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AN EXPLICIT FORM FOR KEROV'S CHARACTER POLYNOMIALS

I. P. GOULDEN AND A. RATTAN

Abstract. Kerov considered the normalized characters of irreducible representations of the symmetric group, evaluated on a cycle, as a polynomial in free cumulants. Biane has proved that this polynomial has integer coefficients, and made various conjectures. Recently, Śniady has proved Biane's conjectured explicit form for the first family of nontrivial terms in this polynomial. In this paper, we give an explicit expression for all terms in Kerov's character polynomials. Our method is through Lagrange inversion.

1. Introduction

1.1. **Background and notation.** A *partition* is a weakly ordered list of positive integers $\lambda = \lambda_1\lambda_2\ldots\lambda_k$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. The integers $\lambda_1, \ldots, \lambda_k$ are called the *parts* of the partition λ, and we denote the number of parts by $l(\lambda) = k$. If $\lambda_1 + \cdots + \lambda_k = d$, then λ is a partition of d, and we write $\lambda \vdash d$. We denote by $\mathcal{P}$ the set of all partitions, including the single partition of 0 (which has no parts). For partitions $\omega, \lambda \vdash n$ let $\chi_\omega(\lambda)$ be the character of the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by ω, and evaluated on the conjugacy class $\mathcal{C}_\lambda$ of $\mathfrak{S}_n$, which consists of all permutations whose disjoint cycle lengths are specified by the parts of λ.

Various scalings of irreducible symmetric group characters have been considered in the recent literature. The *central character* is given by

$$\widetilde{\chi}_\omega(\lambda) = |\mathcal{C}_\lambda| \frac{\chi_\omega(\lambda)}{\chi_\omega(1^n)},$$

where $\chi_\omega(1^n)$ is the *degree* of the irreducible representation indexed by ω. For results about the central character, see, for example, [4, 5, 8]. Related to this scaling, for the conjugacy class $\mathcal{C}_{k1^{n-k}}$ only, is the *normalized character*, given by

$$\widehat{\chi}_\omega(k1^{n-k}) = n(n-1)\cdots(n-k+1)\frac{\chi_\omega(k1^{n-k})}{\chi_\omega(1^n)} = k\widetilde{\chi}_\omega(k1^{n-k}).$$

The subject of this paper is a particular polynomial expression for the normalized character. The statement of this expression requires some notation involving the partition ω of n. We adapt the following description from Biane [1, 2]: consider the Young diagram of ω, in the French convention (see [10, footnote page 2]), and translate it, if necessary, so that the bottom left of the diagram is placed at the origin of an (x, y) plane. Finally, rotate the diagram counter-clockwise by 45° (this

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is often referred to as “Russian notation”). Note that ω is uniquely determined by

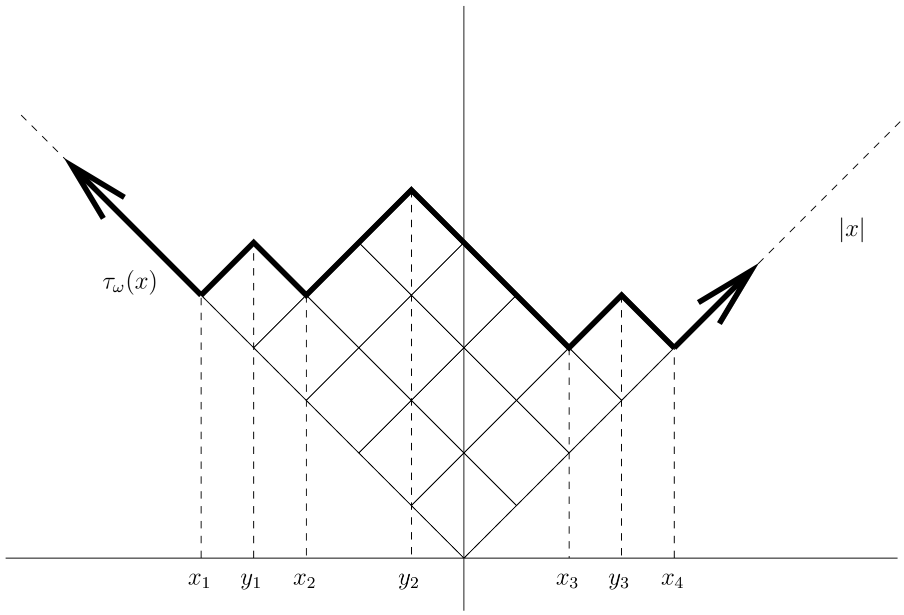

FIGURE 1. The partition (4 3 3 3 1) of 14, drawn in the French convention, and rotated by 45°.

the curve $\tau_\omega(x)$ (see Figure 1). The value of $\tau_\omega(x)$ is equal to $|x|$ for large negative or positive values of x and it is clear that $\tau'_\omega(x) = \pm 1$, where differentiable. The points x_i and y_i are the x-coordinates of the local minima and maxima, respectively, of the curve $\tau_\omega(x)$. We suitably scale the size of the boxes in our Young diagram so that the points x_i and y_i are integers. Setting $\sigma_\omega(x) = (\tau_\omega(x) - |x|)/2$, consider the function

$$G_\omega(z) = \frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{z-x}\ \sigma'_\omega(x)\ dx. \tag{1.1}$$

Carrying out the above integration one obtains

$$G_\omega(z) = \frac{\prod_{i=1}^{m-1}(z-y_i)}{\prod_{i=1}^{m}(z-x_i)},$$

where m is the number of nonempty rows in the Young diagram of ω. Now let $R_i(\omega)$, $i \geq 1$ be defined by

$$z^{-1} + \sum_{i\geq 1} R_i(\omega) z^{i-1} = G_\omega^{\langle -1\rangle}(z),$$

where $\langle -1\rangle$ denotes compositional inverse. Briefly, the origins of the series $G_\omega(z)$, and in fact Kerov's polynomials, come from attempting to answer asymptotic questions about the characters of the symmetric group. The generating series $G_\omega(z)$ is known as the *moment generating series* and is the series

$$\int_{\mathbb{R}} \frac{1}{z-x}\ d(m_\omega)$$

where m_ω is known as the *transition measure* of the diagram ω. The measure is given by

$$m_\omega = \sum_{k=1}^{m} \mu_k \delta_{x_k}$$

where δ_{x_k} is the usual delta function and

$$\mu_k = \frac{\prod_{i=1}^{m-1}(x_k - y_i)}{\prod_{i\neq k}^{m}(x_k - x_i)}.$$

Thus, the coefficient of $(1/z)^k$ in $G_\omega(z)$ is the k^{th} moment of the measure m_ω. In this context, the $R_i(\omega)$'s are known as *free cumulants* of the measure m_ω in free probability theory (see Biane [1, Section 1.2 and Section 3]). Biane uses the theory of free probability to give asymptotic information about the characters of the symmetric group. More specifically, if $\sigma_n \in \mathfrak{S}_n$, $n \geq 1$, is a sequence of permutations (subject to a certain "balanced" restriction on the associated Young diagram) with k_i cycles of length i for $i \geq 2$ and $r = \sum_i ik_i$, then we have

$$\frac{\chi_\omega(\sigma_n)}{\chi_\omega(1^n)} = \prod_{i\geq 2} R_{i+1}^{k_i}(\omega) n^{-r} + O(n^{-\frac{r+1}{2}}).$$

For more information about the asymptotics of characters of the symmetric group (and free cumulants) see, for example, [1, 7, 9].

1.2. **Kerov's character polynomials.** The particular polynomials that are the subject of this paper involve the $R_i(\omega)$'s. They first appeared in Biane [2], where the following result is stated (as Theorem 5.1).

Theorem 1.1. *For $k \geq 1$, there exist universal polynomials Σ_k, with integer coefficients, such that*

$$\widehat{\chi}_\omega(k1^{n-k}) = \Sigma_k(R_2(\omega), R_3(\omega), \ldots, R_{k+1}(\omega)), \tag{1.2}$$

for all $\omega \vdash n$ with $n \geq k$.

Biane attributes Theorem 1.1 to Kerov, who described this result in a talk at an IHP conference in 2000, but a proof first appears in a later paper of Biane [3]. The polynomials Σ_k are known as *Kerov's character polynomials*. They are referred to as "universal polynomials" in Theorem 1.1 to emphasize that they are independent of ω and n, subject only to $n \geq k$. Thus we write them with $R_i(\omega)$ replaced by an indeterminate R_i, $i \geq 2$. In indeterminates R_i, the first six of Kerov's character polynomials, as listed in [2], are given below:

$$\begin{aligned}
\Sigma_1 &= R_2\\
\Sigma_2 &= R_3\\
\Sigma_3 &= R_4 + R_2\\
\Sigma_4 &= R_5 + 5R_3\\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2\\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3
\end{aligned}$$

Note that all coefficients appearing in this list are positive. It is conjectured that this holds in general: that for any $k \geq 1$, all nonzero coefficients in Σ_k are positive.

In Biane [2], this conjecture, which we shall refer to as the *R-positivity conjecture*, is attributed to Kerov. In fact, Biane [1, Section 7] has a conjectured combinatorial interpretation for the coefficients of the R's in Kerov's polynomials pertaining to Cayley graphs. The conjecture there is vague, however, and has been made more precise by Śniady [11] but is still far from being complete. Numerically, the R-positivity conjecture has been verified for k up to 15 by Biane [3], who computed Σ_k for $k \leq 15$, using an implicit formula for Σ_k [1, Theorem 5.1] that he credits to Okounkov (private communication). Biane further comments that "It seems plausible that S. Kerov was aware of this (see especially the account of Kerov's central limit theorem in [7])." The following result gives an adaptation of Biane's formula that appears in Stanley [12].

Theorem 1.2. *Let* $R(x) = 1 + \sum_{i\geq 2} R_i x^i$ *and*

$$F(x) = \frac{x}{R(x)}, \qquad H(x) = \frac{1}{F^{\langle -1\rangle}(x^{-1})}. \tag{1.3}$$

Then, for $k \geq 1$,

$$\Sigma_k = -\frac{1}{k}[x^{-1}]_\infty \prod_{j=0}^{k-1} H(x-j).$$

Theorem 1.2 implicitly determines Σ_k as a polynomial in the R_i's. For explicit formulas, it is convenient to consider separately the graded pieces of Σ_k, defined as follows: let the weight of the monomial $R_{j_1}\cdots R_{j_i}$ be $j_1 + \cdots + j_i$. For $n \geq 0$, we define

$$\Sigma_{k,2n} = [u^{k+1-2n}]\Sigma_k(R_2u^2, \ldots, R_{k+1}u^{k+1}), \tag{1.4}$$

the sum of all terms of weight $k+1-2n$ in Σ_k. (From elementary parity considerations, all other coefficients in Σ_k are 0.) It is immediate that $\Sigma_{k,0} = R_{k+1}$. An explicit formula is known for $\Sigma_{k,2}$, and for the statement of this formula, we introduce polynomials C_m in the R_i's, where $C_0 = 1$, $C_1 = 0$, and

$$C_m = \sum_{\substack{j_2,j_3,\ldots\geq 0 \\ 2j_2+3j_3+\cdots=m}} (j_2+j_3+\cdots)! \prod_{i\geq 2} \frac{((i-1)R_i)^{j_i}}{j_i!}, \quad m \geq 2. \tag{1.5}$$

The following explicit formula for $\Sigma_{k,2}$ was conjectured by Biane [3, Conjecture 6.4], and proved by Śniady [11, Theorem 22]. Śniady's proof was obtained by finding and then solving an equivalent combinatorial problem.

Theorem 1.3. *For* $k \geq 1$,

$$\Sigma_{k,2} = \tfrac{1}{24}(k-1)k(k+1)C_{k-1}.$$

Note that the R-positivity of $\Sigma_{k,2}$ follows immediately from Theorem 1.3, using (1.5).

For $n \geq 2$, only one explicit result is known, given in the following result for the linear coefficient, due to Biane [3] and Stanley [12].

Theorem 1.4. *For* $n \geq 1$, $k \geq 2n-1$, *the coefficient of* R_{k+1-2n} *in* $\Sigma_{k,2n}$ *is equal to the number of* k*-cycles* c *in* $\mathfrak{S}_k$ *such that* $(1\ldots k)c$ *has* $k-2n$ *cycles.*

Finally, for higher order terms when $n = 2$, the following conjecture of Stanley (private communication) has been communicated to us by Biane.

Conjecture 1.5. *For* $i \geq 1$,

$$[R_2^i]\Sigma_{2i+3,4} = \tfrac{1}{540} i(i+1)^3(i+2)^3(i+3)(2i+3).$$

1.3. **Outline of paper.** In this paper, we obtain an explicit formula for $\Sigma_{k,2n}$, where k and n are arbitrary. This is our main result, stated in Section 2 as Theorem 2.1. Variants are given also, as Theorems 2.2 and 2.3. These results are a natural generalization of Theorem 1.3, since they give $\Sigma_{k,2n}$ as a polynomial in the C_m's, with coefficients that are rational polynomials in k. We call such an expression a *C-expansion* for $\Sigma_{k,2n}$. Based on significant amounts of data, we conjecture that $\Sigma_{k,2n}$ is *C-positive* (all nonzero coefficients are positive) for all $n \geq 1$, as Conjecture 2.4. This C-positivity conjecture is stronger than the R-positivity conjecture, immediately from (1.5).

In Section 3, we consider the special cases of our main result for $n = 1$ and $n = 2$. For $n = 1$, this gives another proof of Theorem 1.3. For $n = 2$, the expression for $\Sigma_{k,4}$ that we obtain, in Theorem 3.3, is new. We are able to specialize this expression to prove Conjecture 1.5. Also, we are able to prove the C-positivity conjecture for $\Sigma_{k,4}$, as Corollary 3.5. Finally, we consider the linear terms in the R_i's, for arbitrary n, and obtain another proof of Theorem 1.4.

In general, for $n \geq 3$, we are not able to prove the R-positivity conjecture nor the C-positivity conjecture, perhaps because our methods are not combinatorial. Instead we apply Lagrange inversion to "unwind" the compositional inverse in Theorem 1.2. This is carried out in Section 4, where we give the proof of the main result and variants.

2. The main result

For the partition $\lambda \vdash n$ we denote the *monomial* symmetric function with exponents given by the parts of λ, in indeterminates $x_1, x_2, \ldots$, by m_λ. In this paper, we consider the particular evaluation of the monomial symmetric function at $x_i = i$, for $i = 1, \ldots, k-1$, and $x_i = 0$, for $i \geq k$, and write this as $\hat{m}_\lambda$. Now let $C(t) = \sum_{m \geq 0} C_m t^m$, so from (1.5) we obtain

$$C(t) = \frac{1}{1 - \sum_{i \geq 2}(i-1)R_i t^i}. \tag{2.1}$$

Let $D = t\frac{d}{dt}$, and I be the identity operator, and define

$$P_m(t) = -\frac{1}{m!} C(t)\,(D + (m-2)I)\,C(t) \cdots (D+I)\,C(t) D C(t), \quad m \geq 1. \tag{2.2}$$

For example, we have

$$P_1(t) = -C(t), \quad P_2(t) = -\frac{1}{2} C(t) D C(t),$$

and

$$\begin{aligned} P_3(t) &= -\frac{1}{6} C(t)\,(D+I)\,C(t) D C(t) \\ &= \frac{1}{6}\left(C(t)^2 D C(t) + C(t)\,(DC(t))^2 + C(t)^2 D^2 C(t) \right). \end{aligned}$$

We remind the reader that the differential operator is not commutative; for example, $(DC(t))^2 \neq DC(t)DC(t)$. The latter is properly interpreted as $D\,(C(t)DC(t))$.

Finally, for a partition λ, we write $P_\lambda(t) = \prod_{j=1}^{l(\lambda)} P_{\lambda_j}(t)$. We now state our main result.

Theorem 2.1. *For $n \geq 1$, $k \geq 2n-1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[t^{k+1-2n}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \frac{P_\lambda(t)}{C(t)}.$$

There is a slight modification of this result, given below, in which the term corresponding to the partition with one part is given a simpler (but equivalent) evaluation. This is used mainly for computational purposes as it is easier to compute P_i for smaller i.

Theorem 2.2. *For $n \geq 1$, $k \geq 2n-1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[t^{k+1-2n}] \left(\frac{k-1}{2n} \hat{m}_{2n} P_{2n-1}(t) + \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \geq 2}} \hat{m}_\lambda \frac{P_\lambda(t)}{C(t)} \right).$$

The following result gives a generating function form of the main result.

Theorem 2.3. *For $n \geq 1$, $k \geq 2n-1$,*

$$\Sigma_{k,2n} = -\frac{1}{k}[u^{2n}t^{k+1}] \frac{1}{C(t)} \prod_{j=1}^{k-1} \left(1 + \sum_{i \geq 1} j^i P_i(t) u^i t^i \right),$$

$$\Sigma_k = -\frac{1}{k}[t^{k+1}] \frac{1}{C(t)} \prod_{j=1}^{k-1} \left(1 + \sum_{i \geq 1} j^i P_i(t) t^i \right).$$

Note that, for each $n \geq 1$, these results give $\Sigma_{k,2n}$ as the coefficient of t^{k+1-2n} in a polynomial in $C(t)$ and

$$D^i C(t) = \sum_{m \geq 2} m^i C_m t^m, \quad i \geq 1.$$

Thus $\Sigma_{k,2n}$ is written as a polynomial in the C_m's, with coefficients that are polynomial in k with rational coefficients, so our results give C-expansions for $\Sigma_{k,2n}$, for $n \geq 1$ (the $\hat{m}_\lambda$ are divisible by k for each λ and fixed n, as follows immediately from Propositions 3.1 and 3.2).

Using the above results, with the help of Maple, we have determined the C-expansions and the R-expansions of $\Sigma_{k,2n}$ for all $k \leq 25$ and $n \geq 1$. The R-expansions are in complete agreement with those reported in Biane [3] for $k \leq 11$.

The C-expansions are given below for $k \le 10$:

$$\begin{aligned}
\Sigma_1 - R_2 &= 0\\
\Sigma_2 - R_3 &= 0\\
\Sigma_3 - R_4 &= C_2\\
\Sigma_4 - R_5 &= \tfrac{5}{2}C_3\\
\Sigma_5 - R_6 &= 5\,C_4 + 8\,C_2\\
\Sigma_6 - R_7 &= \tfrac{35}{4}\,C_5 + 42\,C_3\\
\Sigma_7 - R_8 &= 14\,C_6 + \tfrac{469}{3}\,C_4 + \tfrac{203}{3}\,C_2{}^2 + 180\,C_2\\
\Sigma_8 - R_9 &= 21\,C_7 + \tfrac{1869}{4}\,C_5 + \tfrac{819}{2}\,C_3C_2 + 1522\,C_3\\
\Sigma_9 - R_{10} &= 30\,C_8 + 1197\,C_6 + \tfrac{963}{2}\,C_3{}^2 + 1122\,C_4C_2 + 81\,C_2{}^3 + \tfrac{26060}{3}\,C_4\\
&+ \tfrac{17680}{3}\,C_2{}^2 + 8064\,C_2\\
\Sigma_{10} - R_{11} &= \tfrac{165}{4}\,C_9 + \tfrac{5467}{2}\,C_7 + \tfrac{4433}{2}\,C_4C_3 + \tfrac{1133}{2}\,C_3C_2{}^2 + \tfrac{11033}{4}\,C_5C_2\\
&+ 38225\,C_5 + 52580\,C_3C_2 + 96624\,C_3
\end{aligned}$$

Note the form of the data presented above. We have

$$\Sigma_k - \Sigma_{k,0} = \sum_{k\ge 1} \Sigma_{k,2n},$$

where $\Sigma_{k,0} = R_{k+1}$ remains on the left hand side, and we can recover the individual $\Sigma_{k,2n}$ on the right hand side: if the weight of the monomial $C_{m_1}\dots C_{m_i}$ is $m_1 + \cdots + m_i$, then, from (1.5) and (1.4), $\Sigma_{k,2n}$ is the sum of all terms of weight $k+1-2n$.

In the above C-expansions for $k \le 10$, all nonzero coefficients are positive rationals, with apparently small denominators. In fact, this is true for all the data we have computed, up to $k = 25$. We do not have a precise conjecture about the denominators, but conjecture that the positivity holds for all k.

Conjecture 2.4. *For $n \ge 1$, $k \ge 2n-1$, $\Sigma_{k,2n}$ is C-positive.*

This C-positivity conjecture implies the R-positivity conjecture, from (1.5) (so, our data also check the R-positivity conjecture for $k \le 25$). Theorem 1.3 gives an immediate proof that Conjecture 2.4 holds for $n = 1$ and all k. In Corollary 3.5, we are able to prove that Conjecture 2.4 holds for $n = 2$ and all k. We are not able to prove the conjecture for any larger value of n, though of course Theorem 1.4, together with (2.1), proves that the linear terms are C-positive for all n.

The conjecture does not hold for $n = 0$, as described below. We have $\Sigma_{k,0} = R_{k+1}$, and it is straightforward to determine the C-expansion for the R_i's: from (2.1), we obtain

$$\begin{aligned}
1 - \sum_{i\ge 2}(i-1)R_i t^i &= \frac{1}{C(t)}\\
&= \sum_{j_2,j_3,\dots\ge 0} (j_2+j_3+\cdots)! \prod_{m\ge 2} \frac{(-C_m t^m)^{j_m}}{j_m!},
\end{aligned}$$

so we conclude that

$$R_i = \frac{1}{i-1} \sum_{\substack{j_2,j_3,\dots\ge 0\\ 2j_2+3j_3+\cdots=i}} (-1)^{1+j_2+j_3+\cdots}(j_2+j_3+\cdots)! \prod_{m\ge 2} \frac{C_m^{j_m}}{j_m!}, \quad i \ge 2.$$

Thus, terms of negative sign appear in the C-expansion of R_i, for $i \geq 4$. This is the reason that we have presented the data for k up to 10 with R_{k+1} subtracted on the lefthandside. This is also the reason that the R-positivity conjecture does not imply the C-positivity conjecture, so R-positivity and C-positivity are not equivalent.

3. Special cases of the main result

3.1. **Monomial symmetric functions.** To make the expression for $\Sigma_{k,2n}$ that arises from Theorem 2.1 (or Theorem 2.2) explicit, we need to evaluate the $\hat{m}_\lambda$, which are monomial symmetric functions in $1, 2, \ldots, k-1$. For general results about symmetric functions, see Macdonald [10].

Proposition 3.1. *For indeterminates a_i, $i \geq 1$, let $A(x) = 1 + \sum_{i\geq 1} a_i x^i$, and $a_\lambda = \prod_{j=1}^{l(\lambda)} a_{\lambda_j}$, where $\lambda = \lambda_1 \ldots \lambda_{l(\lambda)}$ is a partition. Then*

$$\sum_{\lambda\in\mathcal{P}} \hat{m}_\lambda a_\lambda = \exp \sum_{j\geq 1} \hat{m}_j \sum_{i\geq 1} \frac{(-1)^{i-1}}{i} [x^j](A(x)-1)^i.$$

Proof. We have

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} m_\lambda a_\lambda &= \prod_{n\geq 1} A(x_n) \\ &= \exp \sum_{n\geq 1} \log\left(A(x_n)\right) \\ &= \exp \sum_{n\geq 1} \sum_{i\geq 1} \frac{(-1)^{i-1}}{i} (A(x_n)-1)^i,\end{aligned}$$

and the result follows. □

Proposition 3.1 gives an expression for $\hat{m}_\lambda$ as a polynomial in $\hat{m}_i$, $i \geq 1$, by equating coefficients of a_λ. To evaluate the $\hat{m}_i$, $i \geq 1$, we apply the following result (see, e.g., [10, I 2, Exercise 11] for a proof).

Proposition 3.2. *For $j \geq 1$,*

$$\hat{m}_j = \sum_{i=1}^{j} S(j,i) i! \binom{k}{i+1},$$

where $S(j,i)$, the Stirling numbers of the second kind, *are given by*

$$\sum_{i\geq 0} \sum_{j=0}^{i} S(j,i) u^i \frac{x^j}{j!} = \exp u(e^x - 1).$$

As special cases of this result, we have the following, well-known sums of integer powers.

$$\hat{m}_1 = \tfrac{1}{2}(k-1)k, \quad \hat{m}_2 = \tfrac{1}{6}(k-1)k(2k-1), \quad \hat{m}_3 = \tfrac{1}{4}(k-1)^2 k^2, \tag{3.1}$$

$$\hat{m}_4 = \tfrac{1}{30}(k-1)k(2k-1)(3k^2-3k-1).$$

3.2. **The cases $n = 1, 2$.** We first consider the case $n = 1$ of Theorem 2.2. This immediately gives Biane and Śniady's C-expansion for $\Sigma_{k,2}$, and hence another proof of Theorem 1.3, as shown below.

Proof of Theorem 1.3. From Theorem 2.2, with $n = 1$, we obtain

$$\begin{aligned}\Sigma_{k,2} &= -\frac{1}{k}[t^{k-1}]\left(-\tfrac{1}{2}(k-1)\hat{m}_2 C(t) + \hat{m}_{11}C(t)\right)\\ &= \frac{1}{k}\left(\tfrac{1}{2}(k-1)\hat{m}_2 - \hat{m}_{11}\right)[t^{k-1}]C(t).\end{aligned}$$

But from Proposition 3.1, we obtain

$$\hat{m}_{11} = \tfrac{1}{2}(\hat{m}_1^2 - \hat{m}_2),$$

and the result follows from (3.1), by routine manipulation. □

Next we consider the case $n = 2$ of Theorem 2.2, to obtain an explicit C-expansion for $\Sigma_{k,4}$.

Theorem 3.3. *For $k \geq 3$,*

$$\Sigma_{k,4} = \alpha(k) \sum_{\substack{i,j,m\geq 0\\ i+j+m=k-3}} C_i C_j C_m + \beta(k) \sum_{\substack{i,j,m\geq 0\\ i+j+m=k-3}} i^2 C_i C_j C_m,$$

where

$$\begin{aligned}\alpha(k) &= -\tfrac{1}{17280}(k-3)(k-1)^2k(k+1)(k^2-4k-6),\\ \beta(k) &= \tfrac{1}{2880}(k-1)k(k+1)(2k^2-3).\end{aligned}$$

Proof. From Theorem 2.2, with $n = 2$, letting $b = \frac{1}{6}(\hat{m}_{31} - \frac{1}{4}(k-1)\hat{m}_4)$, we obtain

$$\begin{aligned}\Sigma_{k,4} &= -\frac{1}{k}[t^{k-3}]\Big(b\left(C(t)^2DC(t) + C(t)\left(DC(t)\right)^2 + C(t)^2D^2C(t)\right)\\ &\quad + \tfrac{1}{4}\hat{m}_{22}C(t)\left(DC(t)\right)^2 - \tfrac{1}{2}\hat{m}_{211}C(t)^2DC(t) + \hat{m}_{1111}C(t)^3\Big)\\ &= -\frac{1}{k}[t^{k-3}]\Big(\hat{m}_{1111}C(t)^3 + \left(b - \tfrac{1}{2}\hat{m}_{211}\right)C(t)^2DC(t)\\ &\quad + bC(t)^2D^2C(t) + \left(b + \tfrac{1}{4}\hat{m}_{22}\right)C(t)\left(DC(t)\right)^2\Big)\\ &= -\frac{1}{k}[t^{k-3}]\Big(\hat{m}_{1111}C(t)^3 + \left(b - \tfrac{1}{2}\hat{m}_{211}\right)\tfrac{1}{3}DC(t)^3\\ &\quad + bC(t)^2D^2C(t) + \left(b + \tfrac{1}{4}\hat{m}_{22}\right)\left(\tfrac{1}{6}D^2C(t)^3 - \tfrac{1}{2}C(t)^2D^2C(t)\right)\Big)\\ &= -\frac{1}{k}\left(\hat{m}_{1111} + \tfrac{1}{3}(k-3)\left(b - \tfrac{1}{2}\hat{m}_{211}\right) + \tfrac{1}{6}(k-3)^2\left(b + \tfrac{1}{4}\hat{m}_{22}\right)\right)[t^{k-3}]C(t)^3\\ &\quad - \frac{1}{k}\left(\tfrac{1}{2}b - \tfrac{1}{8}\hat{m}_{22}\right)[t^{k-3}]C(t)^2D^2C(t).\end{aligned}$$

But from Proposition 3.1, we obtain

$$\begin{aligned}\hat{m}_{31} &= \hat{m}_3\hat{m}_1 - \hat{m}_4,\\ \hat{m}_{22} &= \tfrac{1}{2}(\hat{m}_2^2 - \hat{m}_4),\\ \hat{m}_{211} &= \tfrac{1}{2}(\hat{m}_2\hat{m}_1^2 - 2\hat{m}_3\hat{m}_1 - \hat{m}_2^2 + 2\hat{m}_4),\\ \hat{m}_{1111} &= \tfrac{1}{24}(\hat{m}_1^4 - 6\hat{m}_2\hat{m}_1^2 + 8\hat{m}_3\hat{m}_1 + 3\hat{m}_2^2 - 6\hat{m}_4),\end{aligned}$$

so from (3.1), by routine manipulation, we obtain

$$\Sigma_{k,4} = \alpha(k)[t^{k-3}]C(t)^3 + \beta(k)[t^{k-3}]C(t)^2 D^2 C(t), \tag{3.2}$$

where $\alpha(k)$ and $\beta(k)$ are given above. The result follows. □

For monomials in $R_2, R_3, \dots$ that are pure powers of a single R_m, we have the following form of the above result.

Corollary 3.4. *For $m \geq 2$, $i \geq 1$,*

$$\begin{aligned}[R_m^i]\Sigma_{mi+3,4} &= \tfrac{1}{34560}(m-1)^i mi(i+1)(i+2)(mi+2)(mi+3)(mi+4) \\ &\times \left(m^3 i^3 + 2m^2(m+4)i^2 + 4m(3m+5)i + 15m + 18\right).\end{aligned}$$

Proof. From Theorem 3.3, we obtain

$$[R_m^i]\Sigma_{mi+3,4} = \alpha(mi+3)[R_m^i t^{mi}]C(t)^3 + \beta(mi+3)[R_m^i t^{mi}]C(t)^2 D^2 C(t).$$

Now, setting $R_j = 0$ for $j \neq m$, we obtain $C(t) = (1-(m-1)R_m t^m)^{-1}$, so

$$[R_m^i t^{mi}]C(t)^3 = (m-1)^i \binom{i+2}{2}.$$

Also, we have

$$\begin{aligned} D^2 C(t) &= Dm(m-1)R_m t^m \left(1-(m-1)R_m t^m\right)^{-2} \\ &= Dm\left((1-(m-1)R_m t^m)^{-2} - (1-(m-1)R_m t^m)^{-1}\right) \\ &= m^2(m-1)\left(2R_m t^m (1-(m-1)R_m t^m)^{-3}\right. \\ &\qquad \left. - R_m t^m (1-(m-1)R_m t^m)^{-2}\right), \end{aligned}$$

so

$$[R_m^i t^{mi}]C(t)^2 D^2 C(t) = (m-1)^i m^2 \left(2\binom{i+3}{4} - \binom{i+2}{3}\right).$$

The result follows by routine manipulation. □

We now consider the case $m = 2$ of Corollary 3.4, to obtain an immediate proof of Stanley's Conjecture 1.5.

Proof of Conjecture 1.5. We set $m = 2$ in Corollary 3.4. Then the factor that is cubic in i becomes

$$8i^3 + 48i^2 + 88i + 48 = 8(i+1)(i+2)(i+3),$$

and the result follows. □

As the final result of this section, we are able to use the explicit C-expansion given in Theorem 3.3, to prove the C-positivity of $\Sigma_{k,4}$.

Corollary 3.5. *$\Sigma_{k,4}$ is C-positive for all $k \geq 3$.*

Proof. Consider $0 \leq i \leq j \leq m$, with $i + j + m = k - 3$, and let $\gamma = |\mathrm{Aut}(i,j,m)|$. Thus when $k = 12$, for example, $\gamma = 1$ for $(i,j,m) = (2,3,4)$ or $(0,2,7)$, $\gamma = 2$ for $(i,j,m) = (2,2,5)$ or $(1,4,4)$, and $\gamma = 6$ for $(i,j,m) = (3,3,3)$. Then, from Theorem 3.3, we obtain

$$[C_i C_j C_m]\Sigma_{k,4} = \frac{6}{\gamma}\alpha(k) + \frac{2}{\gamma}\left(i^2 + j^2 + m^2\right)\beta(k).$$

Now, the minimum value of $x^2+y^2+z^2$ over the reals, subject to $x+y+z=c$, for any fixed real c, is achieved at $x=y=z=c/3$, so in the above expression we have $i^2+j^2+m^2 \geq \frac{1}{3}(k-3)^2$. But $\beta(k)>0$ for $k\geq 3$, so we obtain

$$\begin{aligned}
&[C_iC_jC_m]\Sigma_{k,4} \geq \frac{2}{\gamma}\left(3\alpha(k)+\tfrac{1}{3}(k-3)^2\beta(k)\right)\\
=\;& \tfrac{1}{8640\gamma}(k-3)(k-1)k(k+1)\left(-3(k-1)(k^2-4k-6)+2(k-3)(2k^2-3)\right)\\
=\;& \tfrac{1}{8640\gamma}(k-3)(k-1)k^3(k+1)(k+3)\geq 0,
\end{aligned}$$

for $k\geq 3$, giving the result. ☐

3.3. **The linear terms.** We now apply Theorem 2.3 to evaluate the linear terms in Σ_k, and thus obtain another proof of Theorem 1.4.

Proof of Theorem 1.4. For $i\geq 1$, let $A^{(i)}(t)$ consist of the terms in $P_i(t)$ that are linear in the C_m's. Also, let $L_{n,k}=[R_{k+1-2n}]\Sigma_{k,2n}$. We apply Theorem 2.3 to determine $L_{n,k}$. From (2.1), we have

$$\begin{aligned}
L_{n,k} &= \left[\frac{C_{k+1-2n}}{k-2n}\right]\Sigma_{k,2n} = \left[\frac{C_{k+1-2n}}{k-2n}\right]\Sigma_k\\
&= -\frac{1}{k}\left[\frac{C_{k+1-2n}}{k-2n}t^{k+1}\right]\frac{1}{C(t)}\prod_{j=1}^{k-1}\left(1-jt+\sum_{i\geq 1}j^iA^{(i)}(t)t^i\right)\\
&= -\frac{1}{k}\left[\frac{C_{k+1-2n}}{k-2n}t^{k+1}\right]\frac{1}{C(t)}\left(\prod_{j=1}^{k-1}(1+\sum_{i\geq 1}\frac{j^iA^{(i)}(t)t^i}{1-jt})\right)\prod_{a=1}^{k-1}(1-at)\\
&= -\frac{1}{k}\left[\frac{C_{k+1-2n}}{k-2n}t^{k+1}\right]\left(1-C(t)+\sum_{j=1}^{k-1}\sum_{i\geq 1}\frac{j^iA^{(i)}(t)t^i}{1-jt}\right)\prod_{a=1}^{k-1}(1-at).
\end{aligned}$$

But

$$A^{(i)}(t)=-\frac{1}{i!}(D+(i-2)I)\cdots(D+I)DC(t)=-\sum_{m\geq 2}\binom{-(m-1)}{i}(-1)^i\frac{C_m}{m-1}t^m,$$

for $i\geq 1$. Now let $\frac{C_m}{m-1}=x^{m-1}$, $m\geq 2$, which gives

$$\begin{aligned}
\sum_{i\geq 1}j^iA^{(i)}(t)t^i &= -\sum_{m\geq 2}\left((1-jt)^{-(m-1)}-1\right)x^{m-1}t^m\\
&= -\frac{t}{1-\frac{xt}{1-jt}}+\frac{t}{1-xt},
\end{aligned}$$

and

$$1-C(t)=-\sum_{m\geq 2}(m-1)x^{m-1}t^m=-\frac{t}{(1-xt)^2}+\frac{t}{1-xt}.$$

Thus we obtain

$$\begin{aligned}
L_{n,k} &= \frac{1}{k}[x^{k-2n}t^{k+1}]\left(\frac{t}{(1-xt)^2}-\frac{t}{1-xt}\right.\\
&\qquad\left.+\sum_{j=1}^{k-1}\left(\frac{t}{1-(j+x)t}-\frac{t}{(1-jt)(1-xt)}\right)\right)\prod_{a=1}^{k-1}(1-at).
\end{aligned}$$

We now finish the proof using the method of Biane [3, Theorem 6.1]: Replace t by t^{-1}, and multiply by t^k, to obtain

$$\begin{aligned} L_{n,k} &= \frac{1}{k}[x^{k-2n}][t^{-1}]_\infty (t)_k \left(\frac{t}{(t-x)^2} - \frac{1}{t-x} \right. \\ &\qquad \left. + \sum_{j=1}^{k-1} \left(\frac{1}{t-j-x} - \frac{t}{(t-j)(t-x)} \right) \right), \end{aligned}$$

where $(t)_k = t(t-1)\cdots(t-k+1)$ is the falling factorial. Now use the fact that the residue is unchanged if we substitute $t+c$ for t, where c is independent of t. Thus, substituting $t+j+x$ for t in the first term of the summation over j, and substituting $t+x$ for t in all other terms, we obtain

$$\begin{aligned} L_{n,k} &= \frac{1}{k}[x^{k-2n}] \left([t](t+x)(t+x)_k - (x)_k + \sum_{j=1}^{k-1} \left((x+j)_k - \frac{x(x)_k}{x-j} \right) \right) \\ &= \frac{1}{k}[x^{k-2n}] \sum_{j=0}^{k-1} (x+j)_k = \frac{1}{k}[x^{k-2n}] \sum_{j=0}^{k-1} (x-j)_k, \end{aligned}$$

where, for the last equality, we have replaced x by $-x$, and multiplied by $(-1)^k$. The result now follows, as shown in Biane [3]. □

4. Lagrange inversion and the proof of the main result

As a first step, we translate Theorem 1.2 into formal power series, using the notation

$$\phi(x) = xH\left(x^{-1}\right), \qquad \Phi(x,u) = \sum_{i\geq 0} \Phi_i(x) u^i = (1-ux)\phi\left(x(1-ux)^{-1}\right), \tag{4.1}$$

where $H(x)$ is defined in (1.3).

Proposition 4.1. *The following two equations hold.*

1) For $k \geq 1$,

$$\Sigma_k = -\frac{1}{k}[x^{k+1}] \prod_{j=0}^{k-1} \Phi(x,j). \tag{4.2}$$

2) For $k, n \geq 1$,

$$\Sigma_{k,2n} = -\frac{1}{k}[u^{2n}x^{k+1}] \prod_{j=0}^{k-1} \Phi(x,ju). \tag{4.3}$$

Proof. For (4.2), we first replace x by x^{-1} in Theorem 1.2, to obtain

$$\Sigma_k = -\frac{1}{k}[x^{k+1}] \prod_{j=0}^{k-1} xH(x^{-1}(1-jx)),$$

and the result follows immediately.

For (4.3), we let ϑ be the substitution operator $R_i \mapsto u^i R_i$, $i \geq 2$. Then, from (1.4), we have

$$\Sigma_{k,2n} = [u^{k+1-2n}]\vartheta\Sigma_k. \tag{4.4}$$

Now, from (1.3), we have

$$\vartheta F(x) = \frac{x}{\vartheta R(x)} = \frac{x}{R(ux)} = \frac{1}{u} F(ux).$$

Applying ϑ to both sides of the equation $x = F(F^{\langle -1 \rangle}(x))$ we obtain

$$\begin{aligned} x &= \vartheta F(\vartheta F^{\langle -1 \rangle}(x)) \\ &= \frac{1}{u} F(u\vartheta F^{\langle -1 \rangle}(x)), \end{aligned}$$

implying

$$\vartheta F^{\langle -1 \rangle}(x) = \frac{1}{u} F^{\langle -1 \rangle}(ux).$$

Thus, combining this with (1.3) and (4.1), we obtain

$$\vartheta \phi(x) = x\vartheta H(x^{-1}) = \frac{x}{\vartheta F^{\langle -1 \rangle}(x)} = \frac{ux}{F^{\langle -1 \rangle}(ux)} = \phi(ux),$$

and then

$$\vartheta \Phi(x, j) = (1 - jx)\phi\left(ux(1 - jx)^{-1}\right) = \Phi\left(ux, ju^{-1}\right).$$

Combining this with (4.4) and (4.2) gives

$$\Sigma_{k,2n} = -\frac{1}{k}[u^{k+1-2n}x^{k+1}] \prod_{j=0}^{k-1} \Phi(ux, ju^{-1})$$

and (4.3) now follows, by substituting first $x = xu^{-1}$, and then $u = u^{-1}$. □

Next, we give an expression for the coefficients Φ_i, $i \geq 0$, defined in (4.1).

Proposition 4.2. *For $i \geq 0$,*

$$\Phi_i(x) = \frac{x}{i!}\left(x^2 \frac{d}{dx}\right)^i \frac{\phi(x)}{x}. \tag{4.5}$$

Note that for $i = 0$, this specializes to $\Phi_0(x) = \phi(x)$.

Proof. From (1.3) and (4.1), we have

$$\phi(x) = 1 + \sum_{j \geq 2} \phi_j x^j,$$

where ϕ_j, $j \geq 2$ are polynomials in the R_i's. For $i = 0$, we have $\Phi_0(x) = \Phi(x, 0) = \phi(x)$. For $i \geq 1$, we have

$$\begin{aligned} \Phi_i(x) &= [u^i]\Phi(x, u) = [u^i]\left(1 - ux + \sum_{j \geq 2} \phi_j x^j (1 - ux)^{1-j}\right) \\ &= -\binom{1}{i} x + \sum_{j \geq 2} \phi_j \binom{j + i - 2}{i} x^{j+i} \\ &= \frac{x}{i!}\left(x^2 \frac{d}{dx}\right)^i \left(\frac{1}{x} + \sum_{j \geq 2} \phi_j x^{j-1}\right), \end{aligned}$$

and the result follows. □

We make use of the following two, closely related, versions of Lagrange's Theorem (see, e.g., [6, Section 1.2], for a proof).

Theorem 4.3. *Suppose ϕ is a formal power series with invertible constant term. Then the functional equation $w = t\phi(w)$ has a unique formal power series solution $w = w(t)$. Moreover,*
1) For a formal Laurent series f and $n \neq 0$, we have

$$\frac{1}{n}[x^{n-1}]\left(\frac{d}{dx}f(x)\right)\phi(x)^n = [t^n]f(w),$$

2) For a formal power series f, and $n \geq 0$, we have

$$[x^n]f(x)\phi(x)^n = [t^n]f(w)\frac{t}{w}\frac{dw}{dt}.$$

Here, we shall consider the functional equation

$$w = t\phi(w), \tag{4.6}$$

where ϕ is the particular series given by (4.1). Then from (1.3) and (4.1), we have

$$w = twH(w^{-1}) = \frac{tw}{F^{\langle -1\rangle}(w)},$$

so $F^{\langle -1\rangle}(w) = t$, and from (1.3) we deduce that

$$t = wR(t). \tag{4.7}$$

We now relate the series $C(t)$ and differential operator D of Section 2 to the variable w.

Proposition 4.4.

$$\frac{Dw}{w} = \frac{1}{R(t)C(t)} \tag{4.8}$$

$$w^2\frac{d}{dw} = tC(t)D \tag{4.9}$$

Proof. From (2.1) and (1.3), we obtain

$$C(t) = \frac{1}{-tD\frac{R(t)}{t}}.$$

But

$$\frac{Dw}{w} = -wD\frac{1}{w} = -\frac{t}{R(t)}D\frac{R(t)}{t},$$

from (4.7), and result (4.8) follows.

Now, (4.8) gives the operator identity

$$w\frac{d}{dw} = R(t)C(t)D,$$

and multiplying by w and using (4.7), we obtain result (4.9). □

Proof of Theorem 2.1. For a partition λ, let $\Phi_\lambda(x) = \prod_{j=1}^{l(\lambda)} \Phi_{\lambda_j}(x)$. Then from (4.3) and (4.5), we have

$$\begin{aligned}
\Sigma_{k,2n} &= -\frac{1}{k}[x^{k+1}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \Phi_\lambda(x)\phi(x)^{k-l(\lambda)} \\
&= -\frac{1}{k}[x^{k+1}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \frac{\Phi_\lambda(x)}{\phi(x)^{l(\lambda)+1}}\phi(x)^{k+1} \\
&= -\frac{1}{k}[t^{k+1}] \sum_{\lambda \vdash 2n} \hat{m}_\lambda \frac{1}{R(t)C(t)} \frac{\Phi_\lambda(w)}{\phi(w)^{l(\lambda)+1}},
\end{aligned}$$

where the last equality follows from Theorem 4.3.2 and (4.8). But, from (4.5), (4.6) and (4.9), for $i \geq 1$ we have

$$\begin{aligned}
\frac{\Phi_i(w)}{\phi(w)} &= \frac{1}{i!}\frac{w}{\phi(w)}\left(w^2 \frac{d}{dw}\right)^i \frac{\phi(w)}{w} \\
&= \frac{t}{i!}\left(tC(t)D\right)^{i-1} tC(t)D\frac{1}{t} \\
&= -\frac{t}{i!}\left(tC(t)D\right)^{i-1} C(t).
\end{aligned}$$

Finally, we prove by induction on $i \geq 1$ that

$$-\frac{1}{i!}\left(tC(t)D\right)^{i-1} C(t) = t^{i-1}P_i(t),$$

where $P_i(t)$ is defined in Section 2. The result is clearly true for $i = 1$. For the induction step, we have

$$\begin{aligned}
-\frac{1}{(i+1)!}(tC(t)D)^i C(t) &= \frac{1}{i+1} tC(t)Dt^{i-1}P_i(t) \\
&= \frac{1}{i+1}\left(t^i C(t)D + (i-1)t^i C(t)I\right) P_i(t) \\
&= t^i P_{i+1}(t),
\end{aligned}$$

as required. Together, these results give

$$\frac{\Phi_i(w)}{\phi(w)} = t^i P_i(t),$$

so

$$\frac{\Phi_\lambda(w)}{\phi(w)^{l(\lambda)+1}} = t^{2n}\frac{P_\lambda(t)}{\phi(w)},$$

since $\lambda \vdash 2n$, and the result follows from (4.6) and (4.7). □

Proof of Theorem 2.2. In the proof of Theorem 2.1, the term in $\Sigma_{k,2n}$ corresponding to the partition with the single part $2n$ can be treated in the following modified way. We obtain

$$\begin{aligned}
-\frac{1}{k}[x^{k+1}]\hat{m}_{2n}\Phi_{2n}(x)\phi(x)^{k-1} &= -\frac{1}{k}[x^{k-2}]\hat{m}_{2n}x^{-3}\Phi_{2n}(x)\phi(x)^{k-1} \\
&= -\frac{1}{k}[x^{k-2}]\hat{m}_{2n}x^{-3}\frac{x}{(2n)!}x^2\frac{d}{dx}\left(x^2\frac{d}{dx}\right)^{2n-1} \\
&\qquad \cdot\frac{\phi(x)^k}{x} \\
&= -\frac{k-1}{k}[t^{k-1}]\hat{m}_{2n}\frac{1}{(2n)!}\left(w^2\frac{d}{dw}\right)^{2n-1}\frac{\phi(w)}{w},
\end{aligned}$$

from Theorem 4.3.1, and the result follows as in the above proof of Theorem 2.1.□

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Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, N2L 3G1, Canada
E-mail address: ipgoulden@math.uwaterloo.ca

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, N2L 3G1, Canada
E-mail address: arattan@math.mit.ca
Current address: Department of Applied Mathematics, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA