

SYMMETRIC FUNCTIONS, CODES OF PARTITIONS AND THE KP HIERARCHY

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ABSTRACT. We consider an operator of Bernstein for symmetric functions, and give an explicit formula for its action on an arbitrary Schur function. This formula is given in a remarkably simple form when written in terms of some notation based on the code of a partition. As an application, we give a new and very simple proof of a classical result for the KP hierarchy, which involves the Plücker relations for Schur function coefficients in a τ -function for the hierarchy. This proof is especially compact because of a restatement that we give for the Plücker relations that is symmetrical in terms of partition code notation.

1. INTRODUCTION

In Macdonald's fundamental book [M, p. 95] on symmetric functions, he considers the operators $B(t)$, B_n given by

$$(1) \quad B(t) := \sum_{n \in \mathbb{Z}} B_n t^n := \exp \left(\sum_{k \geq 1} \frac{t^k}{k} p_k \right) \exp \left(- \sum_{k \geq 1} t^{-k} \frac{\partial}{\partial p_k} \right),$$

which he attributes to Bernstein [Z, p. 69]. Here t is an indeterminate, and p_k is the k th power sum symmetric function of a countable set of indeterminates (in which the power sums are symmetric). Of course, since the power sum symmetric functions p_1, p_2, \dots in a countable set of indeterminates are algebraically independent (see Section 3 for this and other basic results about symmetric functions), we can regard p_1, p_2, \dots as indeterminates. Hence we consider Bernstein's operators $B(t)$, B_n applied to formal power series in p_1, p_2, \dots . Let the *weight* of the monomial $p_1^{i_1} p_2^{i_2} \dots$ be given by $i_1 + 2i_2 + \dots$. Then note that as a series in t , $B(t)$ goes to infinity in both directions, but that when it is applied to a monomial of weight m in p_1, p_2, \dots , the result is a Laurent series with minimum degree $-m$ in t .

The main result of this paper is an explicit expansion for the Laurent series that results from applying $B(t)$ to an arbitrary Schur symmetric function (where the Schur function is written as a polynomial in p_1, p_2, \dots with symmetric group characters as coefficients, as given in (8)). This result, appearing as Theorem 3.4, is given in a remarkably simple form by using an indexing notation for partitions that is natural in terms of the *code* of the partition.

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For an application of our main result, we turn to the KP hierarchy of mathematical physics. By combining our main result and its dual form, we are able to give a new, simple proof (in Theorem 5.2) of the classical result for the KP hierarchy that relates the Schur symmetric function expansion of a τ -function for the hierarchy and the Plücker relations for the coefficients in this expansion. This proof is made especially compact by our restatement (as Theorem 4.1) of the Plücker relations, in a highly symmetrical form in terms of the code notation.

The KP (Kadomtsev-Petviashvili) hierarchy is a completely integrable hierarchy that generalizes the KdV hierarchy (an integrable hierarchy is a family of partial differential equations which are simultaneously solved; for a comprehensive and concise account of the KP hierarchy with an emphasis on the view of physics, see Miwa, Jimbo and Date [MJD]). Over the last two decades, there has been strong interest in the relationship between integrable hierarchies and moduli spaces of curves. This began with Witten's Conjecture [W] for the KdV equations, proved by Kontsevich [Ko] (and more recently by a number of others, including [KL]). Pandharipande [P] conjectured that solutions to the Toda equations arose in a related context, which was proved by Okounkov [Ok], who also proved the more general result that a generating series for what he called *double Hurwitz numbers* satisfies the KP hierarchy. Kazarian and Lando [KL], in their recent proof of Witten's conjecture, showed that it is implied by Okounkov's result for double Hurwitz numbers. More recently, Kazarian [Ka] has given a number of interesting results about the structure of solutions to the KP hierarchy. Combinatorial aspects of this connection have been studied by Goulden and Jackson [GJ].

The structure of this paper is as follows. In Section 2, we describe the notation for partitions (which index symmetric functions), their diagrams and codes (in terms of the physics presentation in [MJD], the code is equivalent to a *Maya diagram* without the location parameter of charge). In Section 3 we describe the algebra of symmetric functions, and give our main result, with a combinatorial proof in terms of the diagrams of partitions. Section 4 contains our symmetrical restatement of the Plücker relations. Finally, in Section 5, we give our new proof of the Schur-Plücker result for the KP hierarchy, together with a description of an equivalent system of partial differential equations for the KP hierarchy in terms of codes of partitions.

2. PARTITIONS AND CODES

We begin with some notation for partitions (see [M], [S]). If $\lambda_1, \dots, \lambda_n$ are integers with $\lambda_1 \geq \dots \geq \lambda_n \geq 1$ and $\lambda_1 + \dots + \lambda_n = d$, then $\lambda = (\lambda_1, \dots, \lambda_n)$ is said to be a *partition* of $|\lambda| := d$ (indicated by writing $\lambda \vdash d$) with $l(\lambda) := n$ *parts*. The empty list ε of integers is to be regarded as a partition of $d = 0$ with $n = 0$ parts, and let \mathcal{P} denote the set of all partitions. If λ has f_j parts equal to j for $j = 1, \dots, d$, then we also write $\lambda = (d^{f_d}, \dots, 1^{f_1})$, where convenient. Also, $\text{Aut } \lambda$ denotes the set of permutations of the n positions that fix λ ; therefore $|\text{Aut } \lambda| = \prod_{j \geq 1} f_j!$. The *conjugate* of λ is the partition $\lambda' = (\lambda'_1, \dots, \lambda'_m)$, where $m = \lambda_1$, and λ'_i is equal to the number of parts λ_j of λ with $\lambda_j \geq i$, $i = 1, \dots, m$. The *diagram* of λ is a left-justified array of juxtaposed unit squares, with λ_i squares in the i th row. For example, the diagram of the partition $(6, 5, 5, 4, 1)$ is given on the left hand side

of Figure 1. The diagram gives a compact geometrical description of the conjugate; the diagram of λ' is obtained by reflecting the diagram of λ about the main diagonal.

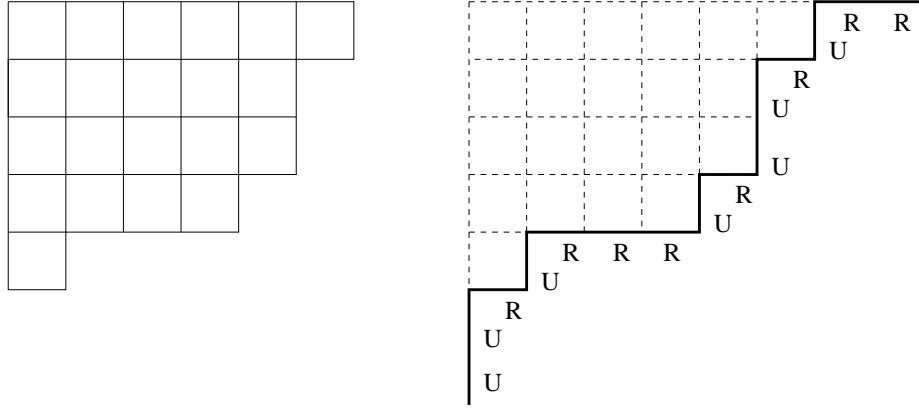


FIGURE 1. The diagram and code for the partition $(6, 5, 5, 4, 1)$.

The *code* of a partition is a two-way infinite binary string in two symbols, say U, R. The string is infinitely U to the left, and infinitely R to the right, and represents a lattice path with unit up-steps (for "U") and unit right-steps (for "R"); the path starts moving up the y -axis towards the origin, and ends moving out the x -axis away from the origin, and forms the lower right hand boundary of the diagram of the partition, when the diagram is placed with its upper left hand corner at the origin. An example is given on the right hand side of Figure 1, which gives us the string ...UURURRRURUURURR... as the code of the partition $(6, 5, 5, 4, 1)$ (see, e.g., [S, p. 467] for more on codes; in [S], the symbols in the code are 0, 1, but we prefer U, R as more mnemonic.)

We let $\lambda^{(i)}$, $i \geq 1$, be the partition whose code is obtained from the code of the partition λ by switching the i th R (from the left) to U. If $\lambda = (\lambda_1, \dots, \lambda_n)$, then we immediately have

$$(2) \quad \lambda^{(i)} = (\lambda_1 - 1, \dots, \lambda_j - 1, i - 1, \lambda_{j+1}, \dots, \lambda_n),$$

where j is chosen (uniquely) from $0, \dots, n$ so that $\lambda_j \geq i > \lambda_{j+1}$ (with the conventions that $\lambda_{n+1} = 0$ and $\lambda_0 = \infty$). Now define $u_i(\lambda)$ to be the number of up-steps U that follow the i th right-step R from the left in the code of λ . Then note that $u_i(\lambda) = j$, and that we have $|\lambda^{(i)}| = |\lambda| - j + i - 1$ from (2), so we can determine $u_i(\lambda) = j$ in terms of i via

$$(3) \quad u_i(\lambda) = |\lambda| - |\lambda^{(i)}| + i - 1.$$

Also note that $u_i(\lambda)$ weakly decreases as i increases, so we also obtain

$$(4) \quad |\lambda| - l(\lambda) = |\lambda^{(1)}| < |\lambda^{(2)}| < \dots, \quad |\lambda^{(i)}| = |\lambda| + i - 1, \quad i > \lambda_1.$$

We also let $\lambda^{(-i)}$, $i \geq 1$, be the partition whose code is obtained from the code of the partition λ by switching the i th U (from the right) to R. If $\lambda = (\lambda_1, \lambda_2, \dots)$, with $\lambda_1 \geq \lambda_2 \dots \geq 0$ (i.e., λ has a finite number, say n , of positive parts, and then append an infinite sequence of 0's), then we immediately have

$$(5) \quad \lambda^{(-i)} = (\lambda_1 + 1, \dots, \lambda_{i-1} + 1, \lambda_{i+1}, \dots).$$

Thus we have $|\lambda^{(-i)}| = |\lambda| - \lambda_i + i - 1$, and since λ_i weakly decreases as i increases, we obtain

$$(6) \quad |\lambda| - \lambda_1 = |\lambda^{(-1)}| < |\lambda^{(-2)}| < \dots, \quad |\lambda^{(-i)}| = |\lambda| + i - 1, \quad i > l(\lambda).$$

Finally, we define skew diagrams. For partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_m)$, with $n \geq m$ and $\lambda_i \geq \mu_i$ for $i = 1, \dots, m$, the *skew diagram* $\lambda - \mu$ is the array of unit squares obtained by removing the squares of the diagram of μ from the diagram of λ . For example, the skew diagram $(6, 5, 5, 4, 1) - (3, 3, 2, 1)$ is given in Figure 2.

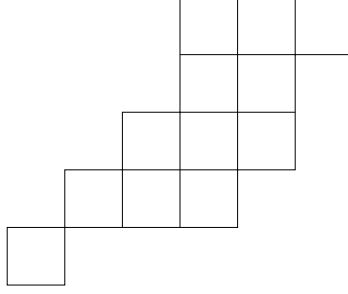


FIGURE 2. The skew diagram $(6, 5, 5, 4, 1) - (3, 3, 2, 1)$.

A skew diagram is a *horizontal strip* if it contains at most one square in every column; it is a *vertical strip* if it contains at most one square in every row. When the total number of squares in a horizontal (or vertical) strip is equal to k , we may refer to this skew diagram as a horizontal (or vertical) k -strip.

3. CODES OF PARTITIONS AND SYMMETRIC FUNCTIONS

3.1. The algebra of symmetric functions. We consider symmetric functions in x_1, x_2, \dots , and refer to [S], [M] for the results that we use in this paper. The i th *power sum* symmetric function is $p_i = \sum_{j \geq 1} x_j^i$, for $i \geq 1$, with $p_0 := 1$. The i th *complete* symmetric function h_i and the i th *elementary* symmetric function are defined by $\sum_{i \geq 0} h_i t^i = \prod_{j \geq 1} (1 - x_j t)^{-1}$ and $\sum_{i \geq 0} e_i t^i = \prod_{j \geq 1} (1 + x_j t)$, respectively, and are related to the power sums through

$$(7) \quad \sum_{i \geq 0} h_i t^i = \exp \sum_{k \geq 1} \frac{p_k}{k} t^k, \quad \sum_{i \geq 0} e_i t^i = \exp \sum_{k \geq 1} \frac{p_k}{k} (-1)^{k-1} t^k.$$

In the algebra $\Lambda_{\mathbb{Q}}$ of symmetric functions in x_1, x_2, \dots over \mathbb{Q} , the $p_i, i \geq 1$ are algebraically independent, the $h_i, i \geq 1$ are algebraically independent, and the $e_i, i \geq 1$ are algebraically independent. This algebra is *graded*, by total degree in the underlying indeterminates x_1, x_2, \dots . Moreover, $\Lambda_{\mathbb{Q}}$ is generated by the p_i 's, the h_i 's or the e_i 's, so we have

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \dots] = \mathbb{Q}[h_1, h_2, \dots] = \mathbb{Q}[e_1, e_2, \dots].$$

Now suppose we define quantities $p_{\lambda}, h_{\lambda}, e_{\lambda}$ indexed by the partition $\lambda = (\lambda_1, \dots, \lambda_n)$, *multiplicatively*, which means that we write $p_{\lambda} = p_{\lambda_1} \cdots p_{\lambda_n}$, $h_{\lambda} = h_{\lambda_1} \cdots h_{\lambda_n}$, $e_{\lambda} = e_{\lambda_1} \cdots e_{\lambda_n}$, with the empty product convention $p_{\varepsilon} = h_{\varepsilon} = e_{\varepsilon} = 1$. Then of course, regarded as a vector space over \mathbb{Q} , $\Lambda_{\mathbb{Q}}$ has (multiplicative) bases $\{p_{\lambda} : \lambda \in \mathcal{P}\}$, $\{h_{\lambda} : \lambda \in \mathcal{P}\}$ and $\{e_{\lambda} : \lambda \in \mathcal{P}\}$.

Another (non-multiplicative) basis is $\{s_\lambda : \lambda \in \mathcal{P}\}$, where s_λ is the *Schur* function of *shape* λ . There are many fascinating aspects about Schur functions, but the one that we shall focus on in this paper is their connection with characters of the symmetric group (we refer to [S] for the results about characters and representations of the symmetric group used in this paper). The *conjugacy classes* of the symmetric group \mathfrak{S}_d on $\{1, \dots, d\}$ are indexed by the partitions of d (and for partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the conjugacy class \mathcal{C}_λ consists of all permutations in \mathfrak{S}_d whose disjoint cycle lengths give the parts of λ in some order). The *irreducible representations* of \mathfrak{S}_d are also indexed by the partitions of d . For partitions λ, μ of d , let χ_μ^λ denote the *character* of the irreducible representation of \mathfrak{S}_d indexed by λ , evaluated at any element of the conjugacy class \mathcal{C}_μ (in general, characters are constant on conjugacy classes); we usually refer to χ_μ^λ generically as an irreducible character. Then, to change bases of $\Lambda_{\mathbb{Q}}$ between $\{p_\lambda : \lambda \in \mathcal{P}\}$, and $\{s_\lambda : \lambda \in \mathcal{P}\}$, we have

$$(8) \quad p_\lambda = \sum_{\mu \vdash d} \chi_\lambda^\mu s_\mu, \quad s_\lambda = \sum_{\mu \vdash d} \frac{|\mathcal{C}_\mu|}{d!} \chi_\mu^\lambda p_\mu, \quad \lambda \vdash d.$$

The endomorphism $\omega : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ defined by $\omega(e_n) = h_n$, $n \geq 1$ is an involution on $\Lambda_{\mathbb{Q}}$. For Schur functions, we have

$$(9) \quad \omega(s_\lambda) = s_{\lambda'}.$$

We have various multiplication rules for Schur functions. For example,

$$(10) \quad h_n s_\lambda = \sum_{\mu} s_\mu,$$

where the sum is over partitions μ such that $\mu - \lambda$ is a horizontal n -strip. In general, when the involution ω is applied to an equation for symmetric functions, we call the resulting equation the *dual*. For example, applying ω to (10), we obtain the dual result

$$(11) \quad e_n s_\lambda = \sum_{\mu} s_\mu,$$

where the sum is over partitions μ such that $\mu - \lambda$ is a vertical n -strip. (Equations (10) and (11) are often referred to as *Pieri* rules.)

Let $\langle \cdot, \cdot \rangle$ be the bilinear form for $\Lambda_{\mathbb{Q}}$ defined by

$$(12) \quad \langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}, \quad \lambda, \mu \in \mathcal{P}.$$

For each symmetric function $f \in \Lambda_{\mathbb{Q}}$, let $f^\perp : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ be the adjoint of multiplication by f , so

$$\langle f^\perp g_1, g_2 \rangle = \langle g_1, f g_2 \rangle, \quad g_1, g_2 \in \Lambda_{\mathbb{Q}}.$$

Then

$$(13) \quad p_n^\perp = n \frac{\partial}{\partial p_n}, \quad n \geq 1.$$

Moreover, from (10) and (12) we have

$$(14) \quad h_n^\perp s_\lambda = \sum_{\mu} s_\mu,$$

where the sum is over partitions μ such that $\lambda - \mu$ is a horizontal n -strip. Similarly, from (11) and (12) we have

$$(15) \quad e_n^\perp s_\lambda = \sum_{\mu} s_{\mu},$$

where the sum is over partitions μ such that $\lambda - \mu$ is a vertical n -strip.

We now consider Bernstein's symmetric function operators $B(t)$, B_n defined in (1). Equivalently, from (7) and (13), together with the fact that the partial differential operators $\frac{\partial}{\partial p_k}$ commute, we have

$$(16) \quad B(t) := \sum_{n \in \mathbb{Z}} B_n t^n := \sum_{k, m \geq 0} (-1)^m t^{k-m} h_k e_m^\perp,$$

(where B_n is an operator on $\Lambda_{\mathbb{Q}}$ for each $n \in \mathbb{Z}$). Then we also have

$$(17) \quad B^\perp(t) = \sum_{n \in \mathbb{Z}} B_n^\perp t^n = \sum_{k, m \geq 0} (-1)^m t^{k-m} e_m h_k^\perp,$$

and we immediately obtain $B_n^\perp = (-1)^n \omega B_{-n} \omega$ (note that this corrects the relation given in [M, p. 96]). In terms of the series B itself, this becomes the operator equation

$$(18) \quad B^\perp(t) = \omega B(-t^{-1}) \omega,$$

where ω applied to a series in t acts linearly, on the coefficients of each monomial in t .

3.2. A combinatorial treatment of Bernstein's operator. We define $\mathcal{R}_{\lambda, \mu}$ to be the set of partitions ν such that $\lambda - \nu$ is a vertical strip, and $\mu - \nu$ is a horizontal strip. Let

$$(19) \quad R_{\lambda, \mu} = \sum_{\nu \in \mathcal{R}_{\lambda, \mu}} (-1)^{|\lambda| - |\nu|},$$

which is 0 when the set $\mathcal{R}_{\lambda, \mu}$ is empty. This sum arises naturally in the action of $B(t)$ on a Schur function s_λ , as given in the following result.

