx + (X_3\, Y_1 \, - X_1\, Y_3\,)dy + (X_1\, Y_2 \, - X_2\, Y_1\,)dz \end{aligned}$$

Now take the sharp of this to get the result □

Note: We will write $*(X^\flat \wedge Y^\flat)^\sharp$ for $(*(X^\flat \wedge Y^\flat))^\sharp$. A common abuse of notation is to write things like $X \wedge Y$ for $X^\flat \wedge Y^\flat$ because the identification of vector fields and one forms is so common, and straightforward, in $\mathbb{R}^3$.

2.3.4 Properties of Grad, Div, and Curl

All three of these operations are linear, and all three have an associated product rule with a scalar function. Suppose $\mathbf{F}, \mathbf{G}$ are vector fields, and suppose f, g are scalar fields, then

$$\nabla(fg) = g\nabla f + f\nabla g$$
$$div(f\mathbf{F}) = f div(\mathbf{F}) + \nabla f \cdot \mathbf{F}$$
$$curl(f\mathbf{F}) = f curl(\mathbf{F}) + \nabla f \times \mathbf{F}$$

along with "zero identities" such as

$$curl(\nabla f) = \mathbf{0}$$
$$div(curl(\mathbf{F})) = 0$$

We leave the product rules for an exercise. Note that since we proved equivalence with our invariant formulas above and these are formulas in $\mathbb{R}^3$, we are free to use either notation to prove the results, but lets prove the zero identities with our invariant formulas.

$$curl(\nabla f) = curl(((df)^\sharp)) = [*(d(((df)^\sharp)^\flat))]^\sharp = [*(d^2 f)]^\sharp = 0$$

since both $\sharp$ and $*$ are linear isomorphisms. Similarly

$$div(curl(\mathbf{F})) = div([*(d\mathbf{F}^\flat)]^\sharp) = *d*[([*(d\mathbf{F}^\flat)]^\sharp)^\flat] = *d*^2(d\mathbf{F}^\flat)$$

but $d\mathbf{F}^\flat$ is a 2 form, and $*^2 = Id$, so we have d^2 again, so this is zero.

There are many, many vector calculus identities, seemingly all of which can be proved by simply cranking it out. For this reason we will mention them when we need them, but not prove all of them.

2.3.5 The Laplacian

The Laplacian is defined as

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and can act on both scalar and vector fields. Notice that as it acts on functions $\nabla^2 = -\Delta$ where Δ is the Laplacian defined in differential geometry. We have the following identities:

$$\nabla(\nabla^2 f) = \nabla^2(\nabla f)$$
$$\nabla \cdot (\nabla^2 \mathbf{F}) = \nabla^2(\nabla \cdot \mathbf{F})$$
$$\nabla \times (\nabla^2 \mathbf{F}) = \nabla^2(\nabla \times \mathbf{F})$$

and we can rewrite this invariantly:

Lemma 2.3.5.1. *If f is a scalar field in $\mathbb{R}^3$, then*

$$\nabla^2 f = *d*df$$

Proof. This is pure application of the definitions:

$$\begin{aligned}
*d*df &= *d*(\frac{\partial f}{\partial x^i}dx^i) \\
&= *d(\frac{\partial f}{\partial x}dy \wedge dz + \frac{\partial f}{\partial y}dz \wedge dx + \frac{\partial f}{\partial z}dx \wedge dy) \\
&= *((\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2})dx \wedge dy \wedge dz) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

□

2.4 Stokes' Theorem in Vector Calculus

We will see that the meat of the theoretical component of AMATH 231 comes from special cases of Stokes' Theorem. Now that we've laid the groundwork, we finish our review of vector calculus 1 with the major theorems presented in that course. The proofs of these theorems given in that course are proofs of special cases, so we do not repeat them here, because we will see that the only special case we need is that of a smooth boundary.

2.4.1 Gauss' Theorem

Gauss' Theorem 2.4.1.1. *Let Ω be a bounded subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a single piecewise smooth oriented closed surface. If $\mathbf{F}$ is of class C^1 on $\Omega \cup \partial\Omega$, then*

$$\iiint\limits_{\Omega} \nabla \cdot \mathbf{F}\, dV = \iint\limits_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega$.

Proof. Assume the boundary is smooth, and note that we can also assume $\Omega \supset \partial\Omega$ since the boundary is a zero set so the integral is unaffected. This means Ω is a closed and bounded subset of $\mathbb{R}^3$ and so compact by Heine-Borel, which means any form on Ω is compactly supported.

On the left side of the equality in the integrand we have

$$\nabla \cdot \mathbf{F}\, dV = div(\mathbf{F}) dx \wedge dy \wedge dz$$

because on $\mathbb{R}^3$ the volume form is $dx \wedge dy \wedge dz$, so Ω has the same volume form restricted to it. By an earlier identity we proved

$$div(\mathbf{F}) dx \wedge dy \wedge dz = *d(*\mathbf{F}^\flat)(dx \wedge dy \wedge dz).$$

but $*(dx \wedge dy \wedge dz) = Id$, and $*^2 = Id$ so we can write

$$div(\mathbf{F}) dx \wedge dy \wedge dz = *^2 d(*\mathbf{F}^\flat) * (dx \wedge dy \wedge dz) = d(*\mathbf{F}^\flat)$$

On the right side, by work above we know

$$\iint\limits_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS = \int\limits_{\partial\Omega} *(\mathbf{F}^\flat)$$

Therefore this theorem says

$$\int\limits_{\Omega} d(*\mathbf{F}^\flat) = \int\limits_{\partial\Omega} *\mathbf{F}^\flat$$

viewing both Ω and $\partial\Omega$ as manifolds. But since Ω is orientable and compact, this is exactly Stokes' Theorem. □

This theorem gives a little more insight into divergence. Suppose A is some physical quantity in Ω such as charge, gas, etc., whose flux density is given by $\mathbf{F}$. This thoerem then says that the rate at which the amount of A in Ω is decreasing is equal the rate at which A is leaving Ω through $\partial\Omega$.

Gauss' Theorem is also known as the Divergence Theorem, in both differential geometry and vector calculus.

2.4.2 The Generalized Divergence Theorem

Generalized Divergence Theorem 2.4.2.1. *Let Ω be a bounded subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a single piecewise smooth oriented closed surface. If $\mathbf{F}$ is of class C^1 on $\Omega \cup \partial\Omega$ except at a point $\mathbf{a} \in \Omega$, then take a sufficiently small ball H centered at $\mathbf{a}$. We can still apply Gauss' Theorem in a modified form:*

$$\iint\limits_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\, dS = \iiint\limits_{\Omega - H} \nabla\cdot\mathbf{F}\, dV + \iint\limits_{\partial H} \mathbf{F}\cdot\mathbf{n}\, dS$$

where $\mathbf{n}$ is the outward normal pointing outward on $\partial\Omega$ and ∂H.

Proof. Consider $\Omega - H$ satisfies Gauss' Theorem with boundary $\partial\Omega \cup \partial H$. If we take the unit normal to point outward on all of the boundary, this reverses the direction of the normal on ∂H, which negates the integral. Therefore by Gauss' Theorem

$$\iiint\limits_{\Omega - H} \nabla\cdot\mathbf{F}\, dV = \iint\limits_{\partial\Omega\cup\partial H} \mathbf{F}\cdot\mathbf{n}\, dS = \iint\limits_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\, dS - \iint\limits_{\partial H} \mathbf{F}\cdot\mathbf{n}\, dS$$

□

It should be noted there is also a generalized divergence theorem for tensors, and that this is not it.

2.4.3 Green's Theorem

Green's Theorem 2.4.3.1. *Let $\mathcal{D}$ be a bounded subset of $\mathbb{R}^2$ (with an interior). Suppose $\partial\mathcal{D}$ is piecewise C^1, a simple closed curve oriented counter-clockwise. Let $\mathbf{F} = (f^1, f^2)$ be a C^1 vector field on $\mathcal{D}$, then*

$$\int_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{x} = \iint\limits_{D} (\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y})\, dx\, dy$$

Proof. Notice here that $\mathcal{D}$ is a 2 dimensional immersed submanifold of $\mathbb{R}^2$. Let μ be its volume form. The metric on $\mathcal{D}$ is $g = \bar{g}\big|_{\mathcal{D}} = (dx)^2 + (dy)^2$, so then $\mu = \sqrt{\det(g)}\, dx \wedge dy$ identifying x and y as the same coordinates on $\mathcal{D}$ under inclusion. However $\bar{g} = I_{2x2}$ as a matrix, so $\sqrt{\det(g)} = 1$ Therefore $\mu = dx \wedge dy$. As a standard abuse of notation, $dx \wedge dy$ is written as $dx\, dy$ in double integrals, with a similar abuse for triple integrals.

Using what we've shown so far, we can write Green's Theorem in terms of integrals over manifolds:

$$\int_{\partial\mathcal{D}} \mathbf{F}^\flat = \int_D (\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y})\, \mu$$

But now $\mathbf{F}^\flat = f^1 dx + f^2 dy$, so

$$d\mathbf{F}^\flat = df^1 \wedge dx + df^2 \wedge dy = \frac{\partial f^1}{\partial y} dy \wedge dx + \frac{\partial f^2}{\partial x} dx \wedge dy = (\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y}) dx \wedge dy = (\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y})\, \mu$$

Therefore Green's Theorem says

$$\int_{\partial\mathcal{D}} \mathbf{F}^\flat = \int_D (\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y})\, \mu = \int_D d\mathbf{F}^\flat$$

But now since a condition on $\mathcal{D}$ was that it was closed and bounded in $\mathbb{R}^2$ by Heine-Borel $\mathcal{D}$ is compact, and so $(\mathbf{F}\big|_{\mathcal{D}})^\flat$ is compactly supported. Therefore Green's Theorem is just Stokes' Theorem in 2 dimensions.

□

The quantity $\frac{\partial f^2}{\partial x} - \frac{\partial f^1}{\partial y}$ is often defined as the vorticity of $\mathbf{F} = (f^1, f^2)$ when $\mathbf{F}$ is the velocity field of a fluid flow in 2 dimensions, meaning a fluid flow in 3 dimensions where one of the dimensions does not change. The vorticity in some sense describes the tendency of the vector field to rotate something dropped in the fluid.

Therefore Green's Theorem says that the work done by a vector field $\mathbf{F}$ on a particle traversing the boundary of a region in $\mathbb{R}^2$ is equal to the sum over the whole region of all the infinitesimal measures of the tendency of $\mathbf{F}$ to rotate (infinitesimally small) objects. This gives us a glimpse into what the exterior derivative does to 1 forms.

2.4.4 Stokes' Theorem

Recall that

$$curl(\mathbf{F}) = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})$$

and that the vorticity of $\mathbf{F}$ is $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$, so we see that the vorticity is the third component of the curl. Letting $\mathbf{k} = (0, 0, 1)$ we can write this as

$$curl(\mathbf{F}) \cdot \mathbf{k} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$$

Green's Theorem says

$$\int_{\partial\mathcal{D}} \mathbf{G} \cdot d\mathbf{x} = \iint\limits_D (\frac{\partial g^2}{\partial x} - \frac{\partial g^1}{\partial y})\, dx\, dy$$

for the planar region $\mathcal{D}$. We can think of $\mathcal{D}$ as a surface in $\mathbb{R}^3$ with unit normal $\mathbf{n} = \mathbf{k}$. Now think of $\mathbf{G}$ as a vector field in $\mathbb{R}^3$ by writing $\mathbf{G} = (g^1, g^2, 0)$. Now we can rewrite Green's Theorem as

$$\int_{\partial\mathcal{D}} \mathbf{G}\cdot d\mathbf{x} = \iint\limits_{D} curl(\mathbf{G})\cdot \mathbf{n}\, dS$$

because we know the volume form of $\mathcal{D}$ in terms of it's parameterization is

$$\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\|\, du \wedge dv$$

However $\mathcal{D}$ is in the $x-y$ plane, so here $\mathbf{p}(x,y) = (x,y,0)$, so that $x = u, y = v$ and

$$\|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\| = \|(1,0,0)\times(0,0,1)\| = \|(0,0,1)\| = 1$$

and so $dS = dx \wedge dy$. This is the heuristic justification of the statement of Stokes' Theorem from Vector Calculus.

Stokes' Theorem (Vector Calculus) 2.4.4.1. *Let Σ be a piecewise C^1 orientable surface, with $\partial\Sigma$ a simple piecewise C^1 closed curve. If $\mathbf{F}$ is of class C^1 on some open set in $\mathbb{R}^3$ containing $\Sigma \cup \partial\Sigma$, then*

$$\iint\limits_{\Sigma} (\nabla \times \mathbf{F})\cdot \mathbf{n} dS = \int\limits_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{x}$$

where the boundary is oriented counter-clockwise when viewed from the side on which the unit normal points.

Proof. Note that the requirement that the boundary be simple means the boundary is a manifold, and since the boundary is closed, we may assume that it is included in Σ because we're dealing with integrals, so that in fact Σ is compact. We will assume that in fact the boundary is smooth.

We know that for a vector field $\mathbf{G}$ we have

$$\iint\limits_{\Sigma} \mathbf{G}\cdot \mathbf{n}\, dS = \int\limits_{\Sigma} *(\mathbf{G}^\flat)$$

so taking $\mathbf{G} = \nabla \times \mathbf{F} = [*(d\mathbf{F}^\flat)]^\sharp$ we have

$$\iint\limits_{\Sigma} (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS = \int\limits_{\Sigma} *(\mathbf{G}^\flat) = \int\limits_{\Sigma} *(([*(d\mathbf{F}^\flat)]^\sharp)^\flat) = \int\limits_{\Sigma} *^2 d\mathbf{F}^\flat) = \int\limits_{\Sigma} d\mathbf{F}^\flat$$

and on the right side of the equation we have

$$\int\limits_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{x} = \int\limits_{\partial\Sigma} \mathbf{F}^\flat$$

so the vector calculus version of Stokes' Theorem says

$$\int\limits_{\Sigma} d\mathbf{F}^\flat = \int\limits_{\partial\Sigma} \mathbf{F}^\flat$$

which is exactly Stokes' Theorem from differential geometry.

□

2.5 Summary of AMATH 231

We have seen that with the proper natural identifications and conventions, that all the notation and theory from AMATH 231 agrees with that of standard differential geometry notation and theory.

- Integration of immersed submanifolds with a global parameterization is carried out through the equations $\int_\gamma \omega = \int_a^b \omega(\gamma(t))$ and $\int\limits_\Sigma \omega = \iint\limits_{\mathcal{D}_{uv}} \omega(\mathbf{p}(u,v))$.
- The volume forms of 1 and 2 dimensional immersed submanifolds are given in global coordinates by $\mu(\gamma(t)) = \|\gamma'(t)\|\,dt$ and $\mu(\mathbf{p}(u,v)) = \|\frac{\partial \mathbf{p}}{\partial u} \times \frac{\partial \mathbf{p}}{\partial v}\|\,du \wedge dv$ respectively.
- The line integral of a scalar field is given by $\int_\gamma f\,ds = \int_\gamma f\mu$ where μ is the volume form of γ. With $f = 1$ we get $vol(\gamma)$ as a manifold.
- The line integral of a vector field is given by $\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_\gamma \mathbf{F}^\flat\big|_\gamma$.

3 AMATH 361

In this section we survey the course notes for AMATH 361 in the same way we did AMATH 231, rewriting definitions and theorems invariantly. Unless otherwise stated, every vector calculus result from this section is taken directly from the AMATH 361 coursenotes for winter 2014 [2], and every differential geometry result is taken from the differential geometry courses I've taken at Waterloo. We will not cover the entirety of [2] because we specifically need the results for fluid dynamics. We begin by justifying our models.

3.1 What is Continuum Mechanics?

Continuum mechanics is concerns with the deformation of matter at scales large compared to intermolecular distances. Typical problems studied include fluid flow and deformation of solids. We will be more concerned with fluids as we make our way towards the Navier-Stokes equations. A fluid is matter which tends to take the shape of the container it's in. Fluids cannot withstand shearing forces and so deform as long as those forces act on them. In contrast solids will deform to a certain degree but then the internal forces that hold the solid together will balance the shear and the deformation will stop. Moreover if the shearing force stops, the solid has a tendency to revert to its previous shape. Liquids do not have this property. For this reason sometimes we say that a fluid has no memory.

My mother is not a mathematician, and when I said that I would be studying fluid mechanics this summer she asked "how is it possible to use math to study fluids?" This is a totally valid question. How would you answer it?

The naive idea is that if we knew the position and velocity of every particle in a fluid we could in theory determine the fluid's flow over time. By the Heisenberg uncertainty principle this is impossible, and besides, such a model would be hopelessly complicated. This is an example where simplifications we assume actually lead us to a better qualitative understanding of a physical model. There's a balance to be struck here: we want a model complex enough to give us a good approximation of the qualities of fluids we wish to study, but simple enough that we can in some sense solve for what we want.

3.1.1 The Continuum Hypothesis

What are some of the fluid flows we might wish to study? Examples include: atmospheric circulation, ocean circulation, surface waves, flow around an object or through a pipe, heating and air conditioning systems, flows of water in rivers, flows of liquid hot magma, blood flow, and even flows of rock in the Earth's mantle over very long time scales. All the models used to study these fluid flows are related to the Navier-Stokes equations, whose derivation is the goal of the first portion of these notes.

All of the problems we just mentioned consider phenomena that occur at scales much larger than intermolecular distances. That is we are not concerned with how molecules move, but how the continuous matter made up of massive aggregates of molecules move. For example a thimble of water contains on the order of 10^{22} molecules of water, and we're studying flows that include more than a thimble full. In continuum mechanics we ignore the molecular structure of matter and treat the matter as continuous, meaning made up of infinitely small pieces. This simplification allows us to pose problems we can actually solve.

We call this assumption the Continuum Hypothesis.

Note that physically speaking, we know for a fact that this assumption is false, but it acts as a good approximation because molecules are so small. We will see that this strikes the balance we were looking for: it makes our equations simple enough that we can study them, but does not simplify so much that we lose the ability to describe the essential physics of fluid flows. This is mainly because the continuum hypothesis allows us to define functions on the fluid, and so to apply calculus methods.

3.2 Kinematics

Kinematics is the study of motion without regard for the causes of that motion. The forces causing motion are not considered in this section, and so apply to all continuous matter.

3.2.1 Particles and Functions

We will take the continuum hypothesis as the justification for why we can use calculus on all of these problems, but we will no longer talk about it. In fact, now that we have these functions, which are defined continuously, and indeed smoothly whenever we find it convenient, we wish to know what they mean physically. This may seem circular but it is not. We have assumed the continuum hypothesis, which is an approximation of reality, and now we consider how to think about these approximations so that they still make physical sense.

To talk about the deformation and motion of matter, we want to be able to discuss physical quantities like density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, and pressure $p(\mathbf{x}, t)$. What do these functions mean physically?

We can talk about the average density of matter easily: if we have a chunk of matter of volume V and it has mass m, then the average density is just

$$\rho = \frac{m}{V}$$

But this is the average density over the volume V, and the functions we've just defined take values at a point, which by definition have no volume. So what do we mean by density at a point $\mathbf{x}$? Consider the following thought experiment:

Take a cube of some material with sides of length l centered at $\mathbf{x}$ and let $M_l(\mathbf{x})$ be the mass of the material in this cube. By definition, the average density of the matter in the cube is then

$$\rho_l(\mathbf{x}) = \frac{M_l(\mathbf{x})}{l^3}$$

Now imagine what happens as $l \to 0$. By continuity ρ_l varies as l decreases, and the more homogeneous the material the slower it varies. When l is small ρ_l is almost constant because it varies so little. However when l is so small that it is comparable to intermolecular distances, ρ_l

varies violently because the number of molecules inside the volume is small, so a change in l that excludes a molecule can result in a massive difference to the average density.

Definition 3.2.1.1. *A particle is a volume of matter small enough that it can be regarded mathematically as a point, but large enough to contain a large number of molecules.*

This definition was given by Euler. What it says is that a particle is a physical point, not just a mathematical point. In particular a particle large enough that we can discuss its mass and consider forces acting on it, and it also makes sense to discuss its motion through space. Using this concept we can now make sense of our functions.

Definition 3.2.1.2. *The density field $\rho(\mathbf{x}, t)$ is the average density of a particle centered at $\mathbf{x}$ at time t.*

Similarly

Definition 3.2.1.3. *The velocity field $\mathbf{u}(\mathbf{x}, t)$ is the average velocity of a particle centered at $\mathbf{x}$ at time t.*

We will describe pressure later.

3.2.2 Pathlines and Streamlines

If the position of a particle is given by $\mathbf{x}(t)$ at time t, then its velocity is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \tag{6}$$

If we are given $\mathbf{u}(\mathbf{x}, t)$, then 6 gives us a set of n ODEs to solve, which is usually impossible to do analytically.

Definition 3.2.2.1. *A pathline is a solution to $\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$*

Physically, a pathline is the trajectory traced out by a particle as it moves. In differential geometry we call these integral curves of the velocity field.

A pathline $\mathbf{x}(t)$ is a curve in space, and we can use the fact that its components are solutions to ODEs to represent $\mathbf{x}$ in terms of an initial point $\mathbf{a}$ and t. That is, we can write $\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$, and the pathline $\mathbf{x}(\mathbf{a}, t)$ describes the point $\mathbf{a}$'s position at time t. Then different values of $\mathbf{a}$ will in general give different pathlines, because we are selecting a different particle and watching it traverse the field over time. In contrast we have

Definition 3.2.2.2. *A streamline is a solution to $\frac{d\mathbf{x}}{dt}(s) = \mathbf{u}(\mathbf{x}(s), t)$ for a fixed time t.*

Here s can be any parameter, not necessarily length along the curve. Physically, a streamline is a curve which is everywhere tangent to the velocity field $\mathbf{u}$ at a fixed time t. The difference between a pathline and streamline is that time varies for the pathline and for the streamline it does not. Therefore a streamline is like a path drawn on a photograph of the velocity field such that it is everywhere tangent to the field, and a pathline would require an animation or video as it describes the motion of a particle over time such that at every instant the tangent to it's trajectory agrees with the velocity field. A pathline is tangent to the velocity field at $\mathbf{x}$ at time t, but not necessarily any other time. If the velocity field is time-independent then steamlines and pathlines are the same concept.

