Algebraic Combinatorics – algebra or combinatorics?

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- who knows? self-avoiding walks in the plane

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- $1 = 1 + 0x + 0x^2 + \cdots$ is a multiplicative identity, and $(1 - 2x)(1 + 2x + 4x^2 + 8x^3 + \ldots) = 1$, (since coefficient of x^n in this product is $2^n - 2 \cdot 2^{n-1} = 0$ for each positive integer n)

Combinatorics of strings and matrix algebra

Simon Newcomb Problem: Consider the generating series

$$R(x_1, \ldots, x_n, u) = \sum_{\sigma \in \{1,\ldots,n\}^*} x_1^{\mathrm{num(1's)}} \cdots x_n^{\mathrm{num(n's)}} u^{\mathrm{num(rises)}},$$

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where a *rise* is a substring *ij* with $i < j$

Matrix encoding: Let

$$A = \begin{pmatrix} 1 & u & \cdots & u & u \\ 1 & 1 & \cdots & u & u \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & u \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

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gives precisely the correct contribution to R for the string $ijklmn$, and that this monomial arises in the ij–entry of the matrix $XAXAXAXAXAX$.

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Sherman-Morrison formula: If P, Q are square matrices of the same size, with P invertible and Q of rank 1, then

$$(P+Q)^{-1} = P^{-1} - \frac{1}{1 + \mathrm{trace} P^{-1}Q} P^{-1}Q,$$

if $1 + \mathrm{trace} P^{-1}Q \neq 0$.

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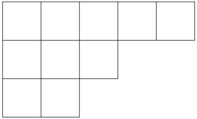
the last equality from Jacobi's identity adapted to formal power series

Symmetric functions and the symmetric group

- A tableau of shape $(5,3,2)$ is given below. Positive integers are placed in each cell so that they are weakly increasing in each row (left to right), and strictly increasing down each column (top to bottom).



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2	3	5		
4	4			

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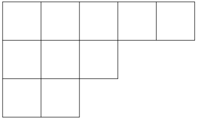

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We call the weakly decreasing list $(5, 3, 2)$ a partition of 10, with *parts* $5, 3, 2$ (e.g., the partitions of 4 are $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$).

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- The Schur function indexed by a partition λ is the generating series

$$s_\lambda(x_1, x_2, \ldots) = \sum_T x_1^{\mathrm{num}(1's)} x_2^{\mathrm{num}(2's)} \cdots ,$$

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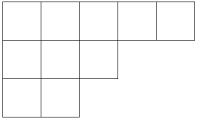

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  summed over all tableaux T of shape λ. Schur functions are symmetric in $x_1,x_2,\ldots$.

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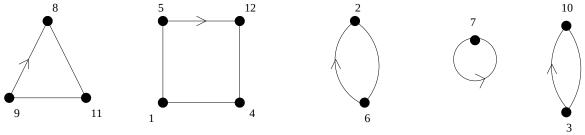

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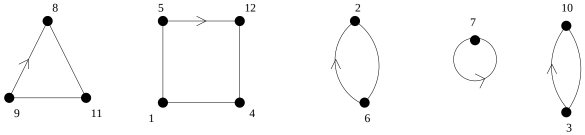

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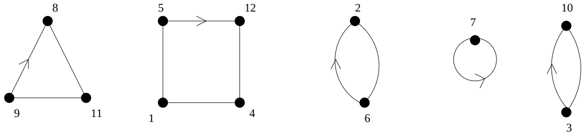

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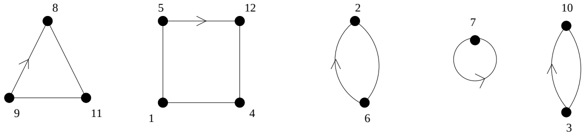

  We often write $\sigma = (1\,5\,12\,4)(2\,6)(3\,10)(7)(8\,11\,9)$. Here the cycles of σ have lengths $(4, 3, 2, 2, 1)$, a partition of 12, and all permutations with cycle lengths specified by a given partition λ form a conjugacy class, denoted by $\mathcal{C}_\lambda$.

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Moreover, there is a basis $\{\mathsf{F}_\theta\}$ of orthogonal idempotents (which means that $\mathsf{F}_\theta \mathsf{F}_\rho = \mathsf{F}_\theta \delta_{\theta\,\rho}$), with

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where 1^n is the partition with n parts, each equal to 1, and $\chi^\lambda(\mu)$ is an irreducible character of the symmetric group.

The combinatorial calculations for multiplying conjugacy classes can be translated into the language of symmetric functions, since

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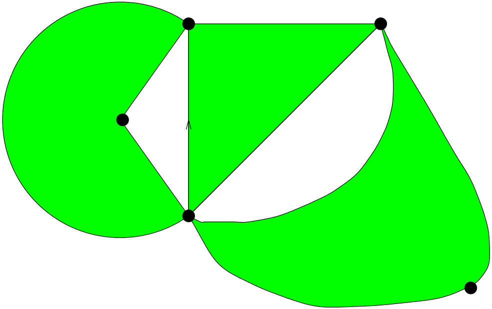
Permutations and rooted hypermaps in orientable surfaces



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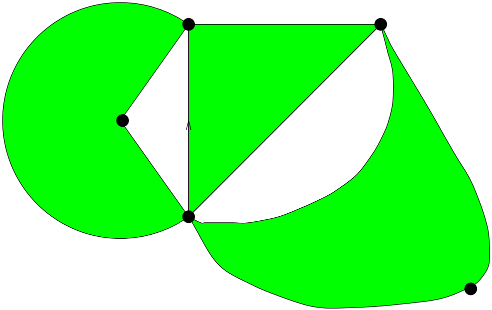
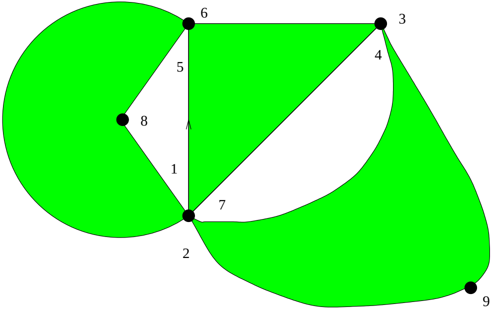


The green faces are hyperedges, the white faces are hyperfaces.


6
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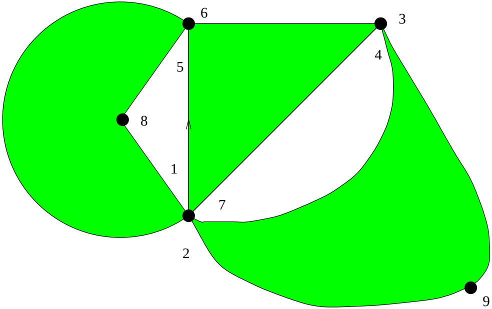
