

Monotone Hurwitz numbers and the HCIZ Integral

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The HCIZ integral

The partition function of the model,

$$I_N(z) = \int e^{zN\text{tr}(AUBU^*)} dU,$$

is known as the *Harish-Chandra-Itzykson-Zuber integral*:

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We consider the *free energy* $F_N(z) = \frac{1}{N^2} \log I_N(z)$.

Theorem

Let $(A_N), (B_N)$ be two sequences of $N \times N$ normal matrices whose spectral radii are uniformly bounded, with least upper bound $M := \sup \{\rho(A_N), \rho(B_N) : N \geq 1\}$, with limiting moments

$$-\phi_k := \lim_{N \rightarrow \infty} \frac{1}{N} \operatorname{tr}(A_N^k), \quad -\psi_k := \lim_{N \rightarrow \infty} \frac{1}{N} \operatorname{tr}(B_N^k)$$

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of all orders. Let $0 \leq r < r_c$, where $r_c = \frac{2}{27}$. Then $F_N(z)$ admits an $N \rightarrow \infty$ asymptotic expansion of the form

$$F_N(z) \sim \sum_{g=0}^{\infty} \frac{C_g(z)}{N^{2g}},$$

uniformly on $\bar{D}(0, rM^{-2})$. Each coefficient $C_g(z)$ is a holomorphic function of z on $D(0, r_c M^{-2})$, with Maclaurin series

$$C_g(z) = \sum_{d=1}^{\infty} C_{g,d} \frac{z^d}{d!}.$$

$$C_g(z) = \sum_{d=1}^{\infty} C_{g,d} \frac{z^d}{d!}, \quad C_{g,d} = \sum_{\alpha, \beta \vdash d} \vec{H}_g(\alpha, \beta) \phi_\alpha \psi_\beta,$$

where $\vec{H}_g(\alpha, \beta) = \#(r+2)$ -tuples $(\sigma, \rho, \tau_1, \dots, \tau_r)$ in S_d :

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1. σ, ρ of cycle type α, β , and the τ_i are transpositions;
2. The product $\sigma\rho\tau_1 \dots \tau_r$ equals the identity permutation;
3. The group $\langle \sigma, \rho, \tau_1, \dots, \tau_r \rangle$ acts transitively on $\{1, \dots, d\}$;
4. $r = 2g - 2 + \ell(\alpha) + \ell(\beta)$;

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Without 5, we have (double) Hurwitz numbers $H_g(\alpha, \beta)$.

Hurwitz numbers

$$\mathbf{H}(z, t, p_1, \dots, q_1, \dots) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{\alpha, \beta \vdash d} H_g(\alpha, \beta) p_{\alpha} q_{\beta}$$

satisfies the **join-cut equation**

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \sum_{i, j \geq 1} (i + j) p_i p_j \frac{\partial \mathbf{H}}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2 \mathbf{H}}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial \mathbf{H}}{\partial p_i} \frac{\partial \mathbf{H}}{\partial p_j},$$

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and is a solution to the $2D$ Toda lattice hierarchy, KP hierarchy, KdV hierarchy. (Witten's conjecture).

Hurwitz numbers and Hodge integrals

The **ELSV Theorem** (Ekedahl, Lando, Shapiro, Vainshtein): $\alpha \vdash d$,

$$H_g((\alpha_1, \dots, \alpha_m), 1^d) = \frac{(2g - 2 + m + d)! d!}{|\text{Aut } \alpha|} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!} P_g(\alpha_1, \dots, \alpha_m),$$

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where $P_g(\alpha_1, \dots, \alpha_m)$ is a polynomial in $\alpha_1, \dots, \alpha_m$ given by

$$P_g(\alpha_1, \dots, \alpha_m) = \int_{\overline{\mathcal{M}}_{g,m}} \frac{1 - \lambda_1 + \dots + (-1)^g \lambda_g}{(1 - \alpha_1 \psi_1) \cdots (1 - \alpha_m \psi_m)},$$

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where $\overline{\mathcal{M}}_{g,m}$ is the (compact) moduli space of stable m -pointed genus g curves, ψ_1, \dots, ψ_m are (complex) codimension 1 classes, λ_k is the (complex codimension k) k th Chern class of the Hodge bundle.

Hurwitz numbers in low genus

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$$H_0(\alpha) = \frac{(d - 2 + \ell(\alpha))! d!}{|\text{Aut } \alpha|} \left(\prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) d^{\ell(\alpha)-3},$$

$$H_1(\alpha) = \frac{(d + \ell(\alpha))! d!}{|\text{Aut } \alpha|} \frac{1}{24} \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \\ \times \left(d^{\ell(\alpha)} - d^{\ell(\alpha)-1} - \sum_{k=2}^{\ell(\alpha)} (k-2)! d^{\ell(\alpha)-k} e_k(\alpha) \right).$$

Hurwitz numbers in low genus - generating series

Consider the functional equation

$$w = ze^{\delta}, \quad \delta = \sum_{m \geq 1} \frac{m^m}{m!} p_m w^m,$$

and let $\phi = \sum_{m \geq 1} \frac{m^{m+1}}{m!} p_m w^m$.

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$$\left(z \frac{d}{dz}\right)^2 \mathbf{H}_0 = \delta, \quad \mathbf{H}_1 = \frac{1}{24} (\log(1 - \phi)^{-1} - \delta),$$

the generating series for single Hurwitz numbers in genus 0, 1.

Hurwitz numbers in genus 2, 3 - generating series

Let $\phi_i = \sum_{m \geq 1} \frac{m^{m+i}}{m!} p_m w^m$, $i \geq 1$ (so $\phi = \phi_0$). Then

$$\mathbf{H}_2 = \frac{1}{5760} \left(\frac{5\phi_3 - 12\phi_2 + 7\phi_1}{(1-\phi)^3} + \frac{29\phi_1\phi_2 - 25\phi_1^2}{(1-\phi)^4} + \frac{28\phi_2^3}{(1-\phi)^5} \right),$$

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(ELSV Theorem, together with **string**, **dilaton** equations.)

Single Hurwitz series and join-cut equation

The **single Hurwitz generating series**

$$\mathbf{H}(z, t, p_1, \dots) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{\alpha \vdash d} H_g(\alpha) p_\alpha$$

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with **initial condition** $\mathbf{H}(z, 0, p_1, \dots) = zp_1$.

Proof:

1. σ of cycle type α and the τ_i are transpositions;
2. The product $\sigma\tau_1 \dots \tau_r$ equals the identity permutation;
3. The group $\langle \sigma, \tau_1, \dots, \tau_r \rangle$ acts transitively on $\{1, \dots, d\}$;
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$$\frac{1}{2t} \left(z \frac{\partial \vec{H}}{\partial z} - z p_1 \right) = \frac{1}{2} \sum_{i,j \geq 1} (i+j) p_i p_j \frac{\partial \vec{H}}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2 \vec{H}}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial \vec{H}}{\partial p_i} \frac{\partial \vec{H}}{\partial p_j}.$$

Integrable hierarchies (2 Toda, KP, KdV)

Let \mathcal{C}_α denote the formal sum of all permutations in the conjugacy class with cycles lengths specified by the parts of α . Then for single Hurwitz numbers, we consider

$$[\mathcal{C}_{1^d}] \mathcal{C}_\alpha \mathcal{C}_{2^{1^{d-2}}}.$$

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Let $\mathcal{J}_m = \sum_{1 \leq i < m} (i \ m)$, the **Jucys-Murphy** elements (so $\mathcal{J}_1 + \dots + \mathcal{J}_d = \mathcal{C}_{2^{1^{d-2}}}$). Then for single **monotone** Hurwitz numbers, we consider

$$[\mathcal{C}_{1^d}] \mathcal{C}_\alpha h_r(\mathcal{J}_1, \dots, \mathcal{J}_d),$$

where h_r is the complete symmetric function of degree r .