Definition 3.2.2.3. *A flow in $\mathbb{R}^n$ is steady if all flow quantities are independent of time with respect to global coordinates.*

So not just the velocity field, but all scalar fields, and every other flow quantity do not vary over time. Of course the same definition applies to a manifold.

3.2.3 The Material Derivative

Consider a fixed fluid particle $\mathbf{a}$ which follows the integral curve $\mathbf{x}(t)$ of the velocity field. We can think of $\mathbf{x}$ as a function of $\mathbf{a}$ and time because $\mathbf{x}$ is the solution to an ODE, and so $\mathbf{a}$ gives the initial conditions for this ODE, meaning without loss of generality we can assume that $\mathbf{x}(0) = \mathbf{a}$. Conceptually, we can think of $\mathbf{a}$ as a moving particle. Suppose $f(\mathbf{x}(t), t)$ is a physical property of the flow such as density or pressure. Then the value of f at the location of the moving particle $\mathbf{a}$, which we'll denote $f_L(\mathbf{a}, t)$, is

$$f_L(\mathbf{a}, t) = f(\mathbf{x}(\mathbf{a}, t), t) = f(x_1(\mathbf{a}, t), x_2(\mathbf{a}, t), x_3(\mathbf{a}, t), t)$$

Now by the chain rule

$$\begin{aligned}
\frac{\partial f_L}{\partial t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} \\
&= \frac{\partial f}{\partial t} + (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}) \cdot (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) \\
&= \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla f
\end{aligned}$$

However recall that by definition of a pathline, $\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t)$, so explicitly showing the dependence of variables, we have

$$\frac{\partial f_L(\mathbf{a}, t)}{\partial t} = \frac{\partial f(\mathbf{x}(\mathbf{a}, t), t)}{\partial t} + \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t) \cdot \nabla f(\mathbf{x}(\mathbf{a}, t), t)$$

That is, this is a derivative along an integral curve of the velocity field. Suppressing the variables, we get

$$\frac{\partial f_L}{\partial t} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f$$

Definition 3.2.3.1. *The material derivative is the differential operator*

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

We define the material derivative to act on vector fields by acting on each component by the above formula. That is, for a vector field $\mathbf{v} = (v^1, v^2, v^3)$,

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{Dv^1}{Dt}, \frac{Dv^2}{Dt}, \frac{Dv^3}{Dt}\right)$$

What does this mean physically? Recall from AMATH 231 that ∇f is a vector pointing in the direction of greatest increase of f. We also have that if $\mathbf{k}$ is a unit vector, then $\nabla f \cdot \mathbf{k}$ is the directional derivative of f in direction $\mathbf{k}$. Since $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ is a unit vector, we have

$$\mathbf{u} \cdot \nabla f = \left(\left(\frac{\mathbf{u}}{\|\mathbf{u}\|} \right) \cdot \nabla f \right) \|\mathbf{u}\|$$

so $\mathbf{u} \cdot \nabla f$ is the rate of change of f with respect to time in the direction of motion multiplied by the speed of the particle $\mathbf{a}$. But then if f is independent of time, this is the rate of change of f with respect to time as measured by the observer who is moving with velocity $\mathbf{v}$.

Then the material derivative is the total rate of change of f with respect to time as measured by an observer moving with the fluid, but expressed in global coordinates. This means that $\frac{Df}{Dt}$ can be nonzero even if f does not change in time, because the particle is following a pathline and so moving over time.

Conceptually the two contributions to the material derivative are $\frac{\partial f}{dt}$, which gives us the rate of change of f at the observer $\mathbf{a}$'s position, and the rate of change of f with respect to time due to motion through space, $\mathbf{u} \cdot \nabla f$.

$\mathbf{u} \cdot \nabla \mathbf{v}$ for vector fields $\mathbf{u}, \mathbf{v}$ looks ambiguous, but it is not. First note that

$$(\mathbf{u} \cdot \nabla)\mathbf{v} = u^i \frac{\partial}{\partial x^i} \mathbf{v}$$

using the standard abuse of treating nabla as a vector of differential operators. Really $\mathbf{u} \cdot \nabla$ is what we call $\mathbf{u}$ in differential geometry. We also have $\nabla \mathbf{v} = \frac{\partial v^i}{\partial x^j} \mathbf{e}_i \otimes \mathbf{e}_j$ is the gradient of the vector field $\mathbf{v}$ by definition. Then

$$\mathbf{u} \cdot (\nabla \mathbf{v}) = \mathbf{u} \cdot \left(\frac{\partial v^i}{\partial x^j} \mathbf{e}_i \otimes \mathbf{e}_j \right)$$

where here $\cdot$ is contraction through the metric, meaning

$$(\mathbf{u} \cdot (\nabla \mathbf{v}))^j = g_{ik} \frac{\partial v^j}{\partial x^i} u^k$$

but here $g_{ik} = \delta_i^k$, so

$$(\mathbf{u} \cdot (\nabla \mathbf{v}))^j = \frac{\partial v^j}{\partial x^i} u^i$$

so we can write

$$\mathbf{u} \cdot (\nabla \mathbf{v}) = \left(u^i \frac{\partial v^1}{\partial x^i}, \ldots, u^i \frac{\partial v^n}{\partial x^i} \right)$$

so in fact $\mathbf{u} \cdot (\nabla \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} = \mathbf{u} \cdot \nabla \mathbf{v}$, so we can unambiguously write

$$\mathbf{u} \cdot \nabla \mathbf{v} = u^i \frac{\partial}{\partial x^i} \mathbf{v} = \left(u^i \frac{\partial v^1}{\partial x^i}, \ldots, u^i \frac{\partial v^n}{\partial x^i} \right)$$

Similarly, for a function f,

$$(\mathbf{u} \cdot \nabla) f = u^i \frac{\partial f}{\partial x^i} = \mathbf{u} \cdot (\nabla f)$$

Therefore $\mathbf{u} \cdot \nabla f$ and $\mathbf{u} \cdot \nabla \mathbf{u}$ are unambiguous expressions. Also note that we have proved that our definition of $\frac{D\mathbf{v}}{Dt}$ as

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{Dv^1}{Dt}, \frac{Dv^2}{Dt}, \frac{Dv^3}{Dt}\right)$$

is now justified, because

$$\begin{aligned}
\frac{D\mathbf{v}}{Dt} &= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{v} \\
&= \left(\frac{\partial v^1}{\partial t}, \dots, \frac{\partial v^n}{\partial t}\right) + \left(u^i\frac{\partial v^1}{\partial x^i}, \dots, u^i\frac{\partial v^n}{\partial x^i}\right) \\
&= \left(\frac{\partial v^1}{\partial t}, \dots, \frac{\partial v^n}{\partial t}\right) + \left(\mathbf{u}\cdot\nabla v^1, \dots, \mathbf{u}\cdot\nabla v^n\right) \\
&= \left(\frac{Dv^1}{Dt}, \frac{Dv^2}{Dt}, \frac{Dv^3}{Dt}\right)
\end{aligned}$$

$\mathbf{u}\cdot\nabla$ can be rewritten invariantly, but we must be careful. Recall that the Euclidean connection on $\mathbb{R}^n$ is also denoted ∇ and is a covariant derivative, meaning it takes as input two vector fields, and gives as output another vector field. The Euclidean connection for two vector fields $\mathbf{X} = X^i\frac{\partial}{\partial x^i}$ and $\mathbf{Y} = Y^j\frac{\partial}{\partial x^j}$ is by definition

$$\nabla_{\mathbf{X}}\mathbf{Y} = (\mathbf{X}Y^j)\frac{\partial}{\partial x^j} = (X^i\frac{\partial Y^1}{\partial x^i}, \dots, X^i\frac{\partial Y^n}{\partial x^i})$$

Therefore $\mathbf{u}\cdot\nabla\mathbf{v} = \nabla_{\mathbf{u}}\mathbf{v}$ where the nabla on the left is the gradient operator, and the nabla on the right is the euclidean connection on $\mathbb{R}^n$. We also know that by definition for f a function,

$$\nabla_{\mathbf{u}}f = \mathbf{u}f = u^i\frac{\partial}{\partial x^i}f = u^i\frac{\partial f}{\partial x^i} = \mathbf{u}\cdot\nabla f$$

Therefore we have shown that

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla_{\mathbf{u}}} \tag{7}$$

So we've written the material derivative invariantly, and we see that in the case of time independent vectors the material derivative is simply the covariant derivative with respect to **u**. Above we defined the material derivative to operate on vectors simply by operating on the component functions and this works because the Christoffel symbols for the Euclidean connection in $\mathbb{R}^n$ vanish, but in general the Christoffel symbols for a connection do not vanish. So for a general manifold we must insist that the material derivative is applied to vector fields using the definitions from differential geometry. That is, in general

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{Dv^1}{Dt}, \frac{Dv^2}{Dt}, \frac{Dv^3}{Dt}\right)$$

does **not** hold. Explicitly,

$$\nabla_{\mathbf{u}}\mathbf{v} = \nabla_{\mathbf{u}}(v^i\frac{\partial}{\partial x^i}) = (\mathbf{u}v^i)\frac{\partial}{\partial x^i} + v^i\nabla_{\mathbf{u}}\left(\frac{\partial}{\partial x^i}\right)$$

and $\nabla_{\mathbf{u}} v^i = \mathbf{u} v^i$. So unless $\nabla_{\mathbf{u}} \left(\frac{\partial}{\partial x^i}\right)$ vanishes, the componentwise formula is incorrect.

Another important point to note is that the material derivative is nonlinear.

3.2.4 Material Volumes

Definition 3.2.4.1. *A material volume is a fixed piece of matter in $\mathbb{R}^3$ which moves with a fluid flow. It is comprised of the same particles for all time. So in coordinates relative to a frame moving with the fluid, a material volume is a fixed region.*

Physically, we can imagine a patch of water in a river dyed red to be our material volume. Of course diffusion will cause this volume to dissipate, but for short time scales where the diffusion is not significant, the red patch moves and deforms with the flow. Therefore in Cartesian coordinates the volume and dimensions of the red patch do change, but as measured from each individual particle, the material volume is a set of particles following their trajectory through the flow, and so remains a constant region. Particles as described above are sometimes called material particles because they consist of the same matter for all time.

Mathematically we can think of a material volume as a fixed set of particles which are moving along their pathlines together. To avoid confusion we repackage some notation:

Definition 3.2.4.2. *Let $\Phi(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping that takes an initial point $\mathbf{a}$ of a particle to its position at time t, meaning $\Phi(\mathbf{a}, t) = \mathbf{x}(\mathbf{a}, t)$. We will also write $\Phi_t(\mathbf{a})$ or $\Phi^{(\mathbf{a})}(t)$ for $\Phi(\mathbf{a}, t)$ as alternate forms.*

This is simply the repackaging of all pathlines into a single global time evolution map. In differential geometry terms, this is the flow of the vector field $\mathbf{u}$, with integral curves $\Phi^{(\mathbf{a})}(t)$, where $\Phi^{(\mathbf{a})}(0) = \mathbf{a}$. We assume that Φ is a diffeomorphism in terms of both $\mathbf{x}$ and t. Physically this means two different particles of matter can't be at the same place at a later time, or that one particle can't split into two.

Notice that since the material derivative is taken along integral curves, the assumption that Φ is a diffeomorphism also means that we can treat any point as a point on a trajectory, and so we can take derivatives at any point in space. There is always a solution to an ODE for small times, but there is no guarantee that solutions exist for all time. We aren't going to worry about this too much, but it's worth mentioning. Unless otherwise stated, we'll assume our flows exist for all time.

Now $\Phi(\mathbf{a}, t) = \Phi^{\mathbf{a}}(t) = \mathbf{x}(\mathbf{a}, t)$, and when we derived the material derivative we pointed out that there was a dependence on $\mathbf{x}(\mathbf{a}, t)$ in each term. Therefore

$$\frac{Df(\Phi^{\mathbf{a}}(t), t)}{Dt} = \frac{\partial f(\Phi^{\mathbf{a}}(t), t)}{\partial t} + \mathbf{u}(\Phi^{\mathbf{a}}(t), t) \cdot \nabla f(\Phi^{\mathbf{a}}(t), t)$$

Lets rewrite the right hand side in a different way. The spacial components are all in terms of $\Phi^{\mathbf{a}}(t)$, so we have function composition in the spatial components. Holding $\mathbf{a}$ fixed, then, we can make the right hand side a function of t only. Define $\Psi(t) = (\Phi^{\mathbf{a}}(t), t)$, then we can write

$$\frac{\partial f(\Psi(t))}{\partial t} + \mathbf{u}(\Psi(t)) \cdot \nabla f(\Psi(t)) = (\Psi)^* \left(\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f\right)$$

which is a function of t. Since $\mathbf{a}$ is fixed, this is a function of t at $\mathbf{a}$ which gives the total rate of change of f at $\mathbf{a}$'s position, at time t. Therefore we have recovered the Lagrangian interpretation of the derivative.

Finally, since we're assuming Φ is a diffeomorphism, we are justified in taking the material derivative at any point, because every point is in the image of Φ, and so is on an integral curve of the fluid flow.

3.2.5 The Transport Theorem

Now take a material volume which is time dependent, say $W(t)$. By definition of a material volume, since material volumes are fixed sets of particles, every particle in $W(0)$ gets mapped to a particle in $W(t)$ and every particle in $W(t)$ has a preimage in $W(0)$. We will write W for $W(0)$ sometimes, so we can write things like $W(t) = \Phi_t(W(0)) = \Phi_t(W)$. If $W(0)$ is a curve, then $W(t)$ is called a material curve, and if $W(0)$ is a surface then $W(t)$ is called a material surface.

Intuitively, you can think of a material curve as a string moving through the fluid, and a material surface as a chunk of material of codimension 1 moving through the fluid.

For now, we will be interested in the time evolution of material volumes, and integrals of the form

$$I(t) = \iiint\limits_{W(t)} f(\mathbf{x}, t)\, dV \tag{8}$$

Here the volume of integration is a time dependent material volume. We wish to find the derivative of $I(t)$. We will use the diffeomorphism Φ and the change of variables theorem. Let $\mathbf{x} = \mathbf{\Phi}(\mathbf{a}, t) = (\Phi_1(\mathbf{a}, t), \Phi_2(\mathbf{a}, t), \Phi_3(\mathbf{a}, t))$. Then by the change of variables theorem

$$\iiint\limits_{W(t)} f(\mathbf{x}, t)\, dV = \iiint\limits_{W(0)} f(\mathbf{\Phi}, t) \det(J(\mathbf{a}, t))\, dV$$

Where here $J(\mathbf{a}, t)$ is the Jacobian matrix of Φ with respect to the variables in $\mathbf{a}$, so $J(\mathbf{a}, t) = \frac{\partial \Phi_i}{\partial a_j}$. Notice at time $t = 0$, $\mathbf{\Phi}(\cdot, 0)$ is the identity map, and so has Jacobian determinant 1. Since we're assuming $\mathbf{\Phi}$ is invertible, the Jacobian determinant of $\mathbf{\Phi}$ is never zero, for any t, and therefore $J(\mathbf{a}, t)$ has positive determinant for all t by continuity of J. This is why we did not have the absolute value of the determinant above, only the determinant, because we knew it would be positive. This also shows us that Φ is an orientation preserving map.

Lemma 3.2.5.1. *Suppose $\mathbf{\Phi}$ is a diffeomorphism and $\mathbf{x} = \mathbf{\Phi}(\mathbf{a}, t) = (\Phi_1(\mathbf{a}, t), \Phi_2(\mathbf{a}, t), \Phi_3(\mathbf{a}, t))$, then*

$$\frac{\partial \det(J)}{\partial t} = \operatorname{div}(\mathbf{u}) \det(J)$$

where $\mathbf{u} = \frac{d\mathbf{x}}{dt}$.

Notice here that $\operatorname{div}(\mathbf{u})$ is actually $\operatorname{div}(\mathbf{u})(\Phi(\mathbf{a}, t)) = (\Phi_t)^* \operatorname{div}(\mathbf{u})$ which will be important for us later, but not now.

Proof. We have $\mathbf{u} = (u_1, u_2, u_3)$, so

$$\text{div}(\mathbf{u}) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

We have $J(\mathbf{a}, t) = \frac{\partial \Phi_i}{\partial a_j}$, write $\Phi_{i,j} = \frac{\partial \Phi_i}{\partial a_j}$ so in 3 dimensions we have

$$\begin{aligned}\det(J(\mathbf{a}, t)) &= \det(\Phi_{i,j}) \\ &= \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \prod_{i=1}^{3} \Phi_{i,j}\end{aligned}$$

Not take the partial derivative with respect to t:

$$\begin{aligned}\frac{\partial \det(J)}{\partial t} &= \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \frac{\partial}{\partial t} \left(\prod_{i=1}^{3} \Phi_{i,\sigma_j} \right) \\ &= \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \left(\left(\frac{\partial}{\partial t} \Phi_{1,\sigma_1} \right) \Phi_{2,\sigma_2} \Phi_{3,\sigma_3} + \Phi_{1,\sigma_1} \left(\frac{\partial}{\partial t} \Phi_{2,\sigma_2} \right) \Phi_{3,\sigma_3} + \Phi_{1,\sigma_1} \Phi_{2,\sigma_2} \left(\frac{\partial}{\partial t} \Phi_{3,\sigma_3} \right) \right)\end{aligned}$$

But now

$$\frac{\partial}{\partial t} \Phi_{i,j} = \frac{\partial^2}{\partial t \partial a^j} = \frac{\partial}{\partial a^j} \frac{\partial}{\partial t} \Phi_i = \frac{\partial}{\partial a^j} u_i(\Phi(\mathbf{a}, t)) = \frac{\partial u_i}{\partial x^k} \frac{\partial \Phi_k}{\partial a^j} = \frac{\partial u_i}{\partial x^k} \Phi_{k,j}$$

so in particular $\frac{\partial}{\partial t} \Phi_{i,\sigma_i} = \frac{\partial u_i}{\partial x^k} \Phi_{k,\sigma_i}$. Putting it all together we have:

$$\frac{\partial \det(J)}{\partial t} = \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \left(\frac{\partial u_1}{\partial x^i} \Phi_{i,\sigma_1} \Phi_{2,\sigma_2} \Phi_{3,\sigma_3} + \frac{\partial u_2}{\partial x^j} \Phi_{1,\sigma_1} \Phi_{j,\sigma_2} \Phi_{3,\sigma_3} + \frac{\partial u_3}{\partial x^k} \Phi_{1,\sigma_1} \Phi_{2,\sigma_2} \Phi_{k,\sigma_3} \right)$$

Look at the first term $\sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \left(\frac{\partial u_1}{\partial x^i} \Phi_{i,\sigma_1} \Phi_{2,\sigma_2} \Phi_{3,\sigma_3} \right)$. expanding the sum we get $\frac{\partial u_1}{\partial x} \left(\sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \Phi_{1,\sigma_1} \Phi_{2,\sigma_2} \Phi_{3,\sigma_3} \right) = \frac{\partial u_1}{\partial x} \det(J)$ plus these two terms:

$$\frac{\partial u_1}{\partial y} \left(\sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \Phi_{2,\sigma_1} \Phi_{2,\sigma_2} \Phi_{3,\sigma_3} \right), \frac{\partial u_1}{\partial z} \left(\sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \Phi_{3,\sigma_1} \Phi_{2,\sigma_2} \Phi_{3,\sigma_3} \right)$$

These bracketed terms correspond to the $\det(J)$ with the first column replaced by the second, and $\det(J)$ with the first column replaced by the second, respectively. However the determinant of a matrix with repeated columns (or rows) is zero, so these two terms vanish, so we get $\frac{\partial u_1}{\partial x} \det(J)$ in the first term of the equation for $\frac{\partial \det(J)}{\partial t}$ above. Expanding the other terms gives similar results, and we find that

$$\frac{\partial \det(J)}{\partial t} = \frac{\partial u_1}{\partial x} \det(J) + \frac{\partial u_2}{\partial y} \det(J) + \frac{\partial u_3}{\partial z} \det(J) = \text{div}(\mathbf{u}) \det(J)$$

□

The Transport Theorem 3.2.5.1. *If Φ is invertible, $\mathbf{u}(\mathbf{x},t)$ is C^1, and $f(\mathbf{x},t)$ is C^1, then*

$$\frac{dI}{dt} = \frac{d}{dt}\iiint\limits_{W(t)} f(\mathbf{x},t)\,dV = \iiint\limits_{W(t)} \left(\frac{Df}{Dt} + f\nabla\cdot\mathbf{u}\right)\,dV$$

Proof. We have

$$\begin{aligned}\frac{dI}{dt} = \frac{d}{dt}\iiint\limits_{W(t)} f(\mathbf{x},t)\,dV &= \frac{d}{dt}\iiint\limits_{W(0)} f(\Phi(\mathbf{a},t),t)\det(J(\mathbf{a},t))\,dV \\ &= \iiint\limits_{W(0)} \left(\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\partial \Phi_i}{\partial t}\right]\det(J) + f\frac{\partial \det(J)}{\partial t}\right)\,dV\end{aligned}$$

But now by definition of Φ, $\Phi_i(\mathbf{a},t) = x_i(\mathbf{a},t)$, so

$$\frac{\partial \Phi_i}{\partial t}(\mathbf{a},t) = u_i(\Phi(\mathbf{a},t),t)$$

Now we use this fact and the lemma we just proved gives us $\frac{\partial \det(J)}{\partial t} = \operatorname{div}(\mathbf{u})\det(J)$, so we have

$$\begin{aligned}\frac{dI}{dt} &= \iiint\limits_{W(0)} \left(\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}u_i\right]\det(J) + f\operatorname{div}(\mathbf{u})\det(J)\right)\,dV \\ &= \iiint\limits_{W(0)} \left(\frac{Df}{Dt}\det(J) + f\operatorname{div}(\mathbf{u})\det(J)\right)\,dV \\ &= \iiint\limits_{W(0)} \left(\frac{Df}{Dt} + f\operatorname{div}(\mathbf{u})\right)\det(J)\,dV \\ &= \iiint\limits_{W(t)} \left(\frac{Df}{Dt} + f\operatorname{div}(\mathbf{u})\right)\,dV\end{aligned}$$

Where here the suppressed variables in the integrand are just $(\mathbf{x},t)$ □

Recall that

$$\operatorname{div}(f\mathbf{u}) = \mathbf{u}\cdot\nabla f + f\operatorname{div}(\mathbf{u})$$

so that

$$\frac{Df}{Dt} + f\operatorname{div}(\mathbf{u}) = \frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f + f\operatorname{div}(\mathbf{u}) = \frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{u})$$

Now we can use the Divergence Theorem (Stokes' Theorem) to get

$$\begin{aligned}\frac{dI}{dt} &= \iiint\limits_{W(t)} \left(\frac{Df}{Dt} + f\nabla\cdot\mathbf{u}\right) dV \\ &= \iiint\limits_{W(t)} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{u})\right) dV \\ &= \iiint\limits_{W(t)} \frac{\partial f}{\partial t}\, dV + \iint\limits_{\partial W(t)} f\mathbf{u}\cdot\mathbf{n}\, dS\end{aligned}$$

Physically, this says that the rate of change of I is equal to the integral of $\frac{\partial f}{\partial t}$ over the constant in time region $W(t)$ plus the flux of f out of $W(t)$.