Proposition 3.1. *For any partition λ , we have*

$$B(t) s_\lambda = \sum_{\mu \in \mathcal{P}} R_{\lambda, \mu} t^{|\mu| - |\lambda|} s_\mu.$$

Proof. The result follows immediately from (16), (10) and (15). □

Now we consider the structure of the set $\mathcal{R}_{\lambda, \mu}$, beginning with the example given in Figure 3, where the diagrams of $\lambda = (5, 4, 2, 2, 2, 1, 1)$ and $\mu = (6, 4, 4, 2, 1, 1)$ are given on the left and right of the diagram, respectively. There are three classes of squares that we shall consider when $\mathcal{R}_{\lambda, \mu}$ is nonempty:

- the squares of μ that are not contained in λ , necessarily bottommost in their column of μ . None of these is contained in any ν in $\mathcal{R}_{\lambda, \mu}$ (in Figure 3, these squares contain "+"). Such squares are contained in the horizontal strip that is added in the multiplication by an h_k ;

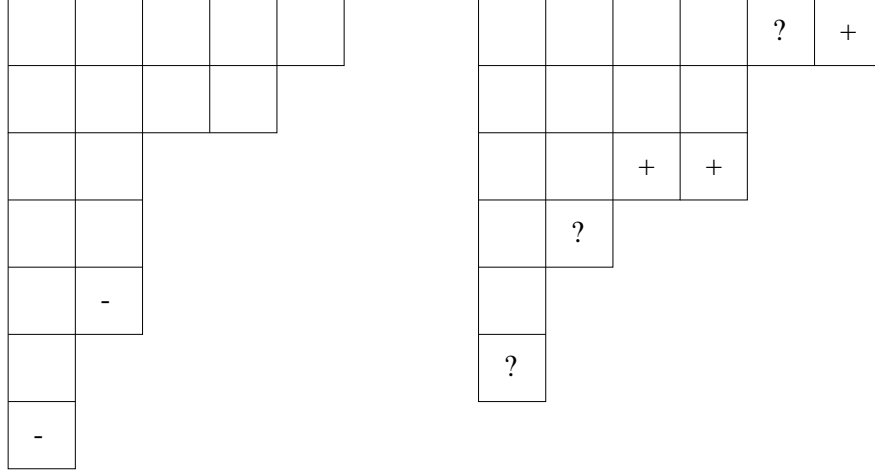


FIGURE 3. Partitions λ and μ with λ -ambiguous squares.

- the squares of λ that are not contained in μ , necessarily rightmost in their row of λ . All of these are contained in every ν in $\mathcal{R}_{\lambda,\mu}$ (in Figure 3, these squares contain “-”) Such squares are contained in the vertical strip that is deleted in the application of an e_m^\perp ;
- the squares that are contained in both λ and μ , that are rightmost in their row of λ , and bottommost in their column of μ . Each of these is contained in some of the ν in $\mathcal{R}_{\lambda,\mu}$ (actually in half of the ν), but not others. We call such a square a λ -ambiguous square of μ (in Figure 3, the λ -ambiguous squares of μ contain “?”). Such squares may have been contained in both a deleted vertical strip and an added horizontal strip, or in neither.

Lemma 3.2. *If μ has any λ -ambiguous squares, then*

$$R_{\lambda,\mu} = 0.$$

Proof. If μ has any λ -ambiguous squares, let c be the rightmost of these (there is at most one λ -ambiguous square in any column of μ , since it can only occur as the bottommost element of a column). Define the mapping

$$\phi : \mathcal{R}_{\lambda,\mu} \rightarrow \mathcal{R}_{\lambda,\mu} : \nu \mapsto \nu'$$

as follows: if ν contains c , then ν' is obtained by removing c from ν ; if ν does not contain c , then ν' is obtained by adding c to ν . This is well-defined, for the following reasons: since c is rightmost in its row of λ and bottommost in its column of μ , every square of λ in the same column as c and below c must belong to the vertical strip $\lambda - \nu$ (and no other squares in this column can belong to this vertical strip), so $\lambda - \nu'$ is a vertical strip whether ν contains c or not; also, every square of μ in the same row as c and to the right of c must belong to the horizontal strip $\mu - \nu$ (and no other squares in this row can belong to this horizontal strip), so $\mu - \nu'$ is a horizontal strip whether ν contains c or not).

Clearly ϕ is an involution on $\mathcal{R}_{\lambda,\mu}$, so it is a bijection, and thus we have

$$(20) \quad R_{\lambda,\mu} = \sum_{\nu \in \mathcal{R}_{\lambda,\mu}} (-1)^{|\lambda|-|\nu|} = - \sum_{\nu' \in \mathcal{R}_{\lambda,\mu}} (-1)^{|\lambda|-|\nu'|} = -R_{\lambda,\mu},$$

where, for the second equality, we have changed summation variables to $\phi(\nu) = \nu'$. The result follows immediately. (In the context of (20), ϕ is often referred to as a *sign-reversing involution*.) \square

If $\mathcal{R}_{\lambda,\mu}$ is nonempty and μ has no λ -ambiguous squares, we call μ a λ -*survivor*. Note that in this case there is a unique ν in $\mathcal{R}_{\lambda,\mu}$; the terminology is chosen since μ “survives” the involution in Lemma 3.2.

Proposition 3.3. *In a λ -survivor μ :*

- (a) *if the rightmost square of row i of λ is not contained in μ , then the rightmost square of row $i - 1$ of λ is not contained in μ ;*
- (b) *if the bottommost square of column i of μ is not contained in λ , then the bottommost square of column $i - 1$ of μ is not contained in λ .*

Proof. For part (a), if the rightmost square of row i of λ is not contained in μ and the rightmost square of row $i - 1$ of λ is contained in μ , then the latter must be bottommost in its column of μ . But that makes it a λ -ambiguous square, impossible in a λ -survivor. For (b), if the bottommost square of column i of μ is not contained in λ and the bottommost square of column $i - 1$ of μ is contained in λ , then the latter is by definition a λ -ambiguous square, impossible in a λ -survivor. \square

3.3. The main result. Now we are able to determine explicitly the action of $B(t)$ on a single Schur function s_λ . The following is our main result.

Theorem 3.4. *For any partition λ , we have*

$$B(t) s_\lambda = \sum_{i \geq 1} (-1)^{|\lambda|-|\lambda^{(i)}|+i-1} t^{|\lambda^{(i)}|-|\lambda|} s_{\lambda^{(i)}}.$$

Proof. From Proposition 3.1 and Lemma 3.2, we have

$$(21) \quad B(t) s_\lambda = \sum_{\mu} R_{\lambda,\mu} t^{|\mu|-|\lambda|} s_\mu,$$

where the summation is over all λ -survivors μ . Now we characterize the λ -survivors. Suppose $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \geq \dots \geq \lambda_n \geq 1$, and we let $\lambda_0 = \infty$, $\lambda_{n+1} = 0$. Then in a λ -survivor μ , from Proposition 3.3(a), the rightmost cells in rows $1, \dots, j$ of λ are not contained in μ , and the rightmost cells of rows $j + 1, \dots, n$ are contained in μ , for some $j = 0, \dots, n$ with $\lambda_j > \lambda_{j+1}$. But, in order to avoid the bottommost cell of column λ_{j+1} in μ being λ -ambiguous, then the bottommost cell of column λ_{j+1} in μ cannot be contained in λ . Thus we conclude from Proposition 3.3(b) that the bottommost cells in columns $1, \dots, \lambda_{j+1}$ of μ are not contained in λ . Also, the bottommost squares in columns $\lambda_{j+1}, \dots, i - 1$ of μ are not contained in λ for some $\lambda_{j+1} < i \leq \lambda_j$. Finally, for each $i \geq 1$, there exists a choice of $j = 0, \dots, n$ for which $\lambda_{j+1} < i \leq \lambda_j$, so the λ -survivor μ described above exists for each $i \geq 1$. This partition μ is obtained from λ by deleting the column

strip consisting of the rightmost squares in rows $1, \dots, j$, and adding the horizontal strip consisting of the bottommost cells in columns $1, \dots, i - 1$. This gives

$$\mu = (\lambda_1 - 1, \dots, \lambda_j - 1, i - 1, \lambda_{j+1}, \dots, \lambda_n) = \lambda^{(i)},$$

where the second equality is from (2), and where $j = u_i(\lambda)$. But we have $R_{\lambda, \lambda^{(i)}} = (-1)^j$, and the result follows from (3) and (21). \square

Note that the right hand side of the above result is a Laurent series in t for each λ , with minimum power of t given by $t^{|\lambda^{(1)}| - |\lambda|} = t^{-l(\lambda)}$.

The following pair of dual corollaries to our main result will be particularly convenient for dealing with the KP hierarchy.

Corollary 3.5. *For scalars a_λ , $\lambda \in \mathcal{P}$, we have*

$$B(t) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda = \sum_{\beta \in \mathcal{P}} s_\beta \sum_{k \geq 1} (-1)^{k-1} t^{|\beta| - |\beta^{(-k)}|} a_{\beta^{(-k)}}.$$

Proof. From Theorem 3.4 and (3), we immediately obtain

$$(22) \quad B(t) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda = \sum_{\lambda \in \mathcal{P}} a_\lambda \sum_{i \geq 1} (-1)^{u_i(\lambda)} t^{|\lambda^{(i)}| - |\lambda|} s_{\lambda^{(i)}}.$$

Now from the code description, it is immediate that $\beta = \lambda^{(i)}$ is equivalent to $\lambda = \beta^{(-k)}$, where $k = u_i(\lambda) + 1$. The result follows immediately, by changing summation variables in (22) from $\lambda \in \mathcal{P}, i \geq 1$ to $\beta \in \mathcal{P}, k \geq 1$. \square

Corollary 3.6. *For scalars a_λ , $\lambda \in \mathcal{P}$, we have*

$$B^\perp(t^{-1}) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda = \sum_{\alpha \in \mathcal{P}} s_\alpha \sum_{m \geq 1} (-1)^{|\alpha| - |\alpha^{(m)}| + m - 1} t^{|\alpha| - |\alpha^{(m)}|} a_{\alpha^{(m)}}.$$

Proof. From Theorem 3.4, (18) and (9), we obtain

$$B^\perp(t^{-1}) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda = \sum_{\lambda \in \mathcal{P}} a_\lambda \omega B(-t) s_{\lambda'} = \sum_{\lambda \in \mathcal{P}} a_\lambda \sum_{i \geq 1} (-1)^{i-1} t^{|\lambda^{(i)}| - |\lambda'|} s_{((\lambda')^{(i)})'}.$$

But from the code description, it is immediate that $(\lambda')^{(i)} = (\lambda^{(-i)})'$, and since $|\mu'| = |\mu|$ for any partition μ , we can simplify the double summation above to obtain

$$(23) \quad B^\perp(t^{-1}) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda = \sum_{\lambda \in \mathcal{P}} a_\lambda \sum_{i \geq 1} (-1)^{i-1} t^{|\lambda^{(-i)}| - |\lambda|} s_{\lambda^{(-i)}}.$$

As in the proof of Corollary 3.5, we have that $\alpha = \lambda^{(-i)}$ is equivalent to $\lambda = \alpha^{(m)}$, where $i = u_m(\alpha) + 1$. The result now follows immediately, by changing summation variables in (22) from $\lambda \in \mathcal{P}, i \geq 1$ to $\alpha \in \mathcal{P}, m \geq 1$, and applying (3) to evaluate $u_m(\alpha)$ (which is the exponent of (-1) when the summation is expressed in terms of α, m). \square

Among the results in [M] and [Z] for Bernstein's operators is

$$B_{\lambda_1} \cdots B_{\lambda_n} 1 = s_\lambda,$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$. This result follows immediately from Theorem 3.4, together with (4). To compose B_i when they are not ordered as in this result, one simply uses

the result that $B_i B_j = -B_{j-1} B_{i+1}$, which follows routinely from Theorem 3.4 and considering what happens when two right-steps are switched to up-steps in the two possible orders.

4. CODES OF PARTITIONS AND THE PLÜCKER RELATIONS

We consider a set $a_{\mathcal{P}} = \{a_{\lambda} : \lambda \in \mathcal{P}\}$ of scalars indexed by the set \mathcal{P} of partitions. Then the *Plücker relations* for $a_{\mathcal{P}}$ are given by the following system of simultaneous quadratic equations: for all $m \geq 1$ and $\alpha = (\alpha_1, \dots, \alpha_{m-1}), \beta = (\beta_1, \dots, \beta_{m+1}) \in \mathcal{P}$ with $l(\alpha) \leq m-1$, $l(\beta) \leq m+1$ (which means that $\alpha_i = 0$ for $i > l(\alpha)$, and $\beta_i = 0$ for $i > l(\beta)$), we have

$$(24) \quad \sum_{k=0}^m (-1)^{k-m+1+\ell} a_{(\alpha_1-1, \dots, \alpha_{\ell}-1, \beta_{k+1}-k+\ell+1, \alpha_{\ell+1}, \dots, \alpha_{m-1})} \cdot a_{(\beta_1+1, \dots, \beta_k+1, \beta_{k+2}, \dots, \beta_{m+1})} = 0,$$

where $\ell = \ell(k)$ is chosen so that $0 \leq \ell \leq m-1$ and

$$(25) \quad \alpha_{\ell} - 1 \geq \beta_{k+1} - k + \ell + 1 \geq \alpha_{\ell+1},$$

with the convention that $\alpha_0 = \infty$, $\alpha_m = -\infty$, and so that $\beta_{k+1} - k + \ell + 1 \geq 0$, (if there is no such choice of ℓ , then the term in the summation indexed by k is identically 0). Note that, for each choice of k , then ℓ (if it exists) is unique; to see this, let $\gamma_i = \alpha_i - i$, $i = 0, \dots, m$. Then $\infty = \gamma_0 > \gamma_1 > \dots > \gamma_m = -\infty$, and so for all real numbers x , there is a unique $0 \leq i \leq m-1$ so that $\gamma_i > x \geq \gamma_{i+1}$. Then, rewriting (25), ℓ is the unique choice of i so that (more restrictively so there may not be such an i) $\gamma_i - 2 \geq x \geq \gamma_{i+1}$, with $x = \beta_{k+1} - k$.

The presentation of the Plücker relations given in (24) above is referred to as “classical” by Fulton [F, p. 133]. In this presentation, each equation is a quadratic alternating summation corresponding to an ordered pair of partitions. Each term in the alternating summation arises from removing a single part from the second partition, and inserting it into the first partition, with some appropriate shift in the remaining parts of both partitions. In our next result, we give a different presentation of the Plücker relations, which is more symmetrical in its form, using the notation developed in Section 2 for codes of partitions.

Theorem 4.1. *The Plücker relations for $a_{\mathcal{P}}$ are given by the following system of simultaneous quadratic equations: for all $\alpha, \beta \in \mathcal{P}$, we have*

$$\sum_{\substack{i, j \geq 1 \\ |\alpha^{(i)}| + |\beta^{(-j)}| = |\alpha| + |\beta| + 1}} (-1)^{|\alpha| - |\alpha^{(i)}| + i + j} a_{\alpha^{(i)}} \cdot a_{\beta^{(-j)}} = 0.$$

Proof. In the Plücker relations, equation (24) is satisfied for each (m, α, β) for $m \geq 1$ and $\alpha, \beta \in \mathcal{P}$ with $l(\alpha) \leq m-1$, $l(\beta) \leq m+1$. Now multiply (24) by $(-1)^{m-1}$, to get equation (24)', and consider equation (24)' for $(m+1, \alpha, \beta)$, where we have $\alpha_m = \beta_{m+2} = 0$. Then on the left hand side, the term indexed by $k = m+1$ in the latter equation is identically 0, since there is no possible choice of ℓ (to see this, we must have $\beta_{k+1} - k + \ell + 1 \geq 0$, so $\ell \geq m$, and since $0 \leq \ell \leq m$, we must uniquely have $\ell = m$; but then we have $\alpha_{\ell} - 1 = -1 < 0 = \beta_{k+1} - k + \ell + 1$, contradicting equation (25)). Thus equation (24)' for (m, α, β) is identical to equation (24)' for $(m+1, \alpha, \beta)$, so there is the following single

equation for each $\alpha = (\alpha_1, \alpha_2, \dots), \beta = (\beta_1, \beta_2, \dots) \in \mathcal{P}$ (which means that $\alpha_i = 0$ for $i > l(\alpha)$, and $\beta_i = 0$ for $i > l(\beta)$):

$$(26) \quad \sum_{k \geq 0} (-1)^{k+\ell} a_{(\alpha_1-1, \dots, \alpha_\ell-1, \beta_{k+1}-k+\ell+1, \alpha_{\ell+1}, \dots)} \cdot a_{(\beta_1+1, \dots, \beta_k+1, \beta_{k+2}, \dots)} = 0,$$

where $\ell = \ell(k)$ is chosen so that

$$\alpha_\ell - 1 \geq \beta_{k+1} - k + \ell + 1 \geq \alpha_{\ell+1},$$

with the convention that $\alpha_0 = \infty$, and so that $\beta_{k+1} - k + \ell + 1 \geq 0$. But, from (2) and (5), equation (26) becomes

$$\sum_{k \geq 0} (-1)^{k+\ell} a_{\alpha^{(\beta_{k+1}-k+\ell+2)}} \cdot a_{\beta^{(-k-1)}} = 0.$$

Finally, note that

$$|\alpha^{(\beta_{k+1}-k+\ell+2)}| + |\beta^{(-k-1)}| = |\alpha| + |\beta| + 1,$$

and the result follows from (3), (4), and (6). \square

To fix ideas, we now give a few examples of Plücker relations. These examples illustrate that there are redundant equations in the Plücker relations.

Example 4.2. For $\alpha = \beta = (1)$, we obtain $\alpha^{(1)} = \varepsilon$, $\alpha^{(2)} = (1, 1)$, $\alpha^{(3)} = (2, 1)$, and $\beta^{(-1)} = \varepsilon$, $\beta^{(-2)} = (2)$, $\beta^{(-3)} = (2, 1)$, so the corresponding quadratic equation is

$$-a_\varepsilon \cdot a_{(2,1)} + a_{(2,1)} \cdot a_\varepsilon = 0.$$

But the left hand side of this equation is identically 0, so the equation is redundant.

For $\alpha = \varepsilon$, $\beta = (1, 1, 1)$, we obtain $\alpha^{(1)} = \varepsilon$, $\alpha^{(2)} = (1)$, $\alpha^{(3)} = (2)$, $\alpha^{(4)} = (3)$, and $\beta^{(-1)} = (1, 1)$, $\beta^{(-2)} = (2, 1)$, $\beta^{(-3)} = (2, 2)$, so the corresponding quadratic equation is

$$(27) \quad a_\varepsilon \cdot a_{(2,2)} - a_{(1)} \cdot a_{(2,1)} + a_{(2)} \cdot a_{(1,1)} = 0.$$

For $\alpha = (2)$, $\beta = (1)$, we obtain $\alpha^{(1)} = (1)$, $\alpha^{(2)} = (1, 1)$, $\alpha^{(3)} = (2, 2)$, and $\beta^{(-1)} = \varepsilon$, $\beta^{(-2)} = (2)$, $\beta^{(-3)} = (2, 1)$, so the corresponding quadratic equation is

$$-a_{(1)} \cdot a_{(2,1)} + a_{(1,1)} \cdot a_{(2)} + a_{(2,2)} \cdot a_\varepsilon = 0,$$

which is the same equation as (27).

5. τ -FUNCTIONS FOR THE KP HIERARCHY AND THE PLÜCKER RELATIONS

There are a number of equivalent descriptions of the KP hierarchy, as is well described in [MJD]. The one that we shall start with in this paper involves the so-called τ -function of the hierarchy, and two independent sets of indeterminates p_1, p_2, \dots and $\widehat{p}_1, \widehat{p}_2, \dots$. A power series is said to be a τ -function for the KP hierarchy if and only if it satisfies the bilinear equation

$$(28) \quad [t^{-1}] \exp \left(\sum_{k \geq 1} \frac{t^k}{k} (p_k - \widehat{p}_k) \right) \exp \left(- \sum_{k \geq 1} t^{-k} \left(\frac{\partial}{\partial p_k} - \frac{\partial}{\partial \widehat{p}_k} \right) \right) \tau(\mathbf{p}) \tau(\widehat{\mathbf{p}}) = 0,$$

where $\mathbf{p} = (p_1, p_2, \dots)$, $\widehat{\mathbf{p}} = (\widehat{p}_1, \widehat{p}_2, \dots)$, and we use the notation $[A]B$ to mean the *coefficient* of A in B .

In considering equation (28), we regard p_1, p_2, \dots as the power sum symmetric functions of an underlying set of variables x_1, x_2, \dots , as in Section 3. Also, we regard $\widehat{p}_1, \widehat{p}_2, \dots$ as the power sum symmetric functions of another set of variables, algebraically independent from x_1, x_2, \dots (one could write this other set of variables as $\widehat{x}_1, \widehat{x}_2, \dots$, say, but the precise names of these variables is irrelevant, since no further explicit mention of either x_1, x_2, \dots or $\widehat{x}_1, \widehat{x}_2, \dots$ will be made). We shall use the notation $\widehat{e}_i, \widehat{h}_i, \widehat{s}_\lambda, \widehat{B}(t)$ to denote the symmetric functions in this other set of variables corresponding to $e_i, h_i, s_\lambda, B(t)$ in x_1, x_2, \dots .

The following result follows immediately by taking this symmetric function view of (28). This view has appeared in a number of earlier works (see, e.g., [JY]), but the results from symmetric functions that have been applied in these works have been different than ours.

Proposition 5.1. *A power series is a τ -function for the KP hierarchy if and only if it satisfies*

$$(29) \quad [t^{-1}] \left(B(t) \tau(\mathbf{p}) \right) \cdot \left(\widehat{B}^\perp(t^{-1}) \tau(\widehat{\mathbf{p}}) \right) = 0.$$

Proof. The result follows immediately from (28), together with (1) and (13). \square

Next we give a new proof of the connection between Schur function coefficients of a τ -function for the KP hierarchy, and the Plücker relations. Our proof is immediate from Corollaries 3.5 and 3.6.

Theorem 5.2 (see, e.g., [MJD, p. 90, (10.3)]). *Let the coefficient of the Schur function of shape λ in a power series be given by a_λ , $\lambda \in \mathcal{P}$. Then the power series is a τ -function for the KP hierarchy if and only if $a_{\mathcal{P}} = \{a_\lambda : \lambda \in \mathcal{P}\}$ satisfies the Plücker relations.*

Proof. We are given $\tau(\mathbf{p}) = \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda$ and $\tau(\widehat{\mathbf{p}}) = \sum_{\lambda \in \mathcal{P}} a_\lambda \widehat{s}_\lambda$. Then, from (29), it is necessary and sufficient that $a_{\mathcal{P}}$ satisfies $S(\mathbf{p}, \widehat{\mathbf{p}}) = 0$, where

$$S(\mathbf{p}, \widehat{\mathbf{p}}) = [t^{-1}] \left(B(t) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda \right) \cdot \left(\widehat{B}^\perp(t^{-1}) \sum_{\mu \in \mathcal{P}} a_\mu \widehat{s}_\mu \right).$$

Now, from Corollaries 3.5 and 3.6, we immediately obtain

$$S(\mathbf{p}, \widehat{\mathbf{p}}) = \sum_{\beta, \alpha \in \mathcal{P}} s_\beta \widehat{s}_\alpha \sum_{\substack{m, k \geq 1 \\ |\alpha^{(m)}| + |\beta^{(-k)}| = |\alpha| + |\beta| + 1}} (-1)^{|\alpha| - |\alpha^{(m)}| + m + k} a_{\alpha^{(m)}} \cdot a_{\beta^{(-k)}}.$$

But $S(\mathbf{p}, \widehat{\mathbf{p}}) = 0$ if and only if $[s_\beta \widehat{s}_\alpha] S(\mathbf{p}, \widehat{\mathbf{p}}) = 0$ for all $\beta, \alpha \in \mathcal{P}$, since the Schur functions form a basis for symmetric functions, and the result follows immediately from Theorem 4.1. \square

Often the KP hierarchy is written as a system of simultaneous quadratic partial differential equations, for τ . In the next result, we apply Theorem 5.2 and the methods of symmetric functions to obtain such a system of partial differential equations, with one equation corresponding to each quadratic equation in the Plücker relations. The result is well known, but we include a simple proof for completeness.

Theorem 5.3 (see, e.g., [MJD, p. 92, Lemma 10.2]). *The power series $\tau(\mathbf{p})$ is a τ -function for the KP hierarchy if and only if the following partial differential equation is satisfied for each pair of partitions α and β :*

$$\sum_{\substack{i,j \geq 1 \\ |\alpha^{(i)}| + |\beta^{(-j)}| = |\alpha| + |\beta| + 1}} (-1)^{|\alpha| - |\alpha^{(i)}| + i + j} (s_{\alpha^{(i)}}(\mathbf{p}^\perp) \tau(\mathbf{p})) \cdot (s_{\beta^{(-j)}}(\mathbf{p}^\perp) \tau(\mathbf{p})) = 0.$$

(Where, e.g., $s_\lambda(\mathbf{p}^\perp)$ is interpreted as the partial differential operator obtained by substituting p_n^\perp for p_n in (8) for each $n \geq 1$, and using (13).)

Proof. Let $\mathbf{q} = (q_1, q_2, \dots)$, where the q_i are independent from the p_j and \widehat{p}_k . We begin the proof by proving that (I): τ satisfies equation (29) if and only if (II): τ satisfies

$$(30) \quad [t^{-1}] \left(B(t) \tau(\mathbf{p} + \mathbf{q}) \right) \cdot \left(\widehat{B}^\perp(t^{-1}) \tau(\widehat{\mathbf{p}} + \mathbf{q}) \right) = 0,$$

for all \mathbf{q} .

It is easy to see that (II) implies (I), by setting $\mathbf{q} = \mathbf{0}$ for $i \geq 1$ (where $\mathbf{0} = (0, 0, \dots)$).

To prove that (I) implies (II), define the operator $\Theta(\mathbf{p}) = \exp\left(\sum_{k \geq 1} q_k \frac{\partial}{\partial p_k}\right)$. Using the multivariate Taylor series expansion of an arbitrary formal power series $f(\mathbf{p})$, we see that

$$(31) \quad f(\mathbf{p} + \mathbf{q}) = \Theta(\mathbf{p}) f(\mathbf{p}).$$

Also define $\Gamma(\mathbf{p}) = \exp\left(\sum_{i \geq 1} \frac{t^i}{i} p_i\right)$, $\Upsilon(\mathbf{p}) = \exp\left(-\sum_{j \geq 1} t^{-j} \frac{\partial}{\partial p_j}\right)$, so $B(t) = \Gamma(\mathbf{p}) \Upsilon(\mathbf{p})$. Then we have

$$B(t) \tau(\mathbf{p} + \mathbf{q}) = \Gamma(\mathbf{p}) \Upsilon(\mathbf{p}) \Theta(\mathbf{p}) \tau(\mathbf{p}) = \Gamma(\mathbf{p}) \Theta(\mathbf{p}) \Upsilon(\mathbf{p}) \tau(\mathbf{p}),$$

from (31) and the trivial fact that $\Upsilon(\mathbf{p})$ commutes with $\Theta(\mathbf{p})$. But using (31) again, we have the operator identity

$$\Theta(\mathbf{p}) \Gamma(\mathbf{p}) = \Gamma(\mathbf{p} + \mathbf{q}) \Theta(\mathbf{p}) = \Gamma(\mathbf{q}) \Gamma(\mathbf{p}) \Theta(\mathbf{p}),$$

and combining these expressions and the fact that $\Gamma(\mathbf{q})^{-1} = \Gamma(-\mathbf{q})$ gives

$$B(t) \tau(\mathbf{p} + \mathbf{q}) = \Gamma(-\mathbf{q}) \Theta(\mathbf{p}) B(t) \tau(\mathbf{p}).$$

Similarly we have $\widehat{B}^\perp(t^{-1}) = \Gamma(-\widehat{\mathbf{p}}) \Upsilon(-\widehat{\mathbf{p}})$, and then obtain

$$\widehat{B}^\perp(t^{-1}) \tau(\widehat{\mathbf{p}} + \mathbf{q}) = \Gamma(\mathbf{q}) \Theta(\widehat{\mathbf{p}}) \widehat{B}^\perp(t^{-1}) \tau(\widehat{\mathbf{p}}).$$

Multiplying these two expressions together, we find that equation (30) becomes

$$\Theta(\mathbf{p}) \Theta(\widehat{\mathbf{p}}) [t^{-1}] \left(B(t) \tau(\mathbf{p}) \right) \cdot \left(\widehat{B}^\perp(t^{-1}) \tau(\widehat{\mathbf{p}}) \right) = 0,$$

since $\Gamma(-\mathbf{q}) \Gamma(\mathbf{q}) = 1$, and $\Theta(\mathbf{p})$, $\Theta(\widehat{\mathbf{p}})$ are independent of t . We conclude that (I) implies (II).

Finally, in order to apply Theorem 5.2, we determine the coefficient of the Schur function of shape λ . This gives

$$[s_\lambda(\mathbf{p})] \tau(\mathbf{p} + \mathbf{q}) = \langle s_\lambda(\mathbf{p}), \tau(\mathbf{p} + \mathbf{q}) \rangle = \langle 1, s_\lambda(\mathbf{p}^\perp) \tau(\mathbf{p} + \mathbf{q}) \rangle = s_\lambda(\mathbf{p}^\perp) \tau(\mathbf{p} + \mathbf{q}) \Big|_{\mathbf{p}=\mathbf{0}} = s_\lambda(\mathbf{q}^\perp) \tau(\mathbf{q}),$$

and the result then follows from Theorem 5.2, replacing \mathbf{q} by \mathbf{p} . \square

As an example of Theorem 5.3, we now give one of the quadratic partial differential equations for a τ -function.

Example 5.4. Consider the Plücker equation (27). Now we have

$$s_\varepsilon = 1, \quad s_{(1)} = p_1, \quad s_{(2)} = \frac{1}{2}(p_1^2 + p_2), \quad s_{(1,1)} = \frac{1}{2}(p_1^2 - p_2), \\ s_{(2,1)} = \frac{1}{3}(p_1^3 - p_3), \quad s_{(2,2)} = \frac{1}{12}(p_1^4 - 4p_1p_3 + 3p_2^2),$$

so from Theorem 5.3, the partial differential equation for τ that corresponds to (27) is given by

$$(32) \quad \frac{1}{12}\tau (\tau_{1111} - 12\tau_{13} + 12\tau_{22}) - \frac{1}{3}\tau_1 (\tau_{111} - 3\tau_3) + \frac{1}{4} (\tau_{11} + 2\tau_2) (\tau_{11} - 2\tau_2) = 0,$$

where we use τ_{i_jk} to denote $\frac{\partial}{\partial p_i} \frac{\partial}{\partial p_j} \frac{\partial}{\partial p_k} \tau$, etc.

Often, in the literature of integrable systems, the series $F = \log \tau$ is used instead of τ itself. This series F is often referred to as a *solution* to the KP hierarchy, where the “KP hierarchy” in this context refers to a system of simultaneous partial differential equations for F . Of course, the system of partial differential equations for τ given in Theorem 5.3 becomes an equivalent system of partial differential equations for F by substituting $\tau = \exp F$ into the equations of Theorem 5.3, and then multiplying the equation by $\exp(-2F)$. For example, when we apply this to (32), we obtain the equation

$$\frac{1}{12}F_{1111} - F_{13} + F_{22} + \frac{1}{2}F_{11}^2 = 0,$$

which is often referred to as “the KP equation”.

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