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For the latter, it is well known that symmetric functions in $\mathcal{J}_1, \dots, \mathcal{J}_d$ are in the centre of S_d (is a linear combination of conjugacy classes):

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$$e_r(\mathcal{J}_1, \dots, \mathcal{J}_d) = \sum_{\ell(\alpha)=d-r} \mathcal{C}_\alpha.$$

Genus 0

$$H_0(\alpha) = \frac{(d-2+\ell(\alpha))!d!}{|\text{Aut } \alpha|} \left(\prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) d^{\ell(\alpha)-3},$$

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$$\vec{H}_0(\alpha) = \frac{d!}{|\text{Aut } \alpha|} \left(\prod_{i=1}^{\ell(\alpha)} \binom{2\alpha_i}{\alpha_i} \right) (2d+1)^{\overline{\ell(\alpha)-3}},$$

where $(x)^{\overline{m}} = x(x+1)\cdots(x+m)$, the rising product with m factors.

Genus 0

$$H_0(\alpha) = \frac{(d - 2 + \ell(\alpha))! d!}{|\text{Aut } \alpha|} \left(\prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) d^{\ell(\alpha)-3},$$

$$\vec{H}_0(\alpha) = \frac{d!}{|\text{Aut } \alpha|} \left(\prod_{i=1}^{\ell(\alpha)} \binom{2\alpha_i}{\alpha_i} \right) (2d + 1)^{\overline{\ell(\alpha)-3}},$$

where $(x)^{\overline{m}} = x(x+1) \cdots (x+m)$, the rising product with m factors. By convention, when $m < 0$,

$$(x)^{\overline{m}} = \frac{1}{(x+m+1)^{\overline{-m}}}.$$

Genus 1

$$H_1(\alpha) = \frac{(d + \ell(\alpha))! d!}{|\text{Aut } \alpha|} \frac{1}{24} \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \left(d^{\ell(\alpha)} - d^{\ell(\alpha)-1} \right. \\ \left. \times - \sum_{k=2}^{\ell(\alpha)} (k-2)! d^{\ell(\alpha)-k} e_k(\alpha) \right),$$

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where $e_k(2\alpha+1)$ is the k th elementary symmetric polynomial in $2\alpha_1+1, 2\alpha_2+1, \dots, 2\alpha_{\ell(\alpha)}+1$.

Generating series in genus 0 and 1

Hurwitz: $w = ze^\delta$,

$$\delta = \sum_{m \geq 1} \frac{m^m}{m!} \rho_m w^m, \quad \phi = \sum_{m \geq 1} \frac{m^{m+1}}{m!} \rho_m w^m,$$

$$\left(z \frac{d}{dz}\right)^2 \mathbf{H}_0 = \delta, \quad \mathbf{H}_1 = \frac{1}{24} \left(\log \frac{1}{1-\phi} - \delta \right).$$

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$$\left(2z \frac{d}{dz} - 1\right) 2z \frac{d}{dz} \vec{\mathbf{H}}_0 = \gamma - \frac{1}{2} \gamma^2, \quad \vec{\mathbf{H}}_1 = \frac{1}{24} \log \frac{1}{1-\eta} - \frac{1}{8} \log \frac{1}{1-\gamma}.$$

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Generating series in genus 2, 3

Hurwitz: Let $\phi_i = \sum_{m \geq 1} \frac{m^{m+i}}{m!} p_m w^m$, $i \geq 1$ (so $\phi = \phi_0$). Then

$$\mathbf{H}_2 = \frac{1}{5760} \left(\frac{5\phi_3 - 12\phi_2 + 7\phi_1}{(1-\phi)^3} + \frac{29\phi_1\phi_2 - 25\phi_1^2}{(1-\phi)^4} + \frac{28\phi_2^3}{(1-\phi)^5} \right).$$

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Let $\eta_i = \sum_{m \geq 1} m^i (2m+1) \binom{2m}{m} p_m s^m$, $i \geq 1$. Then

$$\vec{\mathbf{H}}_2 = \frac{8}{5760} \left(\frac{5\eta_3 - 6\eta_2 - 5\eta_1}{(1-\eta)^3} + \frac{29\eta_2\eta_1 - 10\eta_1^2}{(1-\eta)^4} + \frac{28\eta_1^3}{(1-\eta)^5} - 3 \left(1 - \frac{1}{(1-\eta)^2} \right) \right).$$