Lets repackage this section in differential geometry terms.

The Transport Theorem 3.2.5.2. *Suppose (M, μ) is a manifold with volume form, and $X \in \Gamma(TM)$. Let θ_t be the flow of X. Suppose $f \in C^\infty(M \times \mathbb{R})$ and write $f_t(p)$ for $f(p,t)$. Then if $U \subset M$ is open*

$$\frac{d}{dt}\int_{\theta_t(U)} f_t\,\mu = \int_{\theta_t(U)} \left(\frac{\partial f}{\partial t} + \operatorname{div}_\mu(f_t X)\right)\mu$$

This is the version given in [6]. Here the subscript on the divergence is stressing its the divergence with respect to the volume form μ.

We can see immediately that this is just a change of notation from the vector calculus version of the theorem, by our work done in the AMATH 231 section. We have $\theta_t(U)$ instead of $\Phi_t(W) = W(t)$. Because an open set must have "full dimension", the volume form on $\theta_t(U)$ is just $\mu\big|_{\theta_t(U)}$ which we can just write as μ. The only difference between this version and the version given above, is that this version chooses not to talk about the material derivative, and rewrites the integrand as we did above.

3.3 Derivation of the Governing Equations: Mass, Momentum, and Energy

We will now start developing the equations we need to derive and understand the Navier-Stokes equations. We will make some physical assumptions, and will justify those assumptions. [6]

1. Mass is neither created nor destroyed
2. Newton's 2nd Law: $\mathbf{F} = m\mathbf{a} = \frac{d}{dt}\mathbf{p} = \dot{\mathbf{p}}$ where here $\mathbf{p} = m\mathbf{u}$
3. Energy is neither created nor destroyed

3.3.1 Conservation of Mass and the Continuity Equation

Our first assumption is that mass is neither created nor destroyed. In our integral (8)

$$I(t) = \iiint\limits_{W(t)} f(\mathbf{x}, t)\, dV$$

if we take $f = \rho$, the density field we defined above, then this integral is the mass of $W(t) = \Phi_t(W)$. Define

$$M(t) = \iiint\limits_{W(t)} \rho(\mathbf{x}, t)\, dV$$

the mass of the material volume $W(t)$. Conservation of mass implies $M(t)$ is constant. Therefore, by the transport theorem

$$0 = \frac{dM}{dt} = \iiint\limits_{W(t)} \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \right) dV$$

and this is for any material volume $W(t)$, so by the Dubois-Reymond lemma we have

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0} \tag{9}$$

Which is called the continuity equation, or the differential form of the law of conservation of mass. Note that there is a smoothness assumption required to get to this equation. If we don't have the integrand smooth enough, we must leave it in the integral form.

As an aside, notice that

$$V(t) = \iiint\limits_{W(t)} dV$$

is the volume of $W(t)$, and the time derivative is

$$\frac{dV}{dt} = \iiint\limits_{W(t)} \left(\frac{\partial 1}{\partial t} + \operatorname{div}(1\mathbf{u}) \right) dV = \iiint\limits_{W(t)} \operatorname{div}(\mathbf{u})\, dV$$

This means $\operatorname{div}(\mathbf{u})$ is the rate of change of volume per unit volume.

Getting back to the continuity equation, we can rewrite it as

$$0 = \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = \frac{D\rho}{Dt} + \rho \operatorname{div}(\mathbf{u})$$

which implies, since $\rho > 0$, that we can write

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\operatorname{div}(\mathbf{u})$$

so when $\operatorname{div}(\mathbf{u}) > 0$, the material volume is expanding, and density is decreasing because $\frac{1}{\rho}$ is positive here, as always.

We have a product rule of sorts too, provided mass is conserved:

$$\begin{aligned}
\frac{d}{dt}\iiint\limits_{W(t)} \rho f\, dV &= \iiint\limits_{W(t)} \left(\frac{\partial(\rho f)}{\partial t} + \operatorname{div}((\rho f)\mathbf{u})\right) dV \\
&= \iiint\limits_{W(t)} \left(\rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla(\rho f) + \rho f\operatorname{div}(\mathbf{u})\right) dV \\
&= \iiint\limits_{W(t)} \left(\rho\frac{\partial f}{\partial t} + f\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot f\nabla\rho + \mathbf{u}\cdot\rho\nabla f + \rho f\operatorname{div}(\mathbf{u})\right) dV \\
&= \iiint\limits_{W(t)} \left(\rho\left[\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f\right] + f\left[\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\operatorname{div}(\mathbf{u})\right]\right) dV
\end{aligned}$$

but since we're assuming mass is conserved, by the continuity equation the second term in the integrand is zero, and the first is $\rho\frac{Df}{Dt}$ by definition. Therefore we have shown that

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho f\, dV = \iiint\limits_{W(t)} \rho\frac{Df}{Dt}\, dV$$

but only when mass is conserved. Confusingly, this is also sometimes called the Transport Theorem. Note that this formula applies when f is vector valued as well, since $\frac{D}{Dt}$ acts componentwise.

Notice that so far the metric has not really been needed because we showed we can write the material derivative in terms of the connection, so all the results above really only required a volume form so that we could define divergence.

Lets derive the continuity equation in differential geometry terms:

Lie Derivative Formula for Functions 3.3.1.1. *Suppose $f \in C^r(M)$, $X \in \Gamma^{r-1}(TM)$, and let X have a flow θ_t, then*

$$\frac{d}{dt}(\theta_t)^* f = (\theta_t)^*\mathcal{L}_X f$$

This is a theorem from [6]. Note that if we identify X as a partial derivative as we normally do, then $\mathcal{L}_X f = Xf$ by definition.

Proof. Fix a point $p \in M$. Now we apply the chain rule:

$$\frac{d}{dt}\left((\theta_t)^* f\right)(p) = \frac{d}{dt}(f\circ\theta_t)(p) = \frac{\partial f}{\partial x^i}(\theta_t(p))\frac{d\theta_t^i}{dt}(p) = \nabla f(\theta_t(p))\cdot\frac{d}{dt}\theta_t(p)$$

but by definition $\frac{d}{dt}\theta_t(p) = X_{\theta_t(p)}$. Therefore

$$\frac{d}{dt}\left((\theta_t)^* f\right)(p) = \nabla f(\theta_t(p))\cdot X_{\theta_t(p)} = df(\theta_t(p))X_{\theta_t(p)} = (df(X))(\theta_t(p)) = (Xf)(\theta_t(p)) = \mathcal{L}_X f(\theta_t(p))$$

applying the musical isomorphism definition to $\nabla f = (df)^\sharp$. Finally, by definition of the pullback $\mathcal{L}_X f(\theta_t(p)) = ((\theta_t)^*\mathcal{L}_X f)(p)$

□

Now by the differential geometry version of change of variables, we have that by our observation above, Φ is an orientation preserving diffeomorphism, so for any top form ω on $W(t) = \Phi_t(W)$, we have

$$\int_{\Phi_t(W)} \omega = \int_W (\Phi_t)^* \omega$$

in particular, conservation of mass says

$$\int_{\Phi_t(W)} \rho_t \mu = \int_W \rho_0 \mu$$

for some initial value ρ_0 of $\rho(\mathbf{x}, t)$. So now by the change of variable theorem we have

$$\int_W (\Phi_t)^* \rho_t \mu = \int_{\Phi_t(W)} \rho_t \mu = \int_W \rho_0 \mu$$

so $(\Phi_t)^* \rho_t \mu = \rho_0 \mu$ which is constant. By Du Bois - Reymond and the definition of the pullback this is equivalent to the statement

$$(\Phi_t^* \rho_t) \det(J) = \rho_0$$

where $\det(J)$ is the determinant of the jacobian matrix for Φ as before. Now

$$\begin{aligned} 0 = \frac{d}{dt} ((\Phi_t^* \rho_t) \det(J)) &= \det(J) \frac{d}{dt} (\Phi_t^* \rho_t) + (\Phi_t^* \rho_t) \frac{d}{dt} \det(J) \\ &= \det(J)((\Phi_t)^* \frac{\partial \rho_t}{\partial t} + \Phi_t^* (\mathbf{u} \rho_t)) + \Phi_t^* (\rho_t \operatorname{div}(\mathbf{u})) \det(J) \end{aligned}$$

using our previous lemmas. We also have that $\det(J)$ is never zero, so we have

$$0 = \Phi_t^* \left(\frac{\partial \rho_t}{\partial t} + \mathbf{u} \cdot \nabla \rho_t + \rho_t \operatorname{div}(\mathbf{u}) \right) = \Phi_t^* \left(\frac{\partial \rho_t}{\partial t} + \operatorname{div}(\rho_t \mathbf{u}) \right)$$

which means we've recovered the continuity equation.

3.3.2 Conservation of Linear Momentum and the Stress Tensor

We could also call this section "Balance of Momentum" which is the same concept. Whichever name is chosen, the concept is analagous to Newton's second law. Consider that for a solid, rigid body of mass m, considered as a point with velocity $\mathbf{v}$, Newton's second law states

$$\frac{d}{dt} (m \mathbf{v}) = \mathbf{F}$$

where here $\mathbf{F}$ is the net force acting on the object. This formula does not apply to a continuum, so we need an analogue we can use. Fortunately, in 1776 Euler came up with just such an analogue [2].

1. The total force acting upon a body equals the rate of change of the total linear momentum

2. The total torque acting upon a body equals the rate of change of the total moment of momentum, where both the torque and the moment are taken with respect to the same fixed point.

The first statement is also known as balance of momentum. Euler said these laws apply to all bodies or systems of bodies and to every part of every body whether or not they are treated as a continuum or whether or not they are deforming. It turns out that the second law is not valid for all continuums, but is valid for all the ones we will discuss. The first law is more relevant for us anyway. $\iiint\limits_{W(t)} \rho\mathbf{u}\,dV$ is the total linear momentum of the material volume $W(t)$ by definition and this is a vector valued integral as it should be, since momentum is a vector. The first law then says

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho\mathbf{u}\,dV = \text{ net force acting on W(t)}$$

Note that vector valued integrals are evaluated componentwise.

Now we know that the rate of change of momentum equals the net force acting on a material volume, but what forces can act on a material volume $W(t)$? There are three types

1. Body forces act throughout the volume. One example is the gravitational force

$$\mathbf{F}_g = \iiint\limits_{W(t)} \rho\mathbf{g}\,dV$$

   but we could also have a magnetic field or something. These are external forces. In general we will assume that the body forces can be codified by a force density vector field $\mathbf{b}$ with units of force per unit mass, so that the body forces can be written

$$\iiint\limits_{W(t)} \rho\mathbf{b}\,dV$$

2. Surface forces are forces exerted on $\partial W(t)$ by matter outside of $W(t)$. These forces are due to short range forces between molecules and to movement of molecules across $\partial W(t)$. These forces are called forces of stress as well.

3. Line forces, also known as surface tension, acts on the interface between a liquid and a gas or between two immiscible liquids. We won't worry about these forces because they do not appear in the equations of motion. They do arise in boundary conditions though.

Definition 3.3.2.1. *The stress vector $\mathbf{t}(\mathbf{x}, t, \hat{n})$ is the force per unit area acting on a surface element at $(\mathbf{x}, t)$ with unit outward normal $\hat{n}$. The force is exerted by the material into which $\hat{n}$ points and acts on the material from which $\hat{n}$ points.*

So if dA was a surface element at $(\mathbf{x}, t)$, then $\mathbf{t}(\mathbf{x}, t, \hat{n})dA$ is the force exerted by the matter outside $W(t)$ on dA. This definition allows us to write down the total surface forces as

$$\iint\limits_{\partial W(t)} \mathbf{t}(\mathbf{x}, t, \hat{n})\,dS$$

and since we have the total body forces above, this gives us the balance of momentum equation

$$\boxed{\frac{d}{dt}\iiint\limits_{W(t)} \rho\mathbf{u}\,dV = \iiint\limits_{W(t)} \rho\mathbf{b}\,dV + \iint\limits_{\partial W(t)} \mathbf{t}(\mathbf{x},t,\hat{n})\,dS} \tag{10}$$

Cauchy's Fundamental Theorem for Stress 3.3.2.1. *Balance of momentum implies that* $\mathbf{t}(\mathbf{x},t,\hat{n})$ *is linear in the components of* $\hat{n}$*, meaning*

$$t_j = \sum_i \tau_{ij} n_i$$

for some scalars $\tau_{ij}(\mathbf{x},t)$*.*

We omit the proof of this theorem because although it is straightforward, it is somewhat tedious. It's in the course notes. What this theorem says is that if $\mathbf{t}$ exists and satisfies the balance of momentum equation, then it must be of the form $\tau\cdot\hat{n}$ for a two-tensor τ. Here the notation using $\cdot$ is slightly confusing. Suppose τ has components τ^{ij}, and $\hat{n}$ has components n^k. $\tau\cdot\hat{n}$ is a contraction through the metric g, as follows [6]:

$$(\tau\cdot\hat{n})^i = g_{jk}\tau^{ij}n^k$$

but in $\mathbb{R}^n$, $g_{jk} = \delta^k_j$, so we have

$$(\tau\cdot\hat{n})^i = \sum_j \tau^{ij}n^j$$

Similarly

$$(\hat{n}\cdot\tau)^i = g_{jk}\tau^{ji}n^k = \sum_j \tau^{ji}n^j = t^i$$

by Cauchy's Theorem above. Therefore we have shown that for a two tensor τ,

$$\mathbf{t}(\mathbf{x},t,\hat{n}) = \hat{n}\cdot\tau$$

Definition 3.3.2.2. *Let* $\tau_{jk}(\mathbf{x},t) = t_k(\mathbf{x},t,\hat{i}_j)$*, then* τ_{jk} *form the components of the stress tensor we'll call* τ*. This tensor is called the Cauchy stress tensor.*

Physically, τ_{jk} is the kth component of the force per unit area acting on a surface with outward normal $\hat{i}_j$. Therefore the material into which $\hat{i}_j$ points is exerting the force on the material from which $\hat{i}_j$ points.

This brings us to the derivation of the momentum equations. The balance of linear momentum is

$$\frac{d}{dt}\iiint\limits_{W(t)} \rho\mathbf{u}\,dV = \iiint\limits_{W(t)} \rho\mathbf{b}\,dV + \iint\limits_{\partial W(t)} \mathbf{t}(\mathbf{x},t,\hat{n})\,dS$$

We can use $\mathbf{t}(\mathbf{x},t,\hat{n}) = \hat{n}\cdot\tau$ in the second integral on the right, and the fact that $\frac{d}{dt}\iiint\limits_{W(t)} \rho f\,dV = \iiint\limits_{W(t)} \rho\frac{Df}{Dt}\,dV$ we derived above on the left, to write

$$\iiint\limits_{W(t)} \rho \frac{D\mathbf{u}}{Dt}\, dV = \iiint\limits_{W(t)} \rho \mathbf{b}\, dV + \iint\limits_{\partial W(t)} \hat{n}\cdot\tau\, dS$$

but now $\iint\limits_{\partial W(t)} \hat{n}\cdot\tau\, dS$ is a vector valued integral whose kth component is

$$\iint\limits_{\partial W(t)} \tau_{ik} n_i\, dS$$

since $(\hat{n}\cdot\tau)^k = \sum_j \tau_{jk} n_j$ as we just showed above. Notice also that if we write τ_k for the kth column of τ, that $\sum_j \tau_{jk} n_j = \tau_k \cdot \hat{n}$ where this $\cdot$ is the dot product. Therefore

$$\iint\limits_{\partial W(t)} \tau_{ik} n_i\, dS = \iint\limits_{\partial W(t)} \tau_k \cdot \hat{n}\, dS = \iiint\limits_{W(t)} \mathrm{div}(\tau_k)\, dV$$

by the divergence theorem. We can define the divergence of a two tensor to be

$$\mathrm{div}(\tau) = (\mathrm{div}(\tau_1), \ldots, \mathrm{div}(\tau_n))$$

Note that really what we've done here is contract τ, which is a $1-1$ tensor, with the covariant derivative [11]

$$(\mathrm{div}\,\tau)^j \equiv \nabla_i \tau^i_j$$

This extends our definition invariantly. With this convention we can write

$$\iiint\limits_{W(t)} \rho \frac{D\mathbf{u}}{Dt}\, dV = \iiint\limits_{W(t)} \rho \mathbf{b}\, dV + \iiint\limits_{W(t)} \mathrm{div}(\tau)\, dV$$

so by the Du Bois - Reymond lemma, since $W(t)$ is arbitrary, we have

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{b} + \mathrm{div}(\tau)} \tag{11}$$

and this is called the momentum equation. This is the second basic equation of continuum mechanics after the continuity equation. The ith component of this equation is then

$$\rho \frac{Du_i}{Dt} = \rho b_i + \mathrm{div}(\tau_i)$$

Note that the momentum equation applies to all continuums for which mass is conserved. The form of τ is dependent on the type of matter. It turns out that if the fluid is such that Euler's second law holds, then the stress tensor is symmetric [2]. Therefore there are 6 rather than 9 independent components of the stress tensor. It also turns out that angular momentum is conserved in this case.

Since τ is symmetric, so is it's matrix representation. Recall from linear algebra that a real symmetric matrix is diagonalizable, and as a result the eigenvectors of τ form a basis for $\mathbb{R}^3$. We call the eigenvectors of a symmetric second-order tensor the principal axes. Specifically for the stress tensor τ, the eigenvalues are called the principal stresses.

Now what does this mean physically? Take a reference frame in which τ is diagonal at a fixed point, so

$$\tau = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}$$

Recall that our stress vector $\mathbf{t}$ is the force per unit area acting on a surface with outward unit normal $\hat{n}$, and that by definition of τ, $t_k = \tau_{ik} n_i$. Now since τ is diagonal, this means $t_k = \tau_{kk} n_k$ with no sum over k. This means in this coordinate system where τ is diagonal, surfaces parallel to a coordinate plane have only normal surface forces, so an infinitesimal volume is subjected to normal forces of either compression or tension along the direction of the coordinate axes in which τ is diagonal.

Another fact from linear algebra, is that for a symmetric matrix the eigenvectors corresponding to different eigenvalues are orthogonal, and so by taking an orthonormal basis in each eigenspace we get an orthonormal basis of eigenvectors. This is also intuitively clear because the change of basis matrices that diagonalize a symmetric matrix are orthogonal, meaning they correspond to rotations.

Now in $\mathbb{R}^3$ with τ, consider the surface forces acting on an infinitesimal volume. The surface forces are made up of three compressive or tensile forces acting in three orthogonal directions. Since the eigenvectors of τ, the principal axes, form an orthonormal basis, we see where the name comes from. The principal stresses, the eigenvalues, give us the force per unit area in the direction of the corresponding principal axis. A positive stress indicates a tensile or stretching surface force in the direction of that principal axis, and a negative stress indicates a compressive or ”squishing” surface force in the direction of that principal axis.

This whole construction was done pointwise, so the stress tensor can be thought of as a matrix of scalar fields symmetric at every point, and so diagonalizable at every point. The matrices diagonalizing the stress tensor at a given point correspond to rotations, and in this rotated frame we get a simple description in three independent directions of the forces on a particle at that point.

3.3.3 Conservation of Energy

We now cover the third of our assumptions, the conservation of energy. Taking the dot product of the momentum equation with the velocity gives

$$\sum_i \rho u_i \frac{Du_i}{Dt} = \sum_i [\rho u_i b_i + u_i \operatorname{div}(\tau_i)]$$

We know that $\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla_{\mathbf{v}}$ so $\frac{D}{Dt}$ clearly has a product rule. Now we have

$$\sum_i \rho u_i \frac{Du_i}{Dt} = \sum_i [\rho \frac{D}{Dt}(\frac{1}{2} u_i u_i)] = \sum_i [\frac{D}{Dt}(\frac{\rho}{2} u_i u_i) - \frac{1}{2} u_i u_i \frac{D\rho}{Dt}] = \sum_i [\frac{D}{Dt}(\frac{\rho}{2} u_i u_i) + (\frac{\rho}{2} u_i u_i) \operatorname{div}(\mathbf{u})]$$

by the continuity equation. By linearity of $\frac{D}{Dt}$ we have

$$\sum_i \rho u_i \frac{Du_i}{Dt} = \frac{D}{Dt}(\frac{\rho}{2}\mathbf{u}\cdot\mathbf{u}) + (\frac{\rho}{2}\mathbf{u}\cdot\mathbf{u})\,\mathrm{div}(\mathbf{u})$$

Let $K = \frac{\rho}{2}\mathbf{u}\cdot\mathbf{u}$, called the kinetic energy density, which is the kinetic energy per unit volume. Notice that this value is positive or zero since the dot product is positive definite, and the density is always positive. Now we've shown that

$$\rho\mathbf{u}\cdot\mathbf{b} + \mathbf{u}\cdot\mathrm{div}(\tau) = \sum_i \rho u_i \frac{Du_i}{Dt} = \frac{DK}{Dt} + K\,\mathrm{div}(\mathbf{u})$$

There are lots of physical interpretations to make here:

1. $\frac{DK}{Dt}$ is the rate of change of the kinetic energy density of a particle.
2. Recall that $\mathrm{div}(\mathbf{u})$ being positive indicates an increase in the material volume. Therefore $-K\,\mathrm{div}(\mathbf{u})$ is a negative quantity in this case, and the kinetic energy density , $\frac{DK}{Dt}$, decreases. If $\mathrm{div}(\mathbf{u})$ is negative then $\frac{DK}{Dt}$ increases.
3. $\rho\mathbf{b}$ is the body force per unit volume, so $\rho\mathbf{u}\cdot\mathbf{b}$ is the rate at which work is done, per unit volume, by the body force on the material volume.
4. $\mathbf{u}\cdot\mathrm{div}(\tau)$ is the rate work is done by the surface forces.

Lets define our energy terms, assuming $\mathbf{b} = -\nabla b$ for b a scalar field which is constant in time:

Definition 3.3.3.1. *Define $K = \frac{\rho}{2}\mathbf{u}\cdot\mathbf{u}$ as the kinetic energy density, $P = \rho b$ as the potential energy density, and the total mechanical energy density E as $E = K + P$.*

Now that we have this definition $\frac{DE}{Dt} = \frac{DK}{Dt} + \frac{DP}{Dt}$, and from above we know that

$$\frac{DK}{Dt} = \rho\mathbf{u}\cdot\mathbf{b} + \mathbf{u}\cdot\mathrm{div}(\tau) - K\,\mathrm{div}(\mathbf{u})$$

and we can calculate

$$\frac{DP}{Dt} = \frac{D(\rho b)}{Dt} = \rho\frac{Db}{Dt} + b\frac{D\rho}{Dt} = \rho\frac{\partial b}{\partial t} + \rho\mathbf{u}\cdot\nabla b - b\rho\,\mathrm{div}(\mathbf{u})$$

using the continuity equation. Now $P = b\rho$, $\nabla b = -\mathbf{b}$, and $\frac{\partial b}{\partial t} = 0$ so we have

$$\begin{aligned}\frac{DE}{Dt} &= \frac{DK}{Dt} + \frac{DP}{Dt}\\ &= \rho\mathbf{u}\cdot\mathbf{b} + \mathbf{u}\cdot\mathrm{div}(\tau) - K\,\mathrm{div}(\mathbf{u}) - \rho\mathbf{u}\cdot\mathbf{b} - P\,\mathrm{div}(\mathbf{u})\\ &= -E\,\mathrm{div}(\mathbf{u}) + \mathbf{u}\cdot\mathrm{div}(\tau)\end{aligned}$$

so we have proven

$$\boxed{\frac{DE}{Dt} = -E\,\mathrm{div}(\mathbf{u}) + \mathbf{u}\cdot\mathrm{div}(\tau)} \tag{12}$$

However the total mechanical energy is not conserved because friction causes some of the energy to be transformed into heat. We give heat the name internal energy. It is the energy associated with molecular motion, of which there are 3 types: translational, rotational, and vibrational.

Definition 3.3.3.2. *Let $e(\mathbf{x}, t)$ be the internal energy per unit mass. Then we define ρe to be the internal energy per unit volume.*

This makes sense because in the integral $\rho e\, dV$ corresponds to energy. Now we can discuss balance of energy, because the internal energy plus the total mechanical energy is conserved, meaning that work done by the body force remains constant, because the body force converts potential energy to kinetic energy, leaving E unaffected. Therefore

$$\frac{d}{dt}\iiint_{W(t)} (E + \rho e)\; dV = \text{ rate at which work is done by surface and thermodynamic forces}$$

First the surface forces: We have $\mathbf{t}$ giving force per unit area, and $\mathbf{u}$ is our velocity field. The dot product's units are the product of the units of each vector, so $\mathbf{u}\cdot\mathbf{t}$ has units of force times distance over time times area, meaning the rate of change of work per unit area. Therefore we want to integrate $\mathbf{u}\cdot\mathbf{t}$ over $\partial W(t)$ to get the rate at which work is done by the surface forces:

$$\begin{aligned}\iint_{\partial W(t)} \mathbf{u}\cdot\mathbf{t}\, dV &= \iint_{\partial W(t)} \mathbf{u}\cdot\left(\sum_j \tau_{ji} n_j\right)\, dV \\ &= \iint_{\partial W(t)} \left(\sum_{i,j} u_i \tau_{ji} n_j\right)\, dV\end{aligned}$$

However $\sum_{i,j} u_i\tau_{ji}n_j = (u_i\tau_{1i}, u_i\tau_{2i}, u_i\tau_{3i})\cdot(n_1, n_2, n_3)$ where we're summing over i in each term. Therefore by the divergence theorem

$$\begin{aligned}\iint_{\partial W(t)} \mathbf{u}\cdot\mathbf{t}\, dV &= \iint_{\partial W(t)} \left(\sum_{i,j} u_i \tau_{ji} n_j\right)\, dV \\ &= \iiint_{W(t)} \operatorname{div}((u_i\tau_{1i}, u_i\tau_{2i}, u_i\tau_{3i}))\, dV\end{aligned}$$

but $(\mathbf{u}\cdot\tau)^k = \sum_i \tau_{ik} n_i = \sum_i \tau_{ki} n_i$ by symmetry of τ, so

$$\mathbf{u}\cdot\tau = (u_i\tau_{1i}, u_i\tau_{2i}, u_i\tau_{3i})$$

so we have

$$\begin{aligned}\iint_{\partial W(t)} \mathbf{u}\cdot\mathbf{t}\, dV &= \iiint_{W(t)} \operatorname{div}((u_i\tau_{1i}, u_i\tau_{2i}, u_i\tau_{3i}))\, dV \\ &= \iiint_{W(t)} \operatorname{div}(\mathbf{u}\cdot\tau)\, dV\end{aligned}$$

For the thermodynamic forces, consider that internal energy, or heat, is transferred from hot to cold. Let $\mathbf{q}$ be the local internal flux density. Therefore $\mathbf{q}\cdot\hat{n}dA$ is the heat flux through a surface element of area dA in the direction of the unit normal $\hat{n}$. The rate at which work is done by thermodynamic forces is the rate at which heat is transferred through the surface by definition, represented by the integral

$$\iint_{\partial W(t)} -\mathbf{q}\cdot\hat{n}\,dS = -\iiint_{W(t)} \operatorname{div}(\mathbf{q})\,dV$$

where the minus sign is because we want heat flux into $W(t)$, which is in the opposite direction to the normal. Now we can combine our integrals to get

$$\frac{d}{dt}\iiint_{W(t)} (E+\rho e)\ dV = \iiint_{W(t)} \operatorname{div}(-\mathbf{q}+\mathbf{u}\cdot\tau)\,dV$$

We can rewrite $\frac{d}{dt}\iiint_{W(t)} (E+\rho e)\ dV$ using the transport theorem:

$$\frac{d}{dt}\iiint_{W(t)} (E+\rho e)\ dV = \iiint_{W(t)} \left(\frac{D}{Dt}(E+\rho e) + (E+\rho e)\operatorname{div}(\mathbf{u})\right)\ dV$$

so by Du Bois - Reymond lemma

$$\boxed{\frac{D}{Dt}(E+\rho e) + (E+\rho e)\operatorname{div}(\mathbf{u}) = \operatorname{div}(-\mathbf{q}+\mathbf{u}\cdot\tau)} \tag{13}$$

but we can also rewrite this as

$$\frac{\partial}{\partial t}(E+\rho e) + \operatorname{div}((E+\rho e)\mathbf{u}+\mathbf{q}-\mathbf{u}\cdot\tau) = 0$$

which is the equation written in what's called conservative form. In general if we have a PDE of the form

$$\frac{\partial R}{\partial t} + \operatorname{div}(\mathbf{F}(R)) = 0$$

integrating this equation over the region indicates that the total amount of the function R is conserved. This equation says that R can change only if the amount of R leaving a region is different from that entering the region, meaning the divergence must be zero.

3.3.4 Summary of Equations so far

By assuming conservation of mass, linear momentum, angular momentum, and energy, we have derived the following equations:

1. The Continuity Equation (9) (conservation of mass):

$$\frac{D\rho}{Dt} + \rho \operatorname{div}(\mathbf{u}) = 0$$

2. The Momentum Equation (11) (balance of momentum):

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{b} + \operatorname{div}(\tau)$$

3. The Energy Equation (13) (conservation of energy)

$$\frac{D}{Dt}(E + \rho e) + (E + \rho e) \operatorname{div}(\mathbf{u}) = \operatorname{div}(-\mathbf{q} + \mathbf{u} \cdot \tau)$$

This is five equations in fourteen unknowns:

- the density ρ
- the three velocity components u_i of $\mathbf{u}$
- the six components of τ, since τ is symmetric
- the internal energy density e
- the three components of $\mathbf{q}$, the local internal flux density

3.4 Newtonian Fluids

Up to now, we have been working in the greatest generality possible given our assumptions. To make further progress towards the Navier-Stokes equations we now consider the special case of Newtonian fluids. Experimentally many common liquids such as water and air can be considered Newtonian, which makes this a very important special case.

3.4.1 The Velocity Gradient

Definition 3.4.1.1. *The velocity gradient tensor, denoted* $\nabla \mathbf{u}$ *for the velocity field* $\mathbf{u}(\mathbf{x}, t)$*, is the* $(2, 0)$ *tensor with components*

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}$$

Notice this is simply the Jacobian matrix of $\mathbf{u}$ thought of as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$. As such we can think of it as the linear approximation of $\mathbf{u}$. By definition we know the $(\nabla \mathbf{u})_{ij}$ gives us the rate of change of the component function u_i in the direction $\frac{\partial}{\partial x_j}$. Notice also that the trace of $\nabla \mathbf{u}$ is $\operatorname{div}(\mathbf{u})$.

Lets break up $\nabla \mathbf{u}$ into symmetric and antisymmetric parts:

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Definition 3.4.1.2. *We call the symmetric part of $\nabla \mathbf{u}$, namely $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ the strain rate tensor, and denote it $\mathfrak{e}$. This is also called the Cauchy strain tensor.*

Note that the trace of $\mathfrak{e}$ is $\operatorname{div}(\mathbf{u})$. We can think of this in a different way. We call the term $\frac{\partial u_i}{\partial x_i}$ the linear strain rate of $\mathbf{u}$ in the direction $\frac{\partial}{\partial x_i}$. Physically, this represents the rate of change of length per unit length in the $\frac{\partial}{\partial x_j}$ direction. Therefore the divergence is the sum of the rates of change of length per unit length in three orthogonal directions.

The off diagonal entries of e have a different physical meaning. The shear strain rate is the rate at which the angle between two initially perpendicular lines decreases. This is a measure of the deformation caused by the velocity field, and so describes the rate at which a material volume is changing shape, rather than changing volume, or being translated. If we assume our two lines are parallel to the x_1 and x_2 axes, the shear strain rate is $\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$. This makes intuitive sense because these are the off diagonal terms in a 2 dimensional rotation matrix in the x_i, x_j plane. We can form a shear tensor by taking $s_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ for $i \neq j$ and $s_{ij} = 0$ for $i = j$

Notice for $i \neq j$, $s_{ij} = 2\mathfrak{e}_{ij}$. Now we have a decomposition of the strain rate tensor e in terms of $\nabla(\mathbf{u})$ and the shear s.

$$\mathfrak{e} = \frac{1}{2}s + \begin{pmatrix} (\nabla \mathbf{u})_{11} & 0 & 0 \\ 0 & (\nabla \mathbf{u})_{22} & 0 \\ 0 & 0 & (\nabla \mathbf{u})_{33} \end{pmatrix}$$

So far we have decomposed the symmetric part of $\nabla \mathbf{u}$ into physically meaningful components: s which measures deformation not including volume changes, and the matrix, which measures the rate of change of volume. Therefore e measures the total deformation of a material volume under $\mathbf{u}$.

The antisymmetric part of $\nabla \mathbf{u}$ is $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ and this is directly related to another important vector field.

Definition 3.4.1.3. *The vorticity ω is $\omega = \operatorname{curl}(\mathbf{u})$*

If we define $r_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$, then by definition of the curl we have the rotation tensor

$$r = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

This is called a rotation tensor because if we imagine that $\mathbf{u}$ is a pure rotation, then there is no shear and no change in volume, so e is zero. Therefore

$$(\nabla\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

so $\nabla\mathbf{u} = \frac{1}{2}r$. Note this also means that in the case of a pure rotation, $\frac{\partial u_i}{\partial x_j} = -\frac{\partial u_j}{\partial x_i}$. Now, in general, we can write

$$(\nabla\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

so we have

$$\nabla\mathbf{u} = \mathfrak{e} + \frac{1}{2}r = \frac{1}{2}(s+r) + \begin{pmatrix} (\nabla\mathbf{u})_{11} & 0 & 0 \\ 0 & (\nabla\mathbf{u})_{22} & 0 \\ 0 & 0 & (\nabla\mathbf{u})_{33} \end{pmatrix}$$

Therefore we can think of $\nabla\mathbf{u}$ as the sum of a shear rate, a rotation rate, and a volume dilation rate. If we choose to keep e intact, then $\nabla\mathbf{u}$ is composed of a component that keeps track of rate of orthogonal rotation, and another that keeps track of the total deformation rate. These rates are per unit distance, not per unit time.

3.4.2 Pressure and the Deviatoric Stress Tensor

At this point you may be wondering, in all our talk of fluids so far, there has been no discussion of pressure. We define pressure, as in [6], as a scalar field $p = p(\mathbf{x}, t)$ such that if S is any surface in a fluid with unit normal $\mathbf{n}$, then the force of stress per unit area across S at $\mathbf{x}$ at time t is simply $-p\mathbf{n}$. This is a special form of our stress vector, where the stress is always exerted in the direction of $\mathbf{n}$. So we have $\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = -p\mathbf{n}$, then in particular when $\mathbf{n} = \hat{i}_j$, $\mathbf{t} = -p\hat{i}_j$. Then in the definition of τ we have

$$\tau_{jk} = t_k(\mathbf{x}, t, \hat{i}_j) = -(p\hat{i}_j)_k = -p\delta_j^k$$

Physically this makes sense, because τ_{jk} is the kth component of the force per unit area acting on a a surface with outward normal $\hat{i}_j$. By definition of p we know that this is 0 if $i \neq j$, and $-p$ if $i = j$. A fluid with a scalar field p such that $\tau_{jk} = pg_j^k$ is called an ideal fluid [6].

This was a special case, but now we can define τ in terms of the pressure in general. Recall that our definition of a fluid was not rigorous. Now that we have defined the stress tensor we can make it more exact. Recall the off diagonal terms in the stress tensor correspond to stresses acting on a surface element which are perpendicular to the normal of that surface, so tangent to the surface. These are our shear forces as we saw above. A fluid, then, is a continuum of material that deforms continuously over time whenever shear forces are applied, for as long as they are applied [12]. This is in contrast to a solid, which will deform under shear to a certain extent, but then resist the deformation as it tries to "spring back" to its original shape. Note that both solids and fluids respond to compression forces similarly, but they respond very differently to shear forces.

If a fluid is at rest, then it must not be the subject of any shear forces, because if it were, it would deform continuously by definition, and so would not be at rest. Therefore the Cauchy stress tensor

τ must be diagonal. Moreover τ must be diagonal in any basis we choose to represent it in, because the fluid doesn't care what basis we're looking at it in, and must be at rest no matter how it is described. Recall from linear algebra that the only matrices which are diagonal in every basis are scalar multiples of the identity, so $\tau = -p\text{Id}$ for some scalar p. Therefore we can redefine pressure to be the scalar p at each point [13].

So for fluid at rest, $\tau + p\text{Id}= 0$, but in motion this will not be zero. Therefore we can define a symmetric $(2,0)$ tensor σ by

$$\sigma_{ij} = \tau_{ij} + p\delta_{ij}$$

We call σ the deviatoric stress tensor. Now for any τ we can write

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}$$

Physically, by definition the deviatoric stress tensor is that portion of the stress tensor that is not caused by pressure. Pressure is constant in every direction by construction, but the deviatoric stress tensor encodes all the forces which are not related to compression, which for fluids are forces which cause deformation. Note that the deviatoric stress tensor may still have nonzero entries on the diagonal, because the Cauchy stress tensor may not have $-p$ for its values on the diagonal.

Definition 3.4.2.1. *A Newtonian fluid is a fluid for which the deviatoric stress tensor σ is linearly related to the strain rate tensor e. That is, there exists a $(2,2)$ tensor $K = {K_{ij}}^{kl}(\mathbf{x}, t)$ such that*

$$\sigma_{ij} = {K_{ij}}^{kl}\mathfrak{e}_{kl}$$

Note that this is an assumption, justified by experiments using simple fluids, meaning fluids with simple molecular structure. This assumption leads to very accurate models for important fluids such as air, water, and many oils. We are now losing generality, but the importance of air and water to human existence can hardly be overestimated, so it makes sense to begin our study of special cases with one that models these well.

That said, there are many non-Newtonian fluids including airplane fuel, ketchup, and salad dressing. There are many ways a fluid can be non-Newtonian. If the deviatoric and strain rate tensor are nonlinearly related then by definition the fluid is non-Newtonian. Another way is if a fluid has "memory", meaning its past state influences its time evolution.

We make an additional assumption now that the fluid is isotropic, meaning that its properties are the same in each direction. This implies the stress-strain relationship is independent of the orientation of the orthogonal coordinate system, so that K's components have the same value in every Cartesian coordinate system. It turns out that that any fourth order isotropic tensor ${K_{ij}}^{kl}$ has the form [10]

$${K_{ij}}^{kl} = \lambda\delta_i^j\delta_k^l + \mu\delta_i^k\delta_j^l + \gamma\delta_i^l\delta_j^k$$

but now since σ is symmetric, ${K_{ij}}^{kl}$ is symmetric in i, j as well. Permuting i and j in the above equation makes $\delta_i^k\delta_j^l = +\gamma\delta_i^l\delta_j^k$, so that $\gamma = \mu$. Therefore

$$K_{ij}{}^{kl} = \lambda\delta_i^j\delta_k^l + \mu(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)$$

For scalar fields $\lambda(\mathbf{x}, t)$ and $\mu(\mathbf{x}, t)$. These two scalar fields are regarded as known because they are the result of experiment, as measurable properties of the fluid in question. Now

$$\begin{aligned}
\sigma_{ij} &= K_{ij}{}^{kl}\mathfrak{e}_{kl} \\
&= \left[\lambda\delta_i^j\delta_k^l + \mu(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\right]\mathfrak{e}_{kl} \\
&= \lambda\delta_i^j\delta_k^l\mathfrak{e}_{kl} + \mu(\delta_i^k\delta_j^l\mathfrak{e}_{kl} + \delta_i^l\delta_j^k\mathfrak{e}_{kl}) \\
&= \lambda\delta_i^j\mathfrak{e}_{kk} + \mu(\delta_i^k\mathfrak{e}_{kj} + \delta_i^l\mathfrak{e}_{jl}) \\
&= \lambda\delta_i^j\sum_k\frac{\partial u_k}{\partial x^k} + \mu(\delta_i^k\mathfrak{e}_{kj} + \delta_i^l\mathfrak{e}_{jl}) \\
&= \lambda(\operatorname{div}(\mathbf{u}))\delta_i^j + \mu(\mathfrak{e}_{ij} + \mathfrak{e}_{ji})
\end{aligned}$$

but now e is symmetric, so we can write

$$\sigma_{ij} = \lambda(\operatorname{div}(\mathbf{u}))\delta_i^j + 2\mu\mathfrak{e}_{ij}$$

and therefore

$$\boxed{\tau_{ij} = -p\delta_i^j + \lambda(\operatorname{div}(\mathbf{u}))\delta_i^j + 2\mu\mathfrak{e}_{ij}} \tag{14}$$

Now consider the trace of τ divided by the dimension 3, so the average of the diagonal terms, which gives

$$\frac{1}{3}\sum_i\tau_{ii} = -p + \lambda\operatorname{div}(\mathbf{u}) + \frac{2}{3}\mu\sum_i\mathfrak{e}_{ii}$$

but recall again that $\sum_i\mathfrak{e}_{ii} = \operatorname{div}(\mathbf{u})$, so we have

$$\frac{1}{3}\sum_i\tau_{ii} = -p + \left(\lambda + \frac{2}{3}\mu\right)\operatorname{div}(\mathbf{u})$$

μ is often called the shear viscosity, and is easy to measure experimentally. $\lambda + \frac{2}{3}\mu$ is called the bulk viscosity and is difficult to measure because $\operatorname{div}(\mathbf{u})$ is often very small, and the bulk viscosity is multiplied by it.

3.4.3 The Momentum Equation for a Newtonian Fluid

We will now use (14) to rewrite the momentum equation under our assumption that the fluid in question is Newtonian. Recall from (11) that the ith component of the momentum equation is

$$\rho\frac{Du_i}{Dt} = \rho b_i + \operatorname{div}(\tau_i)$$

where τ_i is the ith column of τ. Therefore $\operatorname{div}(\tau_i) = \frac{\partial\tau_{ki}}{\partial x_k}$ where there is a sum over k here. Now we can use (14) to write, for our Newtonian fluid,

$$\frac{\partial \tau_{ki}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(-p\delta^i_k + \lambda(\operatorname{div}(\mathbf{u}))\delta^i_k + 2\mu\mathfrak{e}_{ki}\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \operatorname{div}(\mathbf{u})\right) + 2\frac{\partial}{\partial x_k}\left(\mu\mathfrak{e}_{ki}\right)$$

Therefore the general momentum equation for a Newtonian fluid is

$$\boxed{\rho\frac{Du_i}{Dt} = \rho b_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \operatorname{div}(\mathbf{u})\right) + 2\frac{\partial}{\partial x_k}\left(\mu\mathfrak{e}_{ki}\right)} \tag{15}$$

This equation has now eliminated τ altogether, expressing the equation in terms of pressure and the two viscosities λ and μ. As we mentioned, these quantities are regarded as known because they are observable properties of the fluid, determined by experiment. They are functions of temperature and pressure, but not of the fluid velocity $\mathbf{u}(\mathbf{x}, t)$. So really, we have introduced only one new variable, the pressure. In practice it often happens that the temperature and pressure variations are sufficiently small that λ and μ can be treated as constant. In this case we can pull out the scalars , and we have

$$2\frac{\partial}{\partial x_k}(\mu\mathfrak{e}_{ki}) = 2\mu\frac{\partial}{\partial x_k}\mathfrak{e}_{ki} = \mu\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) = \mu\left(\frac{\partial^2 u_i}{\partial x_k^2} + \frac{\partial u_k}{\partial x_k \partial x_i}\right) = \mu\frac{\partial}{\partial x_i}\operatorname{div}(\mathbf{u}) + \mu\nabla^2 u_i$$

So now if we have constant viscosities, we can rewrite 15 as

$$\rho\frac{Du_i}{Dt} = \rho b_i - \frac{\partial p}{\partial x_i}(\lambda + \mu)\frac{\partial}{\partial x_i}\operatorname{div}(\mathbf{u}) + \mu\nabla^2 u_i$$

and so in vector form we get

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{b} - \nabla p + (\lambda + \mu)\nabla(\operatorname{div}(\mathbf{u})) + \mu\nabla^2\mathbf{u}$$

where this last expression is because we're in Cartesian coordinates. Otherwise the laplacian of a vector field must be written in terms of the cross product.

3.4.4 The Energy Equation for a Newtonian Fluid

Recall that we have two energy equations:

$$\boxed{\frac{D}{Dt}(E + \rho e) + (E + \rho e)\operatorname{div}(\mathbf{u}) = \operatorname{div}(-\mathbf{q} + \mathbf{u}\cdot\tau)}$$

and

$$\boxed{\frac{DE}{Dt} = -E\operatorname{div}(\mathbf{u}) + \mathbf{u}\cdot\operatorname{div}(\tau)}$$

Subtracting the second from the first gives

$$\frac{D}{Dt}(\rho e) + (\rho e)\operatorname{div}(\mathbf{u}) = \operatorname{div}(-\mathbf{q} + \mathbf{u}\cdot\tau) - \mathbf{u}\cdot\operatorname{div}(\tau))$$

We can rewrite the left hand side:

$$\begin{aligned}\frac{D}{Dt}(\rho e) + (\rho e)\,\mathrm{div}(\mathbf{u}) &= e\frac{D\rho}{Dt} + \rho\frac{De}{Dt} + (\rho e)\,\mathrm{div}(\mathbf{u}) \\ &= e\left(\frac{D\rho}{Dt} + \rho\,\mathrm{div}(\mathbf{u})\right) + \rho\frac{De}{Dt}\end{aligned}$$

which by the continuity equation means the left hand side is just $\rho\frac{De}{Dt}$. Now since τ is symmetric,

$$\begin{aligned}\mathrm{div}(\mathbf{u}\cdot\tau) - \mathbf{u}\cdot\mathrm{div}(\tau) &= \mathrm{div}(\tau\cdot\mathbf{u}) - \mathbf{u}\cdot\mathrm{div}(\tau) \\ &= \mathrm{div}(\tau_i^1 u^i, \tau_i^2 u^i, \tau_i^3 u^i) - u^i\,\mathrm{div}(\tau_i) \\ &= \frac{\partial}{\partial x^j}\left(\tau_i^j u^i\right) - u^i\frac{\partial \tau_i^j}{\partial x^j} \\ &= \frac{\partial u^i}{\partial x^j}\tau_j^i \\ &= \left(\mathfrak{e}_{ij} + \frac{1}{2}\mathbf{r}_{ij}\right)\tau_i^j\end{aligned}$$

But now τ is symmetric and $\mathbf{r}$ is antisymmetric, so their product is zero. For a Newtonian fluid we know

$$\tau_{ij} = -p\delta_i^j + \lambda(\mathrm{div}(\mathbf{u}))\delta_i^j + 2\mu\mathfrak{e}_{ij}$$

and so

$$\begin{aligned}\tau_{ij}\mathfrak{e}_{ij} &= -p\delta_i^j\mathfrak{e}_{ij} + \lambda(\mathrm{div}(\mathbf{u}))\delta_i^j\mathfrak{e}_{ij} + 2\mu\mathfrak{e}_{ij}\mathfrak{e}_{ij} \\ &= -p\,\mathrm{div}(\mathbf{u}) + \lambda(\mathrm{div}(\mathbf{u}))^2 + 2\mu\mathfrak{e}_{ij}\mathfrak{e}_{ij}\end{aligned}$$

because $\delta_i^j\mathfrak{e}_{ij}$ is the trace of $\mathfrak{e}$ as a matrix, which we saw above was $\mathrm{div}(\mathbf{u})$. So we have

$$\rho\frac{De}{Dt} = -\,\mathrm{div}(\mathbf{q}) - p\,\mathrm{div}(\mathbf{u}) + \lambda(\mathrm{div}(\mathbf{u}))^2 + 2\mu\mathfrak{e}_{ij}\mathfrak{e}_{ij}$$

Assumption: Experimental evidence suggests that we can write $\mathbf{q} = -k\nabla T$ where T is temperature, and $k(\mathbf{x}, t)$ is the thermal conductivity. With this assumption we get the internal energy equation:

$$\boxed{\rho\frac{De}{Dt} = \mathrm{div}(k\nabla T) - p\,\mathrm{div}(\mathbf{u}) + \lambda(\mathrm{div}(\mathbf{u}))^2 + 2\mu\mathfrak{e}_{ij}\mathfrak{e}_{ij}} \tag{16}$$

3.4.5 Summary

The continuity equation remains unchanged because it has no τ terms, but we get two new equations under the Newtonian assumption. Using conservation of mass, linear momentum, and angular momentum. The assumption that the fluid is Newtonian and isotropic gives us:

1. The Continuity Equation (9) :
$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}(\mathbf{u}) = 0$$

2. The Momentum Equation (15) for Newtonian fluids :
$$\rho\frac{Du_i}{Dt} = \rho b_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda\,\mathrm{div}(\mathbf{u})\right) + 2\frac{\partial}{\partial x_k}\left(\mu\mathfrak{e}_{ki}\right)$$

   and if λ and μ can be treated as constant,
$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{b} - \nabla p + (\lambda + \mu)\nabla(\mathrm{div}(\mathbf{u})) + \mu\nabla^2\mathbf{u}$$

3. The Internal Energy Equation (16) for Newtonian fluids:
$$\rho\frac{De}{Dt} = \mathrm{div}(k\nabla T) - p\,\mathrm{div}(\mathbf{u}) + \lambda(\mathrm{div}(\mathbf{u}))^2 + 2\mu\mathfrak{e}_{ij}\mathfrak{e}_{ij}$$

We are down to five equations in seven unknowns: ρ, p, e, T, and $\mathbf{u}$. The scalar fields k, λ, and μ are assumed known because they are determined experimentally. We now need some thermodynamics in order to have a well posed system.

If we assume that the State of the fluid is determined by two state variables, meaning there exists a function f such that
$$f(\rho, p, T) = 0$$
where here T is temperature in Kelvin, a scalar field. In this case for some known scalars k, λ, μ, C_p and β which are functions of the thermodynamic variables, the energy equation can be rewritten as:
$$\rho C_p\frac{DT}{Dt} - \beta T\frac{D\rho}{Dt} = \mathrm{div}(k\nabla T) + \lambda(\mathrm{div}(\mathbf{u}))^2 + 2\mu\mathfrak{e}_{ij}\mathfrak{e}_{ij}$$
Now including f, this is six equations in the six unknowns ρ, p, T, and $\mathbf{u}$.

4 The Kamchatnov Paper

We now turn our attention to the paper ”Topological solitons in magnetohydrodynamics” [7]. The first part of this paper, after the introduction, gives a method for associating a vector field to the Hopf fibration from S^3 to S^2. This vector field turns out to satisfy a special case of the Navier-Stokes equations. However, there is a great deal to discuss before we delve into this paper if we wish to understand any of the meaning behind the relatively straight-forward calculations it involves.

4.1 Hopf Fibrations

4.1.1 Spheres and Stereographic projection

We begin by reviewing some differential geometry. Recall that as a set

$$S^n = \left\{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \right\} \subset \mathbb{R}^{n+1}$$

by definition, and S^n can be given the structure of an n dimensional manifold by stereographic projection, which will feature heavily throughout our study of [7]. The point $(0, \ldots, 0, -1) \in S^n$ is commonly referred to as the "south pole" of S^n because its on the "bottom" of the sphere $S^n \subset \mathbb{R}^{n+1}$. Similarly $(0, \ldots, 0, 1) \in S^n$ is called the north pole.

By removing each pole we get an atlas for S^n consisting of just two maps: stereographic projection from the north pole, and stereographic projection from the south pole. Lets look at these maps. Stereographic projection from the south pole, calling the south polse $\mathfrak{s}$, is defined as

$$\begin{aligned} \psi_{\mathfrak{s}} &: S^n \setminus \{\mathfrak{s}\} \to \mathbb{R}^n \\ (x^1, \ldots, x^n, x^{n+1}) &\mapsto \frac{1}{1 + x^{n+1}} (x^1, \ldots, x^n) \end{aligned}$$

and recall this is a diffeomorphism with inverse

$$\begin{aligned} \psi_{\mathfrak{s}}^{-1} &: \mathbb{R}^n \to S^n \setminus \{\mathfrak{s}\} \\ (y^1, \ldots, y^n) &\mapsto \left(\frac{2y^1}{1 + \|y\|^2}, \frac{2y^2}{1 + \|y\|^2}, \ldots, \frac{2y^n}{1 + \|y\|^2}, \frac{1 - \|y\|^2}{1 + \|y\|^2} \right) \end{aligned}$$

Therefore we can view S^n as $S^n = \mathbb{R}^n \cup \{\mathfrak{s}\}$ where $\mathfrak{s}$ represents a single point at infinity. Therefore, topologically speaking S^n is the one point compactification of $\mathbb{R}^n$.

Now lets call the north pole $\mathfrak{n}$. We can define

$$\begin{aligned} \psi_{\mathfrak{n}} &: S^n \setminus \{\mathfrak{n}\} \to \mathbb{R}^n \\ (x^1, \ldots, x^n, x^{n+1}) &\mapsto \frac{1}{1 - x^{n+1}} (x^1, \ldots, x^n) \end{aligned}$$

and recall this is a diffeomorphism with inverse

$$\begin{aligned} \psi_{\mathfrak{n}}^{-1} &: \mathbb{R}^n \to S^n \setminus \{\mathfrak{n}\} \\ (y^1, \ldots, y^n) &\mapsto \left(\frac{2y^1}{1 + \|y\|^2}, \frac{2y^2}{1 + \|y\|^2}, \ldots, \frac{2y^n}{1 + \|y\|^2}, \frac{\|y\|^2 - 1}{1 + \|y\|^2} \right) \end{aligned}$$

Now to show this forms an atlas we need only show that $\psi_{\mathfrak{s}} \circ \psi_{\mathfrak{n}}^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a diffeomorphism. We have $\mathbb{R}^n \setminus \{0\}$ here because

$$\psi_{\mathfrak{s}}(S^n \setminus \{\mathfrak{n}\} \cup S^n \setminus \{\mathfrak{s}\}) = \psi_{\mathfrak{s}}(S^n \setminus \{\mathfrak{n}, \mathfrak{s}\}) = \mathbb{R}^n \setminus \{0\}$$

because $\psi_{\mathfrak{s}}(\mathfrak{n}) = 0 \in \mathbb{R}^n$. Similarly the domain is also $\mathbb{R}^n \setminus \{0\}$. A straightforward calculation shows that

$$\psi_{\mathfrak{s}} \circ \psi_{\mathfrak{n}}^{-1}(y) = \frac{1}{\|y\|^2}(y^1, \ldots, y^n)$$

which is clearly smooth and is its own inverse, because

$$y \mapsto \frac{y}{\|y\|^2} \mapsto \frac{1}{\|\frac{y}{\|y\|^2}\|^2} \frac{y}{\|y\|^2} = \frac{\|y\|^4}{\|y\|^2} \frac{y}{\|y\|^2} = y$$

Therefore S^n is an n dimensional manifold through stereographic projection.

4.1.2 Which spheres are groups?

It should be noted that the next few sections are modelled on a talk given by Spiro Karigiannis, my Master's supervisor, at CUMC in 2010 [19]
Recall [16]:

An n - dimensional normed real division algebra $\mathbb{A}$ satisfies:

1. $\mathbb{A}$ is an n dimensional real vector space, with a norm $|\cdot|$
2. $\mathbb{A}$ has a ring structure with identity, for which any non-zero element is invertible
3. the norm is multiplicative, so for all $a, b \in \mathbb{A}$
$$|ab| = |a||b|$$

$$S^n = \left\{x \in \mathbb{R}^{n+1} : |x|^2 = 1\right\} \subset \mathbb{R}^{n+1}$$

Therefore:

$$S^0 = \left\{x \in \mathbb{R} : |x|^2 = 1\right\} = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$$
$$S^1 = \left\{x \in \mathbb{R}^2 : |x|^2 = 1\right\} \subset \mathbb{R}^2 \cong \mathbb{C}$$

Question: These are both groups. Are all spheres groups?
Answer: No. The reason these two are groups is because the norms are multiplicative, and these are the sets of unit norm elements, which is a group since the identity is included.

Given a normed division algebra $\mathbb{A}$, if $\mathbb{A}$ is associative, we can form

$$S_{\mathbb{A}} = \left\{x \in \mathbb{A} : |x|^2 = 1\right\}$$
$$\Updownarrow$$
$$S_{\mathbb{R}} = \left\{x \in \mathbb{R} : |x|^2 = 1\right\} = S^0$$
$$\Updownarrow$$
$$S_{\mathbb{C}} = \left\{x \in \mathbb{C} : |x|^2 = 1\right\} = S^1$$
$$\Updownarrow$$
$$S_{\mathbb{H}} = \left\{x \in \mathbb{H} : |x|^2 = 1\right\} = S^3$$

all of which are groups. What about if S^{n-1} is a group?

Lemma: 4.1.2.1. *Suppose S^{n-1} is a group, then we can induce an associative normed real division algebra structure on $\mathbb{R}^n$.*

Proof.

$$S^{n-1} = \left\{x \in \mathbb{R}^n : |x|^2 = 1\right\} \subset \mathbb{R}^n$$

take $a, b \in S^{n-1}$, then by the group structure $ab \in S^{n-1}$. Therefore

$$|a||b| = 1 \cdot 1 = 1 = |ab|$$

However every point $x \in \mathbb{R}^n$ is λa for $a \in S^{n-1}$ and $\lambda \in \mathbb{R}$. Therefore, given $x, y \in \mathbb{R}^n$ we can define, assuming $y = \gamma b$ for $b \in S^{n-1}$

$$xy = (\lambda\gamma)\, ab$$

Where the ab is the group product of a and b in S^{n-1}. Then for the norm conditions:

$$\|x\|\|y\| = |\lambda|\|a\||\gamma|\|b\| = |\lambda\gamma|\|ab\| = \|xy\|$$

This shows both that we have a norm, and that it is multiplicative. We take the identity for $\mathbb{R}^n$ as the identity e that must exist because S^{n-1} is assumed to be a group. Now take $x \in \mathbb{R}^n \setminus \{0\}$, we must show it has an inverse. Assume $x = \lambda a$, then $\lambda \neq 0$ so it has an inverse since $\lambda \in \mathbb{R} \setminus \{0\}$ and a has an inverse, so let $y = \lambda^{-1}a^{-1}$, then

$$xy = yx = \lambda\lambda^{-1}aa^{-1} = e$$

□

Notice here that the scalars were not passing through any vectors.

A theorem of Hurwitz [16], says that up to isomorphism the only real normed division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. This lemma says that if S^2 were a group, then $\mathbb{R}^3$ would have a real division algebra structure. By Hurwitz then, since $\mathbb{R}^3$ does not have a real division algebra structure, S^2 is not a group, and indeed any S^k for $k \neq 0, 1, 3$ is not a group.

Now we have

$$\mathbb{A} \text{ is an associative real normed division algebra of dimension n+1}$$

$$\Updownarrow$$

$$S_{\mathbb{A}} = S^n \text{ is a group}$$

Spiro showed last week that there are only three associative real normed division algebras, namely $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. We also have that the only reason $\mathbb{O}$ fail to be a group is because of the failure to be associative.

Therefore: The only spheres that are groups are S^0, S^1, S^3 and almost S^7, corresponding to normed real division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ on $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^4$, and $\mathbb{R}^8$

4.1.3 Projective Spaces and Hopf Fibrations

Def: Let $\mathbb{A}$ be an associative real normed division algebra. We define $\mathbb{A}$ projective space, $\mathbb{A}P^n$ to be the set of all $\mathbb{A}$-lines through the origin in $\mathbb{A}^{n+1}$, a compact, smooth n dimensional manifold.

An $\mathbb{A}$-line through the origin is of the form kp for $k \in \mathbb{A}$, and a fixed $p \in \mathbb{A}^{n+1}$ as k varies.

Formally we define $\sim$ to be an equivalence relation on $\mathbb{A}^{n+1} \setminus \{0\}$ where

$$x \sim y \Leftrightarrow x = \lambda y, \text{ for some } \lambda \in \mathbb{A}^*$$

Then $\mathbb{A}P^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \sim$

However there's a nicer way: Each $\mathbb{A}$-line through the origin intersects a sphere S^k where k depends both on which $\mathbb{A}$ and which n we're in. For example a $\mathbb{C}$ line through the origin in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ intersects S^{2n+1}. Therefore we can restrict $\sim$ to S^k so that as an equivalence relation on S^k

$$x \sim y \Leftrightarrow x = \lambda y, \lambda \in \mathbb{A}, |\lambda| = 1$$

Now if $\pi : S^k \to \mathbb{A}P^n$ is the map that takes each point to it's equivalence class under $\sim$, call it $\pi(x) = [x]$, then what is the fiber at each point $[x]$ in $\mathbb{A}P^n$? Exactly $[x]$ as a set in S^k:

$$\begin{aligned}
\pi^{-1}([x]) &= \big\{y \in S^k : \pi(y) = [x]\big\} \\
&= \big\{y \in S^k : [y] = [x]\big\} \\
&= \big\{y \in S^k : y \sim x\big\} = [x] \\
&= \big\{y \in S^k : \exists \lambda \in \mathbb{A}, |\lambda| = 1 : y = \lambda x\big\} \\
&= \{\lambda x \in \mathbb{H} : |\lambda| = 1\} \\
&\cong \{\lambda \in \mathbb{A} : |\lambda| = 1\} \\
&= S_{\mathbb{A}}
\end{aligned}$$

Therefore the fibres of each of these maps are themselves groups , copies of $S_{\mathbb{A}}$ associated to each x. Also since quotient maps are surjective, we've shown that each S^k can be thought of as a union of fibres, meaning we've shown that S^k can be thought of as $\mathbb{A}P^n$ with a copy of $S_{\mathbb{A}}$ attached at every point. Or equivalently a family of $S_{\mathbb{A}}$s parameterized by $\mathbb{A}P^n$.

We call S^k a fibration over $\mathbb{A}P^n$ with fiber $S_{\mathbb{A}}$ These ARE fiber bundles , trivializations are

$$\begin{aligned}
\pi^{-1}(U) &\to U \times S_{\mathbb{A}} \\
z &\mapsto ([z], z)
\end{aligned}$$

It is natural to try to define $\mathbb{O}P^n$ in an analogous way, but the non-associativity is an obstruction. It is possible to define $\mathbb{O}P^1$ and $\mathbb{O}P^2$, but $\mathbb{O}P^n$ is not defined for $n \geq 3$. For $n = 1$ there's a fibration $\pi : S^{15} \to \mathbb{O}P^1$ with fibre $S^7 = S_{\mathbb{O}}$

Its unsatisfying to have the k in S^k so we can write explicitly the maps we call the **Hopf Fibrations**:

$$\begin{CD} S^0 @>>> S^n \\ @. @VV{\pi}V \\ @. \mathbb{R}P^n \end{CD} \quad \begin{CD} S^1 @>>> S^{2n+1} \\ @. @VV{\pi}V \\ @. \mathbb{C}P^n \end{CD} \quad \begin{CD} S^3 @>>> S^{4n+3} \\ @. @VV{\pi}V \\ @. \mathbb{H}P^n \end{CD} \quad \begin{CD} S^1 @>>> S^{15} \\ @. @VV{\pi}V \\ @. \mathbb{O}P^1 \end{CD}$$

We make use of some exceptional isomorphisms: It can be shown using stereographic projection that

$$\mathbb{R}P^1 \cong S^1 \qquad \mathbb{C}P^1 \cong S^2 \qquad \mathbb{H}P^1 \cong S^4 \qquad \mathbb{O}P^1 \cong S^8$$

So for $n = 1$ we have

$$\begin{CD} S^0 @>>> S^1 \\ @. @VV{\pi}V \\ @. S^1 \end{CD} \quad \begin{CD} S^1 @>>> S^3 \\ @. @VV{\pi}V \\ @. S^2 \end{CD} \quad \begin{CD} S^3 @>>> S^7 \\ @. @VV{\pi}V \\ @. S^4 \end{CD} \quad \begin{CD} S^1 @>>> S^{15} \\ @. @VV{\pi}V \\ @. \mathbb{O}P^1 \end{CD}$$

4.1.4 The Quaternions and Rotations

Sidenote: You may be wondering if the cross product gives the structure of a normed division algebra to $\mathbb{R}^3$. The answer is no, because for example $|a \times b| \neq |a||b|$ in general.

This shows the cross product doesn't work, but we showed above that NO product would work.

Construction: Denote $\{e_1, e_2, e_3, e_4\} = \{1, i, j, k\}$, and define $1q = q1 = q$, $\forall q \in \mathbb{H}$ with the following diagram for rules of multiplication.

Conjugation is defined by $\overline{a + bi + cj + dk} = a - bi - cj - dk$ and it turns out that $\overline{pq} = \bar{q}\bar{p}$ when you do the calculation, which we leave as an exercise for the reader.

Properties:

1. $\mathbb{H}$ is a normed division algebra
2. $\mathbb{H}$ is not commutative, but is associative
3. therefore $S_{\mathbb{H}} = S^3$ is a group, known as $Sp(1)$ or $SU(2)$.

We can write each $q \in \mathbb{H}$ as $q = (t, \mathbf{v})$ for $t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^3$ Then the sum of two quaternions is still carried out componentwise, but the product formula is given by:

$$(t_1, \mathbf{v}_1)(t_2, \mathbf{v}_2) = (t_1 t_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, t_1 \mathbf{v}_2 + t_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

This formula is not a definition, but a direct consequence of the construction above. Here, since real numbers commute with vectors, we have $t_2 \mathbf{v}_1$ in the second component, but actually if we do not allow ourselves to commute, then the calculation yields $\mathbf{v}_1 t_2$. This will be important later when we are defining our differential operators on the quaternions.

Note this formula also means the multiplication of two purely imaginary quaternions is given by

$$(0, \mathbf{v}_1)(0, \mathbf{v}_2) = (-\mathbf{v}_1 \cdot \mathbf{v}_2, \mathbf{v}_1 \times \mathbf{v}_2)$$

Notice also that we can identify scalar multiplication by reals as

$$k(t, \mathbf{v}) = (k, 0)(t, \mathbf{v}) = (kt, k\mathbf{v}) = (t, \mathbf{v})(k, 0) = (t, \mathbf{v})k$$

Now consider the following product. Take $r = (t, \mathbf{v}) \in \mathbb{H} \setminus \{0\}$, and $q = (0, \mathbf{w}) \in Im(\mathbb{H})$, then since $|r|^2 = r\bar{r}$

$$r^{-1} = \frac{\bar{r}}{|r|^2}$$

Therefore

$$\begin{aligned}
rqr^{-1} &= \frac{1}{|r|^2}(t,\mathbf{v})(0,\mathbf{w})(t,-\mathbf{v}) \\
&= \frac{1}{|r|^2}(-\mathbf{v}\cdot\mathbf{w}, t\mathbf{w}+\mathbf{v}\times\mathbf{w})(t,-\mathbf{v}) \\
&= \frac{1}{|r|^2}(-t\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\cdot(t\mathbf{w}+\mathbf{v}\times\mathbf{w}), (\mathbf{v}\cdot\mathbf{w})\mathbf{v}+t(t\mathbf{w}+\mathbf{v}\times\mathbf{w})-(t\mathbf{w}+\mathbf{v}\times\mathbf{w})\times\mathbf{v}) \\
&= \frac{1}{|r|^2}(\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w}), (\mathbf{v}\cdot\mathbf{w})\mathbf{v}+t(t\mathbf{w}+\mathbf{v}\times\mathbf{w})+\mathbf{v}\times(t\mathbf{w}+\mathbf{v}\times\mathbf{w})) \\
&= \frac{1}{|r|^2}(\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w}), (\mathbf{v}\cdot\mathbf{w})\mathbf{v}+t^2\mathbf{w}+2t\mathbf{v}\times\mathbf{w}+\mathbf{v}\times(\mathbf{v}\times\mathbf{w}))
\end{aligned}$$

Now since $\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{v}\times\mathbf{v}) = 0$, we see that $rqr^{-1} \in Im(\mathbb{H})$, and recall that the vector triple product is

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

so

$$\mathbf{v}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{v})\mathbf{w}$$

and so finally

$$rqr^{-1} = \frac{1}{|r|^2}(0, 2(\mathbf{v}\cdot\mathbf{w})\mathbf{v} + (t^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{w} + 2t\mathbf{v}\times\mathbf{w})$$

properties of Ω_r

1. We have $rqr^{-1} \in Im(\mathbb{H})$.
2. Since we are identifying $\mathbb{R}^3 \cong Im(\mathbb{H})$, we can use fact 1 to define the following map as in Lyons-elem: Let $r \in \mathbb{H}$, and let

$$\begin{aligned}
&\Omega_r : \mathbb{R}^3 \to \mathbb{R}^3 \\
&(x,y,z) \mapsto xi+yj+zk \mapsto r(xi+yj+zk)r^{-1} = x'i+y'j+z'k \mapsto (x',y',z')
\end{aligned}$$

   by the above construction, we have shown explicitly that if $r = (t,\mathbf{v})$ is a unit length quaternion, then

$$\Omega_r(\mathbf{w}) = 2(\mathbf{v}\cdot\mathbf{w})\mathbf{v} + (t^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{w} + 2t\mathbf{v}\times\mathbf{w}$$

3. If we replace r with kr for $k \in \mathbb{R}\setminus\{0\}$ we see immediately that we get the same answer, meaning we get the same map for the whole line through the origin in $\mathbb{R}^4$. For this reason we may assume that $|r| = 1$ because every $r' \in \mathbb{H}$ is a real multiple of a unit length quaternion r. This also means that on the line kr in $\mathbb{H} \cong \mathbb{R}^4$ we have two representatives in S^3 that give the same map, antipodal points where the line intersects S^3.

4. $$\Omega_r \circ \Omega_s = \Omega_{rs}$$

$$\Omega_r \circ \Omega_s(p) = \pi(rsp\bar{s}\bar{r}) = \pi((rs)p(\overline{rs})) = \Omega_{rs}(p)$$

5. If $r \in \mathbb{H} \setminus \{0\}$, then by definition of a real division algebra, r has an inverse. We claim that Ω_r is also invertible, and that $(\Omega_r)^{-1} = \Omega_{r^{-1}}$. Well, by fact 3, we may assume $|r| = 1$, in which case $r^{-1} = \bar{r}$. Now

$$\Omega_r\left(\Omega_{r^{-1}}(\mathbf{x})\right) = \pi\left(r\left(r^{-1}(0,\mathbf{x})r\right)r^{-1}\right) = \mathbf{x}$$

where π projects onto the second factor. The other direction is almost the same.

6. By the ring structure of $\mathbb{H}$ we see that Ω_r is linear over $\mathbb{R}$, where here $r = (t, \mathbf{v})$:

$$\begin{aligned}
\Omega_r(a\mathbf{x} + b\mathbf{y}) &= 2(\mathbf{v}\cdot(a\mathbf{x}+b\mathbf{y}))\mathbf{v} + (t^2 - \mathbf{v}\cdot\mathbf{v})(a\mathbf{x}+b\mathbf{y}) + 2t\mathbf{v}\times(a\mathbf{x}+b\mathbf{y}) \\
&= a2(\mathbf{v}\cdot\mathbf{x})\mathbf{v} + 2b(\mathbf{v}\cdot\mathbf{y})\mathbf{v} + a(t^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{x} + b(t^2 - \mathbf{v}\cdot\mathbf{v})\mathbf{y} \\
&+ a2t\mathbf{v}\times\mathbf{x} + b2\mathbf{v}\times\mathbf{y} \\
&= a\Omega_r(\mathbf{x}) + b\Omega_r(\mathbf{y})
\end{aligned}$$

Moreover we can calculate Ω_r's matrix representation by calculating it's value on the basis $\{i, j, k\}$:

7. $$|\Omega_r(p)| = |rpr^{-1}| = |r||p||\frac{\bar{r}}{|r|^2}| = |p|$$

since $|\bar{r}| = |r|$.

8. Each Ω_r is an isometry. This can be seen by the fact that Ω_r is linear, so $|\Omega_r(p) - \Omega_r(q)| = |\Omega_r(p-q)| = |p-q|$, and this implies that $\langle\Omega_r(x), \Omega_r(y)\rangle = \langle x, y\rangle$, assuming $|z|^2 = \langle z, z\rangle$. Therefore, thinking of Ω_r as a matrix under our identification of $e_1, e_2, e_3 = i, j, k$ we have

$$\langle x, y\rangle = \langle\Omega_r(x), \Omega_r(y)\rangle = \langle x, \Omega_r^t\Omega_r(y)\rangle$$

and this is for every x, y, so $\Omega_r^t\Omega_r = I_{3\times 3}$, and Ω_r is orthogonal, and so has determinant ± 1.

What is this map Ω_r for? Note that every rotation in $\mathbb{R}^3$ can be encoded by a scalar representing the angle of rotation, and a vector indicating the axis of rotation. (In fact you could just use three numbers because you could use the length of the vector to indicate the angle, but we're not going to.)

Question: What would you expect the eigenstructure of a rotation to be?
It turns out [18] for $r = (t, \mathbf{v})$, Ω_r has eigenvalue 1, with eigenvector $\mathbf{v}$.

But we showed Ω_r is an isometry with invariant subspace spanned by the imaginary part of r. This means that its a rotation, but what is the angle of rotation θ? Take any vector w perpendicular to $Im(r)$, then

$$cos(\theta) = \frac{w \cdot \Omega_r(w)}{|w|^2} = 2Re(r)^2 - 1 \Rightarrow |Re(r)| = |cos(\frac{\theta}{2})|$$

Therefore the real part keeps track of the angle, and the imaginary part keeps track of the axis of rotation. It seems clear that every rotation can be written this way. However this does not constitute a proof. For an actual proof that every rotation is of this form, see [18]

Recall that $SO(3) = \{A \in SL(3, \mathbb{R}) : A^t A = I\}$. Therefore with what's given above, we can see that the set

$$\{\Omega_r : r \in S_{\mathbb{H}}, \det(\Omega_r) = 1\}$$

is exactly $SO(3)$ if we give it the group operation of composition.

Alternatively: By everything above, we have seen that to every $r \in \mathbb{H}$ we can identify a map Ω_r, and this map has an inverse. Moreover quaternion multiplication corresponds to map composition, so

$$\begin{aligned} S^3 &\to O(3) \\ r &\mapsto \Omega_r \end{aligned}$$

is a continuous (see the Reeder notes on quaternions for explicit construction of the matrix, this shows that the map is continuous.) group homomorphism, and our remark above that every rotation is Ω_r for some r corresponds to this map being surjective. Since S^3 is connected, its image under this map is connected. However $O(3)$ is not connected, and this maps $\pm 1 \mapsto I$, since $\cos(0) = 1$, so the image of this map is in $SO(3) \subset O(3)$. This shows that all of our maps Ω_r have determinant 1. Moreover we claim that the kernel of this map is just $\{\pm 1\}$.

Suppose $\Omega_r = I$, $r = (t, \mathbf{v})$ Let $\mathbf{w} \in \mathbb{R}^3$. Since $\Omega_r(\mathbf{w}) = \mathbf{w}$ for every $\mathbf{w}$, if we suppose $\mathbf{v} \neq 0$, without loss of generality we can take $\mathbf{w}$ nonzero and not colinear with $\mathbf{v}$. Now by definition of Ω_r,

$$\begin{aligned} (0, \mathbf{w}) &= (t, \mathbf{v})(0, \mathbf{w})(t, -\mathbf{v}) \\ (0, \mathbf{w})(t, \mathbf{v}) &= (t, \mathbf{v})(0, \mathbf{w}) \\ (-\mathbf{v} \cdot \mathbf{w}, t\mathbf{w} + \mathbf{w} \times \mathbf{v}) &= (-\mathbf{v} \cdot \mathbf{w}, t\mathbf{w} + \mathbf{v} \times \mathbf{w}) \end{aligned}$$

Now $\mathbf{v} \neq 0$, and $\mathbf{w} \times \mathbf{v} = \mathbf{v} \times \mathbf{w}$, which means $\mathbf{w} \times \mathbf{v} = 0$ which is a contradiction since we chose $\mathbf{w}$ not perpendicular or colinear with $\mathbf{v}$, and both vectors are nonzero.

Therefore $\mathbf{v} = 0$, so $r = \pm 1$. Therefore, since we remarked above that this map is surjective, by the first isomorphism theorem

$$S^3/\mathbb{Z}_2 \cong SO(3)$$

But we saw above that this is exactly $\mathbb{R}P^3$ in the above Hopf fibrations

Now

$$\{\Omega_r : r \in S_{\mathbb{H}}\} \cong SO(3)$$

because we have shown that every Ω_r has positive determinant.

Theorem: 4.1.4.1. $SO(3) \cong \mathbb{R}P^3$

Proof. omitted, but clear. □

4.1.5 The Octonians

Arthur Cayley constructed the Octonians. The method we used last week is called the Cayley - Dickson construction, in which, given an element $p \in \mathbb{O} \cong \mathbb{R}^8$, we associate a pair of quaternions by noting that $p = (s, t)$ for $s, t \in \mathbb{R}^4 \cong \mathbb{H}$, so taking $s, t \in \mathbb{H}$ in the standard way we did above in the previous section, we can think of p as a pair of quaternions. Then by the product we derived last week, we have, for $p = (s, t), q = (u, v) \in \mathbb{O}$,

$$pq = (s, t)(u, v) = (st - \bar{v}u, sv + u\bar{t})$$

where everything inside the brackets on the right is done with the quaternion product and conjugation. Basically what we've done here is map

$$\{e_1, \ldots, e_8\} \mapsto \{(1, 0), (i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k)\}$$

There is another crazy diagram, but its crazy.

4.1.6 The Classical Hopf Fibration

Now consider $S^3 \in \mathbb{R}^4$. As above we can write

$$S^3 = \left\{x \in \mathbb{R}^4 : \|x\| = 1\right\} \subset \mathbb{R}^4$$

or we can identify $\mathbb{R}^4 \cong \mathbb{C}^2$ by $(a, b, c, d) \mapsto (a + bi, c + di)$ in the usual way. The norm on $\mathbb{C}^2$ here is

$$\|(a + bi, c + di)\| = \sqrt{(a + bi)\overline{(a + bi)} + (c + di)\overline{(c + di)}} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

So that we have

$$\|(a, b, c, d)\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \|(a + bi, c + di)\|$$

so that under our identification we may rewrite S^3 as

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : z_1\overline{z_1} + z_2\overline{z_2} = 1\right\}$$

Consider, as in [9], the following equivalence relation on $S^3 \subset \mathbb{C}^2$:

$$(z_1, z_2) \sim (w_1, w_2) \Leftrightarrow (z_1, z_2) = \lambda(w_1, w_2) \text{ for some } \lambda \in \{\alpha \in C : |\alpha| = 1\}$$

Now define $\mathbb{C}P^1$ as the set of equivalence classes of $\mathbb{C}^2 \setminus \{0\}$ under $\sim$, which is a smooth manifold of complex dimension 2. Let $[z_1 : z_2]$ be the equivalence class of (z_1, z_2) under $\sim$, and let $\pi : S^3 \to \mathbb{C}P^1$ be defined by $\pi((z_1, z_2)) = [z_1 : z_2]$. π is a quotient map, and so surjective.

Recall that $\mathbb{C}P^1 \cong S^2$. To see this, recall that one the standard atlas on $\mathbb{C}P^1$ is [15]:

$$[z_1 : z_2] \mapsto \frac{z_1}{z_2}, [z_1 : z_2] \mapsto \frac{z_2}{z_1}$$

where the first map is on the open set where $z_2 \neq 0$ and the second map is on the open set where $z_1 \neq 0$. Consider the image of the first map, we have:

$$\frac{z_1}{z_2} = \frac{z_{11} + z_{12}i}{z_{21} + z_{22}i} = \frac{z_{11} + z_{12}i}{z_{21} + z_{22}i}\frac{z_{21} - z_{22}i}{z_{21} - z_{22}i} = \frac{1}{|z_2|^2}[(z_{11}z_{21} + z_{12}z_{22}) + i(z_{12}z_{21} - z_{11}z_{22})]$$

Which we can identify as an element of $\mathbb{R}^2$, so

$$\frac{z_1}{z_2} \mapsto \frac{1}{|z_2|^2}[z_{11}z_{21} + z_{12}z_{22}, z_{12}z_{21} - z_{11}z_{22}]$$

Now that we're back in $\mathbb{R}^2$, we can use stereographic projection to get to S^2, so under steregographic projection from the north pole

$$\begin{aligned}\frac{1}{|z_2|^2}(z_{11}z_{21} + z_{12}z_{22}, z_{12}z_{21} - z_{11}z_{22}) &\mapsto \left(\frac{2(z_{11}z_{21} + z_{12}z_{22})}{|z_2|^2(1 + \left(\frac{|z_1|}{|z_2|}\right)^2)}, \frac{2(z_{12}z_{21} - z_{11}z_{22})}{|z_2|^2(1 + \left(\frac{|z_1|}{|z_2|}\right)^2)}, \frac{\frac{|z_1|^2}{|z_2|^2} - 1}{1 + \frac{|z_1|^2}{|z_2|^2}}\right) \\ &\mapsto \left(2(z_{11}z_{21} + z_{12}z_{22}), 2(z_{12}z_{21} - z_{11}z_{22}), |z_1|^2 - |z_2|^2\right)\end{aligned}$$

because we started on S^3, so $|z_1|^2 + |z_2|^2 = 1$. So now starting from S^3 and tracing our progress to S^2, we have:

$$\begin{aligned}(a, b, c, d) &\mapsto (a + bi, c + di) \mapsto [(a + bi) : (c + di)] \mapsto \frac{a + bi}{c + di} \\ &\mapsto \frac{1}{c^2 + d^2}(ac + bd, bc - ad) \mapsto (2(ac + bd), 2(bc - ad), a^2 + b^2 - c^2 - d^2)\end{aligned}$$

The keen observer may object that we had a choice here, namely that we used the first of the two charts for $\mathbb{C}P^1$, which requires $z_2 \neq 0$. Suppose $z_2 = 0$, then $z_1 \neq 0$ by definition of $\mathbb{C}P^1$. Then

$$\begin{aligned}(z_1, z_2) &\mapsto \frac{z_2}{z_1} = \frac{1}{|z_1|^2}[(z_{11}z_{21} + z_{12}z_{22}) + i(z_{11}z_{22} - z_{21}z_{12})] \\ &\mapsto \left(2(z_{11}z_{21} + z_{12}z_{22}), 2(z_{11}z_{22} - z_{12}z_{21}), |z_1|^2 - |z_2|^2\right)\end{aligned}$$

which is the same map with the second coordinate multiplied by negative one, where here we used stereographic projection from the south pole.

The point, is that if we define the Hopf fibration

$$\begin{aligned}f : S^3 &\to S^2 \\ (a, b, c, d) &\mapsto (2(ac + bd), 2(bc - ad), a^2 + b^2 - c^2 - d^2)\end{aligned}$$

This is perfectly well defined for every point on S^3. Moreover for $z_2 = c + di = 0$, $f(z_1, z_2) = (0, 0, 1)$. This means that f is surjective because $\pi : S^3 \to \mathbb{C}P^1$ is surjective and the charts for CP^1 are diffeomorphisms from $\mathbb{C}P^1 \setminus \{p\}$ onto $\mathbb{R}^2$ after identification, where p is either $[z_1 : 0]$ or $[0 : z_2]$. If $p = [z_1 : 0]$ then p corresponds to a point at infinity, using stereographic projection from the north pole means we map back to $S^2 \setminus \{\mathfrak{n}\}$ from $\mathbb{R}^2$, which means $\mathfrak{n}$ is the point at infinity on S^2. Therefore composing the chart for $\mathbb{C}P^1$ with (the inverse of) stereographic projection from the north pole on S^2 we have a bijection $\mathbb{C}P^1 \setminus \{[z_1 : 0]\} \leftrightarrow S^2 \setminus \{\mathfrak{n}\}$ which we extend to a bijection $\mathbb{C}P^1 \leftrightarrow S^2$ by mapping $[z_1 : 0] \mapsto \mathfrak{n}$. f is clearly smooth since all of its component functions are smooth.

4.2 Kamchatnov's Algorithm

Kamchatnov's paper does not deal directly with the Navier-Stokes equations. Instead, the solution he induces to a special case of the Navier-Stokes equations arises in the form of a vector potential for a magnetic field, by which we mean a vector field $B \in \Gamma(T\mathbb{R}^3)$ such that

$$H = \mathrm{curl}(B)$$

is the magnetic field with the properties he desires. We will not discuss magnetic fields, and rather than H, we will soon show that it is this vector potential B that interests us. Lets follow Kamchatnov's Algorithm for constructing this vector potential.

Lemma 4.2.0.1. *Let $A \in \Omega^1(\mathbb{R}^3)$, $\psi : S^3 \setminus \{\mathfrak{s}\} \to \mathbb{R}^3$ be stereographic projection from the south pole, and let $\widehat{A} = \psi^* A$. Assume that $\widehat{A}^\sharp$ is tangent to S^3. Then the components of $\widehat{A} = \sum_{i=1}^{4} \widehat{A}_i du_i$ and $A = \sum_{j=1}^{3} A_j dx_j$ satisfy:*

$$\widehat{A}_i = \frac{1}{2}(1 + |x|^2)A_i - x_i \sum_{j=1}^{3} x_j A_j \text{ for } i = 1, 2, 3$$

$$\widehat{A}_4 = -\sum_{j=1}^{3} x_j A_j$$

$$A_i = (1 + u_4)\widehat{A}_i - u_i \widehat{A}_4 \text{ for } i = 1, 2, 3$$

where here $\{u_1, u_2, u_3, u_4\}$ are the the standard coordinates on $\mathbb{R}^4$ and $\{x_1, x_2, x_3\}$ are the standard coordinates on $\mathbb{R}^3$.

Proof. From the previous section we know ψ is a diffeomorphism we can pull back A by ψ:

$$\psi^* A = \psi^*(\sum_{j=1}^{3} A_j dx_j) = \sum_{j=1}^{3} (A_j \circ \psi)\, d(x_j \circ \psi)$$

Let $L = \{(u^1, u^2, u^3, -1) \in \mathbb{R}^4 : u^i \in \mathbb{R}\}$ which is clearly closed by taking sequences. As a map from $\mathbb{R}^4 \to \mathbb{R}^3$, we actually have $\psi : \mathbb{R}^4 \setminus \{L\} \to \mathbb{R}^3$ given by

$$\psi(u_1, u_2, u_3, u_4) = \frac{1}{1 + u^4}(u_1, u_2, u_3)$$

We have that $\mathbb{R}^4 \setminus \{L\} =: M$ is an open subset of $\mathbb{R}^4$ since L is closed, so u^1, u^2, u^3, u^4 are the natural coordinates to take on M. Notice $L \cap S^3 = S^3 \setminus \{\mathfrak{s}\}$. So we know $\psi^* A \in \Omega^1(M)$, so we can write

$$\psi^* A = \sum_{j=1}^{3} \widehat{A}_j du^j + \widehat{A}_4 du^4$$

for $B_i \in C^\infty(M)$. Now we wish to restrict this form to S^3, but restriction is by definition pullback by the inclusion, so we need the inclusion explicitly. Well because

$$S^3 = \left\{ u \in \mathbb{R}^4 : \|u\| = 1 \right\} \subset \mathbb{R}^4$$

we can take u^1, u^2, u^3 as independent coordinates on S^3, and write $u^4 = \pm\sqrt{1-(u^1)^2-(u^2)^2-(u^3)^2}$. Then the inclusion map has the form

$$\begin{aligned} \iota : S^3 &\hookrightarrow \mathbb{R}^4 \\ (u^1, u^2, u^3) &\mapsto (u^1, u^2, u^3, \pm\sqrt{1-(u^1)^2-(u^2)^2-(u^3)^2}) \end{aligned}$$

depending which square root we pick. Now

$$\iota^*(\psi^* A) = \sum_{j=1}^{3} \widehat{A}_j du^j + \widehat{A}_4 d(u^4 \circ \iota)$$

because ι does nothing to the first three coordinates by construction. For the fourth we have

$$\begin{aligned} d(u^4 \circ \iota) &= \pm d(\sqrt{1-(u^1)^2-(u^2)^2-(u^3)^2}) \\ &= \pm \sum_{i=1}^{3} \frac{\partial}{\partial u^i}(\sqrt{1-(u^1)^2-(u^2)^2-(u^3)^2}) du^i \\ &= \pm \sum_{i=1}^{3} \frac{1}{2}\left(1-(u^1)^2-(u^2)^2-(u^3)^2\right)^{-\frac{1}{2}}(-2u^i) du^i \\ &= \mp \sum_{i=1}^{3} \frac{u^i}{u^4} du^i \end{aligned}$$

Therefore

$$\begin{aligned} \iota^*(\psi^* A) &= \sum_{i=1}^{3} \widehat{A}_i du^i - \widehat{A}_4 \sum_{i=1}^{3} \frac{u^i}{u^4} du^i \\ &= \sum_{i=1}^{3} \left(\widehat{A}_i - \widehat{A}_4 \frac{u^i}{u^4} \right) du^i \end{aligned}$$

on the other hand, we can calculate $\iota^*(\psi^* A)$ directly, taking the first three u^i as the independent coordinates:

$$\begin{aligned} \iota^*(\psi^* A) &= \iota^*(\psi^* \sum_{i=1}^{3} A_i dx^i) \\ &= \sum_{i=1}^{3} (A_i \circ \psi \circ \iota)\, d\left(x^i \circ \psi \circ \iota\right) \end{aligned}$$

We suppress the $\circ\psi \circ \iota$ for the component functions A_i in what follows for clarity of notation. Now

$$d\left(x^i\circ\psi\circ\iota\right)=\sum_{j=1}^{3}\frac{\partial}{\partial u^j}\left(\frac{u^i}{1+u^4}\right)du^j$$
$$=\sum_{j=1}^{3}\left(\frac{\delta_i^j}{1+u^4}+\frac{u^iu^j}{u^4(1+u^4)^2}\right)du^j$$

where here u^4 is not a coordinate, but a function of the first three u^i. Putting it all together, we have

$$\iota^*(\psi^*A)=\iota^*(\psi^*\sum_{i=1}^{3}A_idx^i)$$
$$=\sum_{i=1}^{3}A_i\sum_{j=1}^{3}\left(\frac{\delta_i^j}{1+u^4}+\frac{u^iu^j}{u^4(1+u^4)^2}\right)du^j$$
$$=\sum_{j=1}^{3}\left(\frac{A_j}{1+u^4}+\sum_{i=1}^{3}\frac{A_iu^iu^j}{u^4(1+u^4)^2}\right)du^j$$

And now we equate our two versions of $\iota^*(\psi^*A)$ which of course must be equal:

$$\sum_{j=1}^{3}\left(\widehat{A}_j-\widehat{A}_4\frac{u^j}{u^4}\right)du^j=\iota^*(\psi^*A)=\sum_{j=1}^{3}\left(\frac{A_j}{1+u^4}+\sum_{i=1}^{3}\frac{A_iu^iu^j}{u^4(1+u^4)^2}\right)du^j$$

Equating coefficients gives us:

$$\widehat{A}_j-\widehat{A}_4\frac{u^j}{u^4}=\frac{A_j}{1+u^4}+\frac{u^j}{u^4}\frac{\sum_{i=1}^{3}A_iu^i}{(1+u^4)^2},\ \text{for } j=1,2,3$$

We still haven't used tangency of $\widehat{A}^\sharp$ to S^3. Notice the normal of S^3 at a point $(u^1,u^2,u^3,u^4)\in S^3$ is exactly the vector $(u^1,u^2,u^3,u^4)\in\mathbb{R}^4$. Therefore we can check that $\widehat{A}$ is tangent to S^3 simply by checking that

$$\sum_{i=1}^{4}\left(\widehat{A}_iu^i\right)=0$$

at every point on S^3. The keen observer will now notice that we have 4 equations, and so we can rewrite the $\widehat{A}_i$ in terms of the A_j and vice versa. Multiply both sides of our equation by u_j and sum over $j=1,2,3$, then we use the fact that $\sum_{j=1}^{4}(u^j)^2=1$ and the tangency condition:

$$\widehat{A}_j u_j - \widehat{A}_4 \frac{(u^j)^2}{u^4} = \frac{A_j u^j}{1+u^4} + \frac{(u^j)^2}{u^4} \frac{\sum_{i=1}^{3} A_i u^i}{(1+u^4)^2}$$

$$\sum_{j=1}^{3} \widehat{A}_j u_j - \widehat{A}_4 \frac{1-(u^4)^2}{u^4} = \frac{\sum_{j=1}^{3} A_j u^j}{1+u^4} + \frac{1-(u^4)^2}{u^4} \frac{\sum_{i=1}^{3} A_i u^i}{(1+u^4)^2}$$

$$\sum_{j=1}^{4} \widehat{A}_j u_j - \widehat{A}_4 u_4 - \widehat{A}_4 \frac{1-(u^4)^2}{u^4} = \frac{\sum_{j=1}^{3} A_j u^j}{1+u^4} + \frac{1-(u^4)^2}{u^4} \frac{\sum_{i=1}^{3} A_i u^i}{(1+u^4)^2}$$

Now the first term on the left is 0 by tangency. Multiply both sides by u^4 to get

$$-\widehat{A}_4 (u_4)^2 - \widehat{A}_4 (1-(u^4)^2) = \sum_{j=1}^{3} A_j u^j \left(\frac{u^4}{1+u^4} + \frac{1-(u^4)^2}{(1+u^4)^2} \right)$$

$$\widehat{A}_4 = -\sum_{j=1}^{3} A_j u^j \left(\frac{u^4}{1+u^4} + \frac{1-(u^4)^2}{(1+u^4)^2} \right)$$

$$\widehat{A}_4 = -\sum_{j=1}^{3} A_j u^j \left(\frac{u^4+(u^4)^2}{(1+u^4)^2} + \frac{1-(u^4)^2}{(1+u^4)^2} \right)$$

$$\widehat{A}_4 = -\frac{\sum_{j=1}^{3} A_j u^j}{(1+u^4)}$$

but now $x^i \circ \psi \circ \iota = \frac{u^i}{1+u^4}$, so suppressing the $\circ\psi \circ \iota$ for both A_j and x^j we have

$$\widehat{A}_4 = -\sum_{j=1}^{3} A_j x^j$$

We still need two equations. Consider again our coordinate equations

$$\widehat{A}_j - \widehat{A}_4 \frac{u^j}{u^4} = \frac{A_j}{1+u^4} + \frac{u^j}{u^4} \frac{\sum_{i=1}^{3} A_i u^i}{(1+u^4)^2}$$

We can now use our equation for $\widehat{A}_4$, and the fact that a short calculation shows that $\frac{1}{1+u^4} = \frac{1}{2}\left(1+|x|^2\right)$ and some simplification to get

$$\widehat{A}_j + \sum_{i=1}^{3} A_i x^i \frac{u^j}{u^4} = \frac{1}{2}\left(1+|x|^2\right) A_j + \sum_{i=1}^{3} A_i u^i \left(\frac{u^j}{u^4}\frac{1}{(1+u^4)^2}\right)$$

$$\widehat{A}_j + \sum_{i=1}^{3} A_i x^i \frac{u^j}{u^4} = \frac{1}{2}\left(1+|x|^2\right) A_j + \sum_{i=1}^{3} A_i x^i \left(\frac{u^j}{u^4}\frac{1}{(1+u^4)}\right)$$

$$\widehat{A}_j = \frac{1}{2}\left(1+|x|^2\right) A_j + \sum_{i=1}^{3} A_i x^i \frac{u^j}{u^4}\left(\frac{1}{(1+u^4)} - 1\right)$$

$$\widehat{A}_j = \frac{1}{2}\left(1+|x|^2\right) A_j + \sum_{i=1}^{3} A_i x^i \frac{u^j}{u^4}\left(\frac{-u^4}{1+u^4}\right)$$

$$\widehat{A}_j = \frac{1}{2}\left(1+|x|^2\right) A_j - x^j \sum_{i=1}^{3} A_i x^i$$

Which is another equation we wanted. For the third equation, we again use that $\widehat{A}_4 = -\sum_{j=1}^{3} A_j x^j$, this time to rewrite the right hand side of our coordinate equation:

$$\widehat{A}_j - \widehat{A}_4 \frac{u^j}{u^4} = \frac{A_j}{1+u^4} + \frac{u^j}{u^4}\frac{\sum_{i=1}^{3} A_i u^i}{(1+u^4)^2}$$

$$\widehat{A}_j - \widehat{A}_4 \frac{u^j}{u^4} = \frac{A_j}{1+u^4} - \frac{u^j}{u^4}\frac{\sum_{i=1}^{3} A_i x^i}{(1+u^4)}$$

$$\widehat{A}_j(1+u^4) - \widehat{A}_4 \frac{u^j}{u^4}(1+u^4) = A_j - \widehat{A}_4 \frac{u^j}{u^4}$$

$$A_j = \widehat{A}_j(1+u^4) + \widehat{A}_4\left(\frac{u^j}{u^4} - \frac{u^j}{u^4}(1+u^4)\right)$$

$$A_j = \widehat{A}_j(1+u^4) - u^j \widehat{A}_4$$

Which is the last equation we wanted.

□

5 Translating to 7 dimensions

5.1 Div, Curl, Grad, and Laplacian Revisited

5.1.1 Forms on Quaternions

In this section we use what was developed in section 2.3 to define analogous operators on forms, and to rewrite the exterior derivative in terms of those operators for each of the sets of $0, 1, 2$ and 3 forms.

For time dependent forms on $\mathbb{R}^3$, what we really mean is forms on the product manifold $\mathbb{R} \times \mathbb{R}^3$ where the first copy of $\mathbb{R}$ is a global time coordinate. Note this distinction is important, because if we passed to a general manifold M, by time dependent vector field on M, we mean a vector field on $\mathbb{R} \times M$, so that the first coordinate is global, while the coordinates on M are local, and we take the subset of forms or vector fields which take a zero coefficient on terms containing dt or $\frac{\partial}{\partial t}$. Denote time dependent vector fields on M by $\Gamma_t(TM)$ and time dependent forms on M by $\Omega_t^k(M)$

In our case, sticking with $\mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4 \cong \mathbb{H}$, we can think of every time dependent form as a form on the quaternions $\mathbb{H}$. Therefore

$$\begin{aligned}
\Omega^0(\mathbb{H}) &\ni f = f(t,x,y,z) \\
\Omega^1(\mathbb{H}) &\ni \alpha = f(t,x,y,z)dt + \alpha_i(t,x,y,z)dx^i \\
\Omega^2(\mathbb{H}) &\ni \alpha = \alpha_i(t,x,y,z)dt \wedge dx^i + \beta_{ij}(t,x,y,z)dx^i \wedge dx^j \\
\Omega^3(\mathbb{H}) &\ni \alpha = f(t,x,y,z)dx \wedge dy \wedge dz + \beta_{ij}(t,x,y,z)dt \wedge dx^i \wedge dx^j \\
\Omega^4(\mathbb{H}) &\ni \alpha = f(t,x,y,z)dt \wedge dx \wedge dy \wedge dz
\end{aligned}$$

It is clear that all of these addition signs correspond to L_2 orthogonal decompositions of the space because for example in the sum in $\Omega^1(\mathbb{H})$, $*dx^i$ contains a dt, so wedging with dt in the inner product will give zero, and similarly for the other two.

We can also immediately pick out some isomorphisms.

1. Clearly through $*$ we have

$$\Omega^4(\mathbb{H}) \cong \Omega^0(\mathbb{H}) = C^\infty(\mathbb{H})$$

2. In $\Omega^1(\mathbb{H})$, since there's no sum in $f(t,x,y,z)dt$, all 1-forms of this type are isomorphic to $\Omega^0(\mathbb{H})$ as well. Clearly the set of all forms like $\alpha_i(t,x,y,z)dx^i$ is just $\Omega_t^1(\mathbb{R}^3)$, giving us

$$\Omega^1(\mathbb{H}) \cong C^\infty(\mathbb{H}) \bigoplus_{\perp} \Omega_t^1(\mathbb{R}^3)$$

3. In $\Omega^2(\mathbb{H})$, we claim that

$$\begin{aligned}
\Omega_t^1(\mathbb{R}) &\to \Omega^2(\mathbb{H}) \\
\alpha &\mapsto dt \wedge \alpha
\end{aligned}$$

is a linear injective map. Linear is clear because $\wedge$ is bilinear. For injectivity, suppose $dt \wedge \alpha = dt \wedge \beta$, then $dt \wedge (\alpha - \beta) = 0$, but neither α or β has a factor which includes dt, so $\alpha = \beta$.

Now the set of all forms of the type $\alpha_i(t,x,y,z)dt \wedge dx^i$ is clearly the image of this map, and since injective linear maps are isomorphic to their images, the set of all forms of this type is isomorphic to $\Omega_t^1(\mathbb{R}^3)$.

Through $*$ for $\mathbb{R}^3$ we see that the set of forms of the type $\beta_{ij}(t,x,y,z)dx^i \wedge dx^j$ is isomorphic to $\Omega_t^1(\mathbb{R}^3)$. Putting it all together we have

$$\Omega^2(\mathbb{H}) \cong \Omega^1_t(\mathbb{R}^3) \underset{\perp}{\oplus} \Omega^1_t(\mathbb{R}^3)$$

4. In $\Omega^3(\mathbb{H})$ clearly the set of forms of the type $f(t,x,y,z)dx \wedge dy \wedge dz$ is isomorphic to $\Omega^0(\mathbb{H})$.

   For forms of the type $\beta_{ij}(t,x,y,z)dt \wedge dx^i \wedge dx^j$, using $*$ for $\mathbb{R}^4$ gives us $\Omega^1_t(\mathbb{R}^3)$. Therfore

$$\Omega^3(\mathbb{H}) \cong C^\infty(\mathbb{H}) \underset{\perp}{\oplus} \Omega^1_t(\mathbb{R}^3)$$

Now to summarize:

$$\begin{aligned}
\Omega^0(\mathbb{H}) &\cong \Omega^4(\mathbb{H}) = C^\infty(\mathbb{H}) \\
\Omega^1(\mathbb{H}) &\cong C^\infty(\mathbb{H}) \underset{\perp}{\oplus} \Omega^1_t(\mathbb{R}^3) \\
\Omega^2(\mathbb{H}) &\cong \Omega^1_t(\mathbb{R}^3) \underset{\perp}{\oplus} \Omega^1_t(\mathbb{R}^3) \\
\Omega^3(\mathbb{H}) &\cong C^\infty(\mathbb{H}) \underset{\perp}{\oplus} \Omega^1_t(\mathbb{R}^3)
\end{aligned}$$

5.1.2 The Exterior Derivative on Quaternions

Now we can write the exterior derivative

$$d : \Omega^k(\mathbb{H}) \to \Omega^{k+1}(\mathbb{H})$$

in terms of $\text{div}, \text{grad}, \text{curl}$ on time dependent forms of $\mathbb{R}^3$, which we've repackaged as $\mathbb{H}$. Note that all the coefficient functions below are functions of t, x, y, z, but we've suppressed this so we can fit it all in one pdf.

1. Let $f \in \Omega^0(\mathbb{H})$, then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x^i}dx^i$$

2. Let $\alpha = fdt + \alpha_i dx^i \in \Omega^1(\mathbb{H})$, then

$$\begin{aligned}
d\alpha &= d(f dt + \alpha_i dx^i) \\
&= df \wedge dt + d\alpha_i \wedge dx^i \\
&= \left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x^i} dx^i\right) \wedge dt + \left(\frac{\partial \alpha_i}{\partial t} dt + \frac{\partial \alpha_i}{\partial x^j} dx^j\right) \wedge dx^i \\
&= \frac{\partial f}{\partial x^i} dx^i \wedge dt + \sum_i \frac{\partial \alpha_i}{\partial t} dt \wedge dx^i + \sum_i \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i \\
&= \operatorname{grad} f \wedge dt + dt \wedge \left(\frac{\partial}{\partial t} \Im(\alpha)\right) + \sum_i \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i \\
&= \left(\operatorname{grad} f - \frac{\partial}{\partial t} \Im(\alpha)\right) \wedge dt + \sum_i \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i \\
&= \left(\operatorname{grad} f - \frac{\partial}{\partial t} \Im(\alpha)\right) \wedge dt + * \operatorname{curl} \Im(\alpha)
\end{aligned}$$

Because for the second term, we mentioned above that this sum is $* \operatorname{curl} \Im(\alpha)$

3. Let $\alpha = \alpha_i dt \wedge dx^i + \beta_{ij} dx^i \wedge dx^j \in \Omega^2(\mathbb{H})$, then we can rewrite this as $\alpha' = \alpha_i dx^i \in \Omega^1_t(\mathbb{R}^3)$, $\beta = \beta_{ij} dx^i \wedge dx^j \in \Omega^2_t(\mathbb{R}^3)$, so $\alpha = dt \wedge \alpha' + \beta$

$$\begin{aligned}
d\alpha &= d(\alpha_i dt \wedge dx^i + \beta_{ij} dx^i \wedge dx^j) \\
&= (d\alpha_i) \wedge dt \wedge dx^i + (d\beta_{ij}) \wedge dx^i \wedge dx^j \\
&= \left(\frac{\partial \alpha_i}{\partial t} dt + \frac{\partial \alpha_i}{\partial x^k} dx^k\right) \wedge dt \wedge dx^i + \left(\frac{\partial \beta_{ij}}{\partial t} dt + \frac{\partial \beta_{ij}}{\partial x^k} dx^k\right) \wedge dx^i \wedge dx^j \\
&= \frac{\partial \alpha_i}{\partial x^k} dx^k \wedge dt \wedge dx^i + \frac{\partial \beta_{ij}}{\partial t} dt \wedge dx^i \wedge dx^j + \frac{\partial \beta_{ij}}{\partial x^k} dx^i \wedge dx^j \wedge dx^k \\
&= dt \wedge \left(\frac{\partial \alpha_i}{\partial x^k} dx^i \wedge dx^k\right) + dt \wedge \left(\frac{\partial \beta_{ij}}{\partial t} dx^i \wedge dx^j\right) + \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \frac{\partial \beta_{\sigma(1)\sigma(2)}}{\partial x^{\sigma(3)}} dx^1 \wedge dx^2 \wedge dx^3 \\
&= dt \wedge \left(* \operatorname{curl}(\alpha') + \frac{\partial}{\partial t} \beta\right) + \left(\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \frac{\partial \beta_{\sigma(1)\sigma(2)}}{\partial x^{\sigma(3)}}\right) dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}$$

But now

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \frac{\partial \beta_{\sigma(1)\sigma(2)}}{\partial x^{\sigma(3)}} = \operatorname{div}(\beta_{12} - \beta_{21}, \beta_{23} - \beta_{32}, \beta_{31} - \beta_{13}) dx^1 \wedge dx^2 \wedge dx^3$$

Let $\beta' = (\beta_{12} - \beta_{21}, \beta_{23} - \beta_{32}, \beta_{31} - \beta_{13})$, then we can write

$$d\alpha = dt \wedge \left(* \operatorname{curl}(\alpha') + \frac{\partial}{\partial t} \beta\right) + \operatorname{div}(\beta') dx^1 \wedge dx^2 \wedge dx^3$$

4. Let $\alpha = f dx \wedge dy \wedge dz + \beta_{ij} dt \wedge dx^i \wedge dx^j \in \Omega^3(\mathbb{H})$ then

$$\begin{aligned}
d\alpha &= d\left(f dx \wedge dy \wedge dz + \beta_{ij} dt \wedge dx^i \wedge dx^j\right) \\
&= d\left(f dx \wedge dy \wedge dz\right) + d\left(\beta_{ij} dt \wedge dx^i \wedge dx^j\right) \\
&= df \wedge dx \wedge dy \wedge dz + (d\beta_{ij}) \wedge dt \wedge dx^i \wedge dx^j \\
&= \left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x^i} dx^i\right) \wedge dx \wedge dy \wedge dz + \left(\frac{\partial \beta_{ij}}{\partial t} dt + \frac{\partial \beta_{ij}}{\partial x^k} dx^k\right) \wedge dt \wedge dx^i \wedge dx^j \\
&= \frac{\partial f}{\partial t} dt \wedge dx \wedge dy \wedge dz + \frac{\partial \beta_{ij}}{\partial x^k} dt \wedge dx^i \wedge dx^k \wedge dx^j \\
&= \frac{\partial f}{\partial t} dt \wedge dx \wedge dy \wedge dz - dt \wedge \left(\frac{\partial \beta_{ij}}{\partial x^k} dx^i \wedge dx^j \wedge dx^k\right) \\
&= \frac{\partial f}{\partial t} dt \wedge dx \wedge dy \wedge dz - dt \wedge \left(div(\beta') dx \wedge dy \wedge dz\right) \\
&= dt \wedge \left(\left(\frac{\partial f}{\partial t} - div(\beta')\right) dx \wedge dy \wedge dz\right)
\end{aligned}$$

5.1.3 Newtonian Navier-Stokes Revisited

Recall we had the following equations for Navier-Stokes, assuming we have a Newtonian fluid with constant viscocities :

1. The Continuity Equation (9) :
$$\frac{D\rho}{Dt} + \rho \operatorname{div}(\mathbf{u}) = 0$$
2. The Momentum Equation (15) for Newtonian fluids :
$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{b} - \nabla p + (\lambda + \mu)\nabla(\operatorname{div}(\mathbf{u})) + \mu \nabla^2 \mathbf{u}$$
3. The Internal Energy Equation (16) for Newtonian fluids:
$$\rho \frac{De}{Dt} = \operatorname{div}(k \nabla T) - p \operatorname{div}(\mathbf{u}) + \lambda(\operatorname{div}(\mathbf{u}))^2 + 2\mu \mathfrak{e}_{ij}\mathfrak{e}_{ij}$$

Recall also that $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho$ for our fixed velocity field $\mathbf{u}$, so the continuity equation is just $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0$.

Let $w = (\rho, \mathbf{u}) \in \mathbb{H}$, then we can write the continuity equation as $\frac{\partial}{\partial t} \Re w + \operatorname{div}(\Re w \Im w) = 0$

For the momentum equation we use the fact that $\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$. Recall that for a vector field A,

$$\frac{1}{2}\nabla(A \cdot A) = A \times \operatorname{curl}(A) + A \cdot \nabla A \Rightarrow A \cdot \nabla A = \frac{1}{2}\nabla(A \cdot A) + \operatorname{curl}(A) \times A$$

5.2 Quaternionic Navier-Stokes

5.2.1 Quaternionic Formulation of Maxwell's Equations

In this section we review the content of [5], reformulating the Maxwell equations of electromagnetism in terms of the complex quaternions. Unless otherwise noted, all information in this section is from that book. We wish to study this reformulation in order to show the inspiration for adapting this method to reformulate the Navier-Stokes equations in terms of the complex quaternions.

Let

$$A = \mathbb{H} \otimes \mathbb{C} = \mathbb{H} + i\mathbb{H}$$

where the tensor product is over $\mathbb{R}$. We must be careful what we mean here. This i is not the i from the basis for $\mathbb{H}$ as a vector space, but the i from the copy of $\mathbb{C}$ we tensored with. For this reason we will adopt the convention that if we need to mention the basis $\{1, i, j, k\}$ for $\mathbb{H}$, we will write these as $\{1, v_1, v_2, v_3\}$, and we'll call $\text{span}\{v_1, v_2, v_3\} = V \cong \mathbb{R}^3$

We must define the interaction of i with $\mathbb{H}$, but we do so by declaring that

$$i(qr) = (iq)(r) = (q)(ir) = (qr)i$$

We also define

$$B = \mathbb{R} \oplus iV$$

Compare this with $\mathbb{H} = \mathbb{R} \oplus V$, and it is clear B is a linear subspace of A over $\mathbb{R}$. Note also that $\{1, iv_1, iv_2, iv_3\}$ is a basis for B.

We define the complex conjugate of $a = q + ir \in A$ as

$$\overline{a} = q - ir$$

and we note that $\overline{ad} = \overline{a}\overline{d}$, whose proof is left as an exercise.

Take coordinates $(t, x_1, x_2, x_3) = (t, x, y, z)$ on $\mathbb{R}^4$. Define the operator

$$D = \left(\frac{\partial}{\partial t}, i\nabla\right)$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}$$

is the differential operator familiar from vector calculus. We'll call D the Hamilton-Maxwell operator. Clearly D can act on mappings of the form $\theta : \mathbb{R}^4 \to A$.

In particular, let $E, H : \mathbb{R}^4 \to V \subset B \subset A$ be electric and magnetic fields respectively. Set $\theta = E + iH$, then $\theta : \mathbb{R}^4 \to V + iV \subset A$. This means we can write $\theta = (0, E + iH)$ in complex quaternionic form, because E, H take values in V.

Recall that the product on $\mathbb{H}$ is defined as

$$(t_1, \mathbf{v}_1)(t_2, \mathbf{v}_2) = (t_1 t_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, t_1 \mathbf{v}_2 + \mathbf{v}_1 t_2 + \mathbf{v}_1 \times \mathbf{v}_2)$$

Now suppose $D(\theta) = 0$, then

$$\begin{aligned} 0 &= D(\theta) \\ &= \left(\frac{\partial}{\partial t}, i\nabla\right)(0, E + iH) \\ &= \left(-i \operatorname{div} E - i^2 \operatorname{div} H, \frac{\partial E}{\partial t} + i\frac{\partial H}{\partial t} + i \operatorname{curl} E + i^2 \operatorname{curl} H\right) \\ &= \left(-i \operatorname{div} E + \operatorname{div} H, \frac{\partial E}{\partial t} + i\frac{\partial H}{\partial t} + i \operatorname{curl} E - \operatorname{curl} H\right) \end{aligned}$$

But now we can equate real and imaginary parts in both the scalar and vector components to get

$$\begin{aligned} \frac{\partial E}{\partial t} &= \operatorname{curl}(H) \\ \frac{\partial H}{\partial t} &= -\operatorname{curl}(E) \\ \operatorname{div} E &= 0 \\ \operatorname{div} H &= 0 \end{aligned}$$

which are the free Maxwell equations. Lets add a source term, that is, consider $D(\theta) = (-i\rho, -J)$ for ρ the charge density and J the electric current. Then comparing real and imaginary parts in

$$\left(-i \operatorname{div} E + \operatorname{div} H, \frac{\partial E}{\partial t} + i\frac{\partial H}{\partial t} + i \operatorname{curl} E - \operatorname{curl} H\right) = (-i\rho, -J)$$

gives us

$$\begin{aligned} \frac{\partial E}{\partial t} &= \operatorname{curl}(H) - J \\ \frac{\partial H}{\partial t} &= -\operatorname{curl}(E) \\ \operatorname{div} E &= \rho \\ \operatorname{div} H &= 0 \end{aligned}$$

which are the maxwell equations with an electric, but no magnetic, source.

5.3 Maxwell's Equations in Differential Forms

In this section we look at Maxwell's equations as described by differential forms, as discussed in [4]. Unless otherwise stated ever result we use in this section is from that excellent book.

5.3.1 Some Preliminary Facts

Take a manifold (M, η) where η is a metric of index $(3, 1)$, called a Lorentz metric. In particular we take η of the form

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is called the Minkowski metric. Note that it is not positive definite, so in fact its not a metric at all, but it is still called a metric everywhere, and we do not break with that regrettable tradition here. Now (M, η) is called a Lorentz manifold. We now list some facts from [4] that we will need.

Lemma: 1. *If M is a Lorentz manifold of dimension n, then*

1. *Let $d : \Omega^{k-1}(M) \to \Omega^k(M)$ be the exterior derivative, then the adjoint of d is*

$$d^* : \Omega^k(M) \to \Omega^{k-1}(M)$$
$$d^* = (-1)^{n(k+1)} * d*$$

2. *The hodge star, $*$, has the property that*

$$*^2 = (-1)^{1+k(n-k)} Id$$

We now specialize to the case where $M = \mathbb{R}^4$, with coordinates (t, x, y, z). In this case our lemma gives us

1. $d^* = *d*$
2. $*^2 = (-1)^{k+1} Id$

because $n = 4$, and one can check easily that $(-1)^{k+1} = (-1)^{1+k(4-k)}$ for $k = 0, 1, 2, 3, 4$.

5.3.2 The Electromagnetic Tensor

Now let $A = \phi dt + A_i dx^i \in \Omega^1(\mathbb{R}^4)$, where $i = 1, 2, 3$. Define $F = dA$, the electromagnetic tensor, so

$$\begin{aligned} F &= dA \\ &= d\left(\phi dt + A_i dx^i\right) \\ &= d\phi \wedge dt + dA_i \wedge dx^i \\ &= \left(\frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x^i} dx^i\right) \wedge dt + \left(\frac{\partial A_i}{\partial t} dt + \frac{\partial A_i}{\partial x^j} dx^j\right) \wedge dx^i \\ &= -dt \wedge \left(\frac{\partial \phi}{\partial x^i} dx^i\right) + dt \wedge \left(\frac{\partial A_i}{\partial t} dx^i\right) + \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \\ &= dt \wedge \left(\left[\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x^i}\right] dx^i\right) + \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \end{aligned}$$

Now use the curl map on one forms defined above, and define

$$B = \text{curl}(A) = *_3 d_3 A = *_3 \left(\frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \right)$$

since $*_3^2 = Id$, we have

$$*_3 B = \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i$$

Let $E = -\left[\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x^i}\right] dx^i$, so $E^\sharp = \nabla \phi - \frac{\partial A}{\partial t}$. Then we can rewrite F as

$$F = dA = -dt \wedge E + *_3 B$$

but now since $d^2 = 0$, $dF = d^2 A = 0$. On the other hand

$$\begin{aligned} dF &= d\left(-dt \wedge E + *_3 B\right) \\ &= dt \wedge dE + d\left(\frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i\right) \\ &= dt \wedge dE + \frac{\partial}{\partial t}\frac{\partial A_i}{\partial x^j} dt \wedge dx^j \wedge dx^i + \frac{\partial}{\partial x^k}\frac{\partial A_i}{\partial x^j} dx^k \wedge dx^j \wedge dx^i \end{aligned}$$

but notice in the first term that $d = d_4$ has a dt term which will drop out since we're wedging with dt, so $dt \wedge dE = dt \wedge d_3 E$. We also have that the last term is $d_3 *_3 B$. Now

$$\begin{aligned} 0 = dF &= dt \wedge (d_3 E) + dt \wedge \left(\frac{\partial}{\partial t}\frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i\right) + d_3 *_3 B \\ &= dt \wedge \left(\frac{\partial *_3 B}{\partial t} + d_3 E\right) + d_3 *_3 B \end{aligned}$$

Which means

$$\begin{aligned} \frac{\partial *_3 B}{\partial t} + d_3 E = 0 \\ d_3 *_3 B = 0 \end{aligned}$$

Notice that $\frac{\partial}{\partial t} *_3 = *_3 \frac{\partial}{\partial t}$ because $*_3$ is linear. Therefore we can take $*_3$ of the first equation to get

$$\frac{\partial B}{\partial t} = - *_3 d_3 E = -\,\text{curl}(E)$$

So we can identify these 1 forms with their musical isomorphism dual vector fields.

In the second equation, take $*_3$ of both sides, and use the fact that for a vector field X, $\text{div}(X) = *_3 d_3 *_3 X^\flat$, we see that

$$\text{div}(B^\sharp) = 0$$

However since we're in $\mathbb{R}^3$, we can identify B with $B^\sharp$ since their coefficient functions are identical, and so $\text{div}(B) = 0$.

Therefore, since $d^2 = 0$, $dF = 0$ implies

$$\begin{aligned} \operatorname{div}(B) &= 0 \\ \frac{\partial B}{\partial t} &= -\operatorname{curl}(E) \end{aligned}$$

which are two of the four Maxwell equations.

5.3.3 The Current One-Form

We want the other two of course. Define the current one-form $J = -\rho dt + j_i dx^i \in \Omega^1(\mathbb{R}^4)$ with ρ the electric charge density and (j_1, j_2, j_3) the electric current density. Suppose that $d^*F = J$, which is really a condition on A, then since J is a 1 form, we know that $*_4^2 J = J$ by the lemma. Therefore

$$d^*F = J \Leftrightarrow *_4 d *_4 F = J \Leftrightarrow d *_4 F = *_4 J$$

where we've used the fact that $d * F$ is a 3 form, and so again $*_4^2 = Id$. Now we calculate:

$$\begin{aligned} d *_4 F &= d *_4 (-dt \wedge E + *_3 B) \\ &= d *_4 \left(-dt \wedge E + \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \right) \\ &= d \left(- *_4 dt \wedge E + *_4 \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \right) \end{aligned}$$

Lets do each term separately:

$$*_4 dt \wedge \left(\left[\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x^i} \right] dx^i \right) = \left[\frac{\partial A_i}{\partial t} - \frac{\partial \phi}{\partial x^i} \right] *_4 dt \wedge dx^i$$

Now a quick calculation shows that

$$\begin{aligned} *_4 dt \wedge dx &= dz \wedge dy = - *_3 dx \\ *_4 dt \wedge dy &= dx \wedge dz = - *_3 dy \\ *_4 dt \wedge dz &= dy \wedge dx = - *_3 dz \end{aligned}$$

so that $*_4 dt \wedge dx^i = - *_3 dx^i$ Therefore if $\alpha = \alpha_i dx^i$,

$$\begin{aligned} *_4(dt \wedge \alpha) &= *_4(\alpha_i dt \wedge dx^i) \\ &= \alpha_i *_4 dt \wedge dx^i \\ &= \alpha_i(- *_3 dx^i) \\ &= - *_3 \alpha \end{aligned}$$

Now we can use this fact:

$$*_4 dt \wedge E = - *_3 E$$

So now we have the first term. For the second, another calculation shows that

$$\begin{aligned}
*_4 dx \wedge dy &= dt \wedge dz \\
*_4 dy \wedge dz &= dt \wedge dx \\
*_4 dx \wedge dz &= dy \wedge dt
\end{aligned}$$

Therefore

$$\begin{aligned}
&*_4 \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \\
&= \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) *_4 dx \wedge dy + \left(\frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3}\right) *_4 dx \wedge dz + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) *_4 dy \wedge dz \\
&= \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) dt \wedge dz + \left(\frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3}\right) dy \wedge dt + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) dt \wedge dx \\
&= dt \wedge \left(\left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) dx + \left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right) dy + \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) dz\right)
\end{aligned}$$

However we also have that

$$\begin{aligned}
&*_3 \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i \\
&= \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) *_3 dx \wedge dy + \left(\frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3}\right) *_3 dx \wedge dz + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) *_3 dy \wedge dz \\
&= \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) dz + \left(\frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3}\right) (-dy) + \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right) dx
\end{aligned}$$

Therefore

$$*_4 *_3 B = *_4 \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i = dt \wedge \left(*_3 \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i\right) = dt \wedge B$$

since $*_3^2 = Id$. Putting it all together, we have shown that

$$*_4 F = *_3 E + dt \wedge B$$

Now we need to take $d = d_4$ of this.

$$\begin{aligned}
d *_4 F &= d *_3 E + d(dt \wedge B) \\
&= d *_3 E - dt \wedge d_3 B
\end{aligned}$$

where the second term is using d_3 since a term with dt is zero when wedged with dt. For the first term, we can explicitly write out $d *_3 E$:

$$
\begin{aligned}
d *_3 E &= d *_3 \left(E_i dx^i\right) \\
&= d\left(E_i *_3 dx^i\right) \\
&= d\left(E_1 dy \wedge dz + E_2 dz \wedge dx + E_3 dx \wedge dy\right)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
d(E_1 dy \wedge dz) &= \left(\frac{\partial E_1}{\partial t} dt + \frac{\partial E_1}{\partial x^i} dx^i\right) \wedge dy \wedge dz \\
&= \frac{\partial E_1}{\partial t} dt \wedge dy \wedge dz + \frac{\partial E_1}{\partial x} dx \wedge dy \wedge dz \\
&= dt \wedge *_3 \frac{\partial E_1}{\partial t} dx + \frac{\partial E_1}{\partial x} dx \wedge dy \wedge dz \\
d(E_2 dz \wedge dx) &= \left(\frac{\partial E_2}{\partial t} dt + \frac{\partial E_2}{\partial x^i} dx^i\right) \wedge dz \wedge dx \\
&= \frac{\partial E_2}{\partial t} dt \wedge dz \wedge dx + \frac{\partial E_2}{\partial y} dx \wedge dy \wedge dz \\
&= dt \wedge *_3 \frac{\partial E_2}{\partial t} dy + \frac{\partial E_2}{\partial y} dx \wedge dy \wedge dz \\
d(E_3 dx \wedge dy) &= \left(\frac{\partial E_2}{\partial t} dt + \frac{\partial E_3}{\partial x^i} dx^i\right) \wedge dx \wedge dy \\
&= \frac{\partial E_3}{\partial t} dt \wedge dx \wedge dy + \frac{\partial E_3}{\partial z} dx \wedge dy \wedge dz \\
&= dt \wedge *_3 \frac{\partial E_3}{\partial t} dz + \frac{\partial E_3}{\partial z} dx \wedge dy \wedge dz
\end{aligned}
$$

Therefore

$$
d *_3 E = dt \wedge *_3 \frac{\partial E}{\partial t} + \left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right) dx \wedge dy \wedge dz
$$

However

$$
*_3 \left(\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}\right) dx \wedge dy \wedge dz = \operatorname{div}(E) = *_3 d_3 *_3 E
$$

so the second term is $d_3 *_3 E$, and so

$$
d *_3 E = dt \wedge *_3 \frac{\partial E}{\partial t} + d_3 *_3 E
$$

Now we have shown that

$$
d *_4 F = dt \wedge *_3 \frac{\partial E}{\partial t} + d_3 *_3 E - dt \wedge d_3 B
$$

We have assumed that $d *_4 F = *_4 J$, so what s $*_4 J$?

$$\begin{aligned}
*_4 J &= -\rho *_4 dt + j_i *_4 dx^i \\
&= \rho dx \wedge dy \wedge dz + j_1 dt \wedge dz \wedge dy + j_2 dt \wedge dx \wedge dz + j_3 dt \wedge dy \wedge dx \\
&= \rho dx \wedge dy \wedge dz - dt \wedge j_1 *_3 dx - dt \wedge j_2 *_3 dy - dt \wedge j_3 *_3 dz \\
&= \rho dx \wedge dy \wedge dz - dt \wedge (*_3 j_i dx^i)
\end{aligned}$$

Now we have

$$dt \wedge *_3 \frac{\partial E}{\partial t} + d_3 *_3 E - dt \wedge d_3 B = \rho dx \wedge dy \wedge dz - dt \wedge (*_3 j_i dx^i)$$

which gives us

$$\begin{aligned}
*_3 \frac{\partial E}{\partial t} - d_3 B &= - *_3 j_i dx^i \\
-\frac{\partial E}{\partial t} + *_3 d_3 B &= j_i dx^i
\end{aligned}$$

but recall that$*_3 d_3 B = \text{curl}(B)$, so we have the Maxwell equation

$$-\frac{\partial E}{\partial t} + \text{curl}(B) = (j_1, j_2, j_3)$$

and we also have

$$d_3 *_3 E = \rho dx \wedge dy \wedge dz$$

and taking $*_3$ of both sides gives us the other Maxwell equation we're missing, namely

$$\text{div}(E) = \rho$$

So now lets summarize: We let $A = \phi dt + A_i dx^i \in \Omega^1(\mathbb{R}^4)$. We let $F = dA = -dt \wedge E + *_3 B$. Since F is exact its closed, so the equation $dF = 0$ gave us the first two of Maxwell's equations below.

We then demanded that A was chosen such that $d^* F = J = -\rho dt + j_i dx^i \in \Omega^1(\mathbb{R}^4)$. This equation gave us the second two of the Maxwell's equation below.

$$\begin{aligned}
\text{div}(B) &= 0 \\
\frac{\partial B}{\partial t} &= -\,\text{curl}(E) \\
-\frac{\partial E}{\partial t} + \text{curl}(B) &= (j_1, j_2, j_3) \\
\text{div}(E) &= \rho
\end{aligned}$$

5.3.4 The Charge Conservation Equation

Notice that $d^*F = J$ means that $d^*J = (d^*)^2 F = 0$. We know from the lemma above that $d^* = *d*$, so we have

$$*d*J = 0 \Leftrightarrow d*J = 0$$

Well we found $*J = \rho dx \wedge dy \wedge dz - dt \wedge (*_3 j_i dx^i)$ in the previous section. Now we take $d = d_4$

$$\begin{aligned} 0 &= d*J \\ &= d\left(\rho dx \wedge dy \wedge dz - dt \wedge (*_3 j_i dx^i)\right) \\ &= \frac{\partial \rho}{\partial t} dt \wedge dx \wedge dy \wedge dz + dt \wedge (d *_3 j_i dx^i) \end{aligned}$$

However in the second term we can take d_3 instead of d_4 because we're wedging with dt.

$$\begin{aligned} d_3 *_3 j_i dx^i &= d_3(j_1 dy \wedge dz + j_2 dz \wedge dx + j_3 dx \wedge dy) \\ &= \frac{\partial j_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial j_2}{\partial y} dx \wedge dy \wedge dz + \frac{\partial j_3}{\partial z} dx \wedge dy \wedge dz \\ &= \operatorname{div}((j_1, j_2, j_3)) dx \wedge dy \wedge dz \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= d*J \\ &= \frac{\partial \rho}{\partial t} dt \wedge dx \wedge dy \wedge dz + \operatorname{div}((j_1, j_2, j_3)) dt \wedge dx \wedge dy \wedge dz \end{aligned}$$

Now we take $*_4$ of both sides to get

$$\frac{\partial \rho}{\partial t} + \operatorname{div}((j_1, j_2, j_3)) = 0$$

which is the continuity equation from fluid dynamics if we let $(j_1, j_2, j_3) = \rho \mathbf{u}$ for $\mathbf{u}$ the fluid flow and ρ the density. In electromagnetism this is called the charge conservation equation.

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