Maps and Branched Covers - Combinatorics, Geometry and Physics

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Outline

- KP hierarchy, integrable systems;

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- transitive factorizations in the symmetric group, combinatorics; branched covers, Hurwitz numbers, geometry
- rooted hypermaps, rooted maps, rooted triangulations; asymptotics of maps

The KP (Kadomtsev-Petviashvili) hierarchy is a system of partial differential equations for an unknown function $F(p_1, p_2, \ldots)$. There is a countable number of equations in the hierarchy, and each is quadratic. The first few equations are:

$$F_{2,2} - F_{3,1} + \tfrac{1}{12}F_{1,1,1,1} + \tfrac{1}{2}F_{1,1}^2 = 0,$$

$$F_{3,2} - F_{4,1} + \tfrac{1}{6}F_{2,1,1,1} + F_{1,1}F_{2,1} = 0,$$

$$F_{4,2} - F_{5,1} + \tfrac{1}{4}F_{3,1,1,1} - \tfrac{1}{120}F_{1,1,1,1,1,1} + F_{1,1}F_{3,1} + \tfrac{1}{2}F_{2,1}^2 - \tfrac{1}{8}F_{1,1,1}^2 - \tfrac{1}{12}F_{1,1}F_{1,1,1,1} = 0.$$

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$$- \tfrac{1}{8} F_{1,1,1}^2 - \tfrac{1}{12} F_{1,1} F_{1,1,1,1} = 0.$$

For any solution F, $\tau = e^F$ is called a τ-function for the hierarchy.

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- Miwa, Jimbo, Date, 2000, ”Solitons: Differential equations, symmetries and ∞-dimensional algebras”

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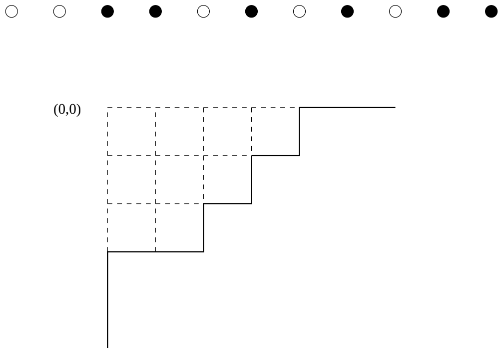

Figure: A Maya diagram and associated Young diagram

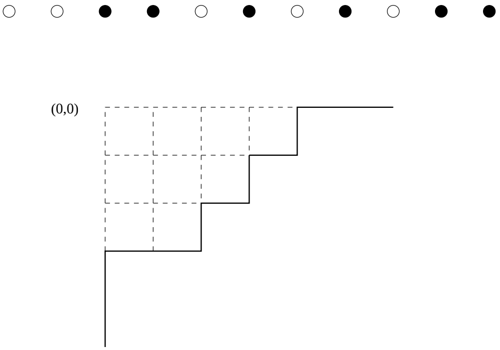

Figure: A Maya diagram and associated Young diagram

This is the Young diagram for the partition $(4, 3, 2)$

Theorem

$$\sum_{\theta \in \mathcal{P}} b_\theta \, s_\theta(p_1, p_2, \ldots)$$

is a τ*-function for the KP hierarchy if and only if*

$$\{ b_\theta : \theta \in \mathcal{P} \}$$

satisfies the Plücker relations.

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Here $s_\theta(p_1, p_2, \ldots)$ is the Schur symmetric function written in terms of the power sum symmetric functions $p_1, p_2, \ldots$ (which are the variables of the KP hierarchy).

Suppose $\alpha_1 \geq \cdots \geq \alpha_{m-1} \geq 0$ and $\beta_1 \geq \cdots \geq \beta_{m+1} \geq 0$. Then we say that $\{b_\lambda \in \mathbb{Q}[u_1, u_2, \ldots] : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations if it satisfies the equation

$$\sum_{k=0}^{m} (-1)^k b_{(\alpha_1-1,\ldots,\alpha_{m-1}-1,\beta_{k+1}+m-k)} \cdot b_{(\beta_1+1,\ldots,\beta_k+1,\beta_{k+2},\ldots,\beta_{m+1})} = 0,$$

for each such pair of weak partitions α, β.

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for each such pair of weak partitions α, β.

Conventions: $b_{(\ldots,\, i, j,\ldots)} = b_{(\ldots,\, j-1,\, i+1,\ldots)}$, and $b_{(\ldots,\, 0)} = b_{(\ldots)}$.

Example: $\alpha = (2, 1)$ and $\beta = (4, 3, 2, 1)$ (so $m = 3$). Then we have

$$+b_{(1,0,7)} \cdot b_{(3,2,1)} - b_{(1,0,5)} \cdot b_{(5,2,1)} + b_{(1,0,3)} \cdot b_{(5,4,1)} - b_{(1,0,1)} \cdot b_{(5,4,3)} = 0$$

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Example: $\alpha = (0)$ and $\beta = (1, 1, 1)$ (so $m = 2$). Then we have

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1	2	4	4	4	5
2	4	5	6		
4	5				

For example, $s_{(6,4,2)} = \ldots + x_1^1 x_2^2 x_3^0 x_4^5 x_5^3 x_6^1 + \ldots$

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For example, $s_{(6,4,2)} = \ldots + x_1^1 x_2^2 x_3^0 x_4^5 x_5^3 x_6^1 + \ldots$ This is a symmetric function of $x_1, x_2, \ldots$

For partition $\theta = (\theta_1, \ldots, \theta_m)$, we have

$$s_\theta = \det\left(h_{\theta_i - i + j}\right)_{i,j=1,\ldots,m},$$

the Jacobi-Trudi identity, where h is the homogeneous or complete symmetric function given by

$$h_k = \sum x_1^{a_1} x_2^{a_2} \cdots ,$$

summed over all nonnegative integers $a_1, a_2, \ldots$ with $a_1 + a_2 + \cdots = k$, for $k \geq 1$, with $h_0 = 1$ and $h_k = 0$ for $k < 0$.

$$s_{(\ldots,i,j,\ldots)} \quad = \quad \begin{pmatrix} \ddots & \ldots & \ldots & \ldots \\ \ldots & h_i & h_{i+1} & \ldots \\ \ldots & h_{j-1} & h_j & \ldots \\ \ldots & \ldots & \ldots & \ddots \end{pmatrix}$$

$$
\begin{aligned}
s_{(\ldots,i,j,\ldots)} &= \begin{pmatrix} \ddots & \ldots & \ldots & \ldots \\ \ldots & h_i & h_{i+1} & \ldots \\ \ldots & h_{j-1} & h_j & \ldots \\ \ldots & \ldots & \ldots & \ddots \end{pmatrix} \\
&= -\begin{pmatrix} \ddots & \ldots & \ldots & \ldots \\ \ldots & h_{j-1} & h_j & \ldots \\ \ldots & h_i & h_{i+1} & \ldots \\ \ldots & \ldots & \ldots & \ddots \end{pmatrix},
\end{aligned}
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so we have $s_{(\dots,i,j,\dots)} = -s_{(\dots,j-1,i+1,\dots)}$.

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so we have $s_{(\dots,0)} = -s_{(\dots)}$. Thus the Schur function s_θ satisfies the Conventions for b_θ in the Plücker relations.

The following $2m \times 2m$ matrix has determinant equal to 0:

$$
\left(\begin{array}{ccc|ccc}
h_{\alpha_1-1} & \dots & h_{\alpha_1+m-2} & 0 & \dots & 0 \\
\dots & \ddots & \dots & \vdots & \ddots & \vdots \\
h_{\alpha_{m-1}-m+1} & \dots & h_{\alpha_{m-1}} & 0 & \dots & 0 \\
\hline
h_{\beta_1+1} & \dots & h_{\beta_1+m} & h_{\beta_1+1} & \dots & h_{\beta_1+m} \\
\dots & \ddots & \dots & \vdots & \ddots & \vdots \\
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h_{\alpha_{m-1}-m+1} & \dots & h_{\alpha_{m-1}} & 0 & \dots & 0 \\
\hline
h_{\beta_1+1} & \dots & h_{\beta_1+m} & h_{\beta_1+1} & \dots & h_{\beta_1+m} \\
\dots & \ddots & \dots & \vdots & \ddots & \vdots \\
h_{\beta_{m+1}+1} & \dots & h_{\beta_{m+1}} & h_{\beta_{m+1}+1} & \dots & h_{\beta_{m+1}}
\end{array}\right),$$

and applying the Laplace expansion with columns partitioned into the first m and last m gives the Plücker relations for the Schur function.

0	1	2	3
-1	0	1	2
-2	-1	0	
-3			

Above, each box in the Young diagram of the partition $(4, 4, 3, 1)$ contains its content (denoted by $c(t)$ for box t)

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Above, each box in the Young diagram of the partition $(4, 4, 3, 1)$ contains its content (denoted by $c(t)$ for box t)

Theorem

$$\left\{\left(\prod_{t\in\theta} y_{c(t)}\right) s_\theta : \theta \in \mathcal{P}\right\}$$

satisfies the Plücker relations.

A Plücker relation for Schur functions:

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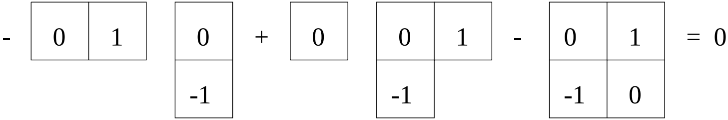

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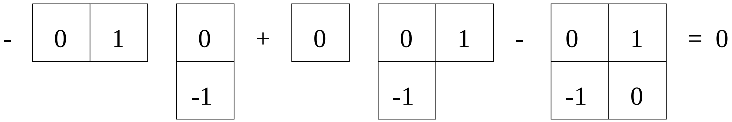

Note that the product of the content variables for the cells in the above three pairs of Young diagrams is a constant:

$$y_0 y_1 \cdot y_0 y_{-1} = y_0 \cdot y_{-1} y_0 y_1 = y_{-1} y_0^2 y_1$$

Theorem

$$\left\{\prod_{t\in\theta}\left(y_{c(t)}\right) s_\theta : \theta \in \mathcal{P}\right\}$$

satisfies the Plücker relations.

Corollary

$$\sum_{\theta\in\mathcal{P}}\left(\prod_{t\in\theta} y_{c(t)}\right) s_\theta \cdot s_\theta(p_1, p_2, \ldots)$$

is a τ-function for the KP hierarchy.

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Corollary

$$\log \sum_{\theta\in\mathcal{P}}\left(\prod_{t\in\theta} y_{c(t)}\right) s_\theta \cdot s_\theta(p_1, p_2, \ldots)$$

is a solution to the KP hierarchy.

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$$s_\theta = \sum_\mu \frac{|\mathcal{C}_\mu|}{d!} \chi^\theta_\mu p_\mu,$$

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where the sum is over partitions μ of d, $|\mathcal{C}_\mu|$ is the size of the conjugacy class $\mathcal{C}_\mu$ in S_d, χ^θ_μ is an irreducible character of S_d, and $p_\mu = p_{\mu_1} p_{\mu_2} \cdots$.

For partitions α, β of $d \geq 1$, and nonnegative integers $a_1, a_2, \ldots$, define $b_{\alpha,\beta}^{(a_1,a_2,\ldots)}$ to be the number of tuples $(\sigma, \gamma, \pi_1, \pi_2, \ldots)$ of permutations in S_d such that

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In geometry, such a transitive product of permutations corresponds to a connected branched cover of the sphere by a Riemann surface (with d sheets). The Riemann-Hurwitz Theorem gives

$$a_1 + a_2 + \cdots = l(\alpha) + l(\beta) + 2g - 2,$$

where g is the genus of the surface.

Theorem

The generating series

$$B = \sum \frac{b_{\alpha,\beta}^{(a_1,a_2,\dots)}}{d!} p_\alpha q_\beta u_1^{a_1} u_2^{a_2} \cdots$$

is a solution to the KP hierarchy.

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is a solution to the KP hierarchy.

Proof.

By standard exponential generating series methods, and character theory for the symmetric group, we have

$$\log \sum_{\theta \in \mathcal{P}} \left(\prod_{t \in \theta} y_{c(t)} \right) s_\theta(q_1, q_2, \dots) \cdot s_\theta(p_1, p_2, \dots)$$

where, for any integer j, $y_j = \prod_{i \geq 1}(1 + u_i j)$.

□

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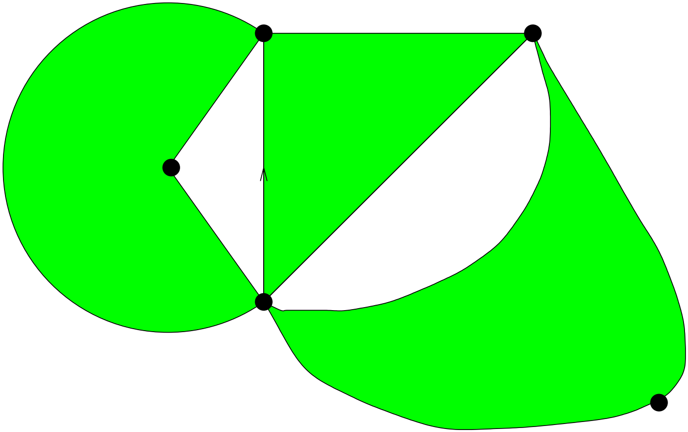
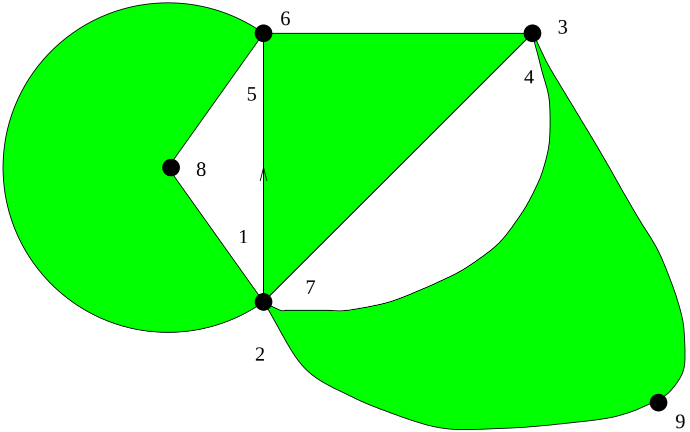
When $a_1 + \cdots + a_m = l(\alpha) + l(\beta) + 2g - 2$, for fixed m (and $g = 0$, β is the partition of all 1's), an explicit formula for these numbers has been given by Bousquet-Mélou and Schaeffer.





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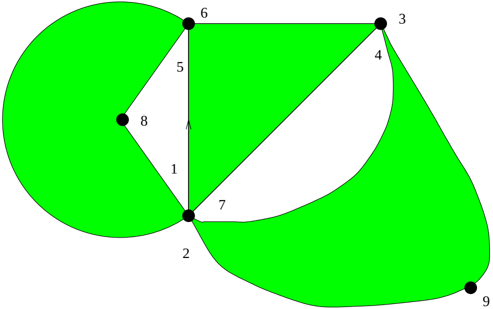

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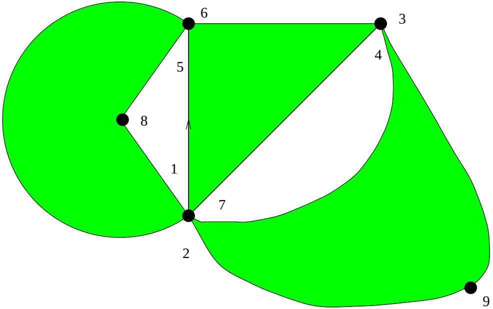
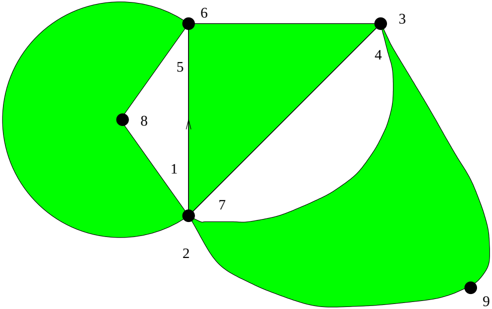
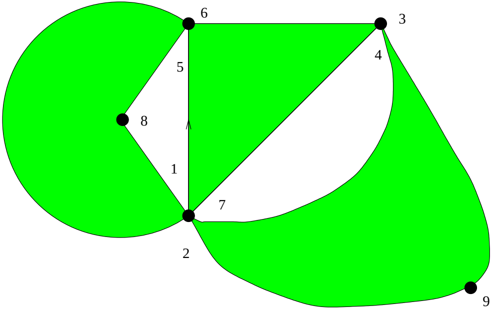


$$V = (1\,2\,7)(3\,4)(5\,6)(8)(9),$$



$$V = (1\,2\,7)(3\,4)(5\,6)(8)(9), \qquad G = (1\,8\,6)(2\,9\,4)(3\,5\,7),$$



$V = (1\,2\,7)(3\,4)(5\,6)(8)(9), \quad G = (1\,8\,6)(2\,9\,4)(3\,5\,7),$
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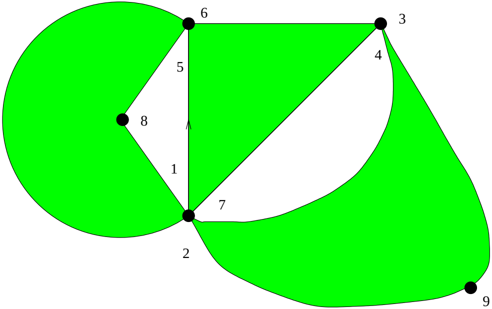

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Corollary

Let $H^g_{\alpha,\beta}$ be the number of rooted hypermaps in a surface of genus g, with vertex degrees specified by the parts of α, hyperedge degrees specified by the parts of β, and

$$H = \sum \frac{H^g_{\alpha,\beta}}{d} p_\alpha q_\beta z^{l(\alpha)+l(\beta)+2g-2}.$$

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Then H is a solution to the KP hierarchy. (We determine the number of hyperfaces by Euler's formula.)

Proof.

We have $u_1 = z$, $u_2 = u_3 = \cdots = 0$, and $\frac{(d-1)!}{d!} = \frac{1}{d}$. □

For example, H satisfies the pde

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But, rooted maps with vertices of degree at most 3 can be uniquely constructed from rooted maps with all vertices of degree 3 (cubic maps).

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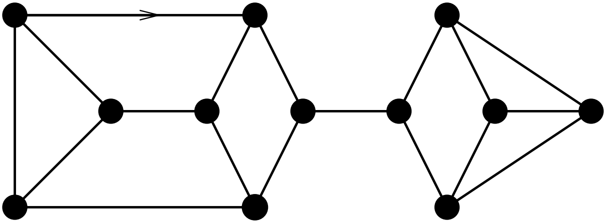
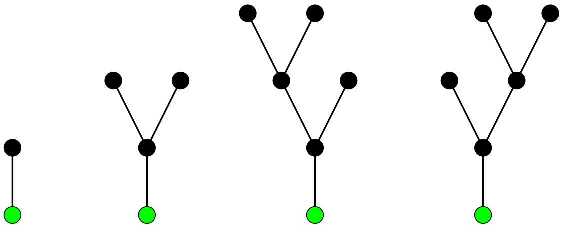
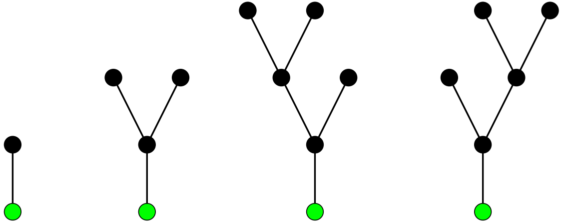


A rooted cubic map in the plane with 8 faces.



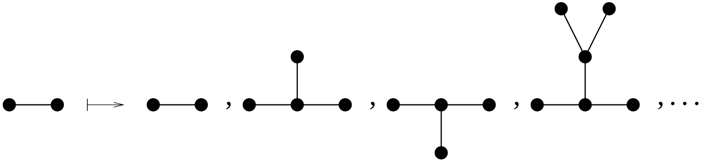
Rooted cubic trees, in which all vertices have degrees 1 or 3.

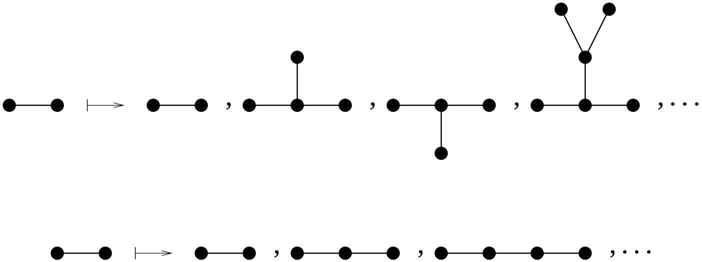


Rooted cubic trees, in which all vertices have degrees 1 or 3.

The generating series $T(x)$, with respect to non-root vertices, satisfies quadratic equation:

$$T = x + xT^2.$$





Let $\mathcal{S} = \{(n, g)\colon n \geq -1, 0 \leq g \leq \frac{1}{2}(n+1)\} = \{(-1,0),(0,0),$ $(1,0),(1,1),\ldots\}$, Let $\mathcal{S} = \{(n, g) \colon n \geq -1, 0 \leq g \leq \frac{1}{2}(n+1)\} = \{(-1,0), (0,0), (1,0), (1,1), \ldots\}$, and $T(n, g)$ be the number of rooted triangulations in a surface of genus g, with $2n$ faces, Let $\mathcal{S} = \{(n,g)\colon n \geq -1, 0 \leq g \leq \frac{1}{2}(n+1)\} = \{(-1,0),(0,0),(1,0),(1,1),\ldots\}$, and $T(n,g)$ be the number of rooted triangulations in a surface of genus g, with $2n$ faces, and $c(n,g) = (3n+2)T(n,g)$.

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$$c(n,g) = \frac{4(3n+2)}{n+1}\left(n(3n-2)c(n-2,g-1) + \sum c(i,h)c(j,k)\right),$$

for $(n,g) \in \mathcal{S} \setminus \{(-1,0)\}$, where the sum is over $\mathcal{S} \times \mathcal{S}$, with $i+j = n-2$ and $h+k = g$.

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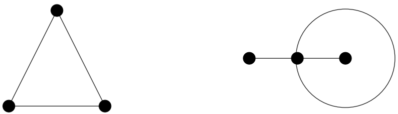
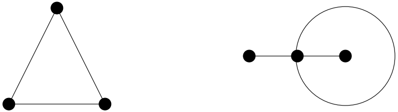
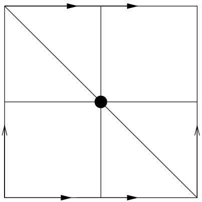
for $(n,g) \in \mathcal{S} \setminus \{(-1,0)\}$, where the sum is over $\mathcal{S} \times \mathcal{S}$, with $i+j = n-2$ and $h+k = g$. The initial conditions are $c(-1,0) = \frac{1}{2}$ and $c(n,g) = 0$ for $(n,g) \notin \mathcal{S}$.



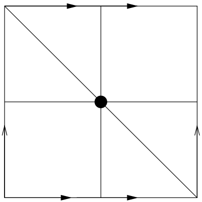
For example, $T(1,0) = 4$,



For example, $T(1,0) = 4$, obtained by the $1+3$ rootings of the maps above.



For example, $T(1,1) = 1$,



For example, $T(1,1) = 1$, since there is only 1 rooting of the map above.

It is known that the asymptotic number of maps in many classes, rooted and unrooted (including rooted triangulations), is given by

$$\alpha\, t_g (\beta N)^{\frac{5}{2}(g-1)} \gamma^N,$$

where N is the number of edges, α, β, γ are constants, and t_g is determined implicitly.

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Recently, our quadratic recurrence for triangulations has been analyzed by Bender, Gao, Richmond, and they have been able to give explicit asymptotics, and hence an explicit form for t_g.

Problems

- Give a combinatorial proof that the generating series for rooted maps with vertices of degrees at most 3 satisfies the pde
$$F_{2,2} - F_{3,1} + \frac{1}{12}F_{1,1,1,1} + \frac{1}{2}F_{1,1}^2 = 0.$$

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- Give a combinatorial proof that the number $T(n,g)$ of rooted triangulations in a surface of genus g with $2n$ faces satisfies the recurrence
$$c(n,g) = \frac{4(3n+2)}{n+1}\Big(n(3n-2)c(n-2,g-1) + \sum c(i,h)c(j,k)\Big),$$
where $c(n,g) = (3n+2)T(n,g)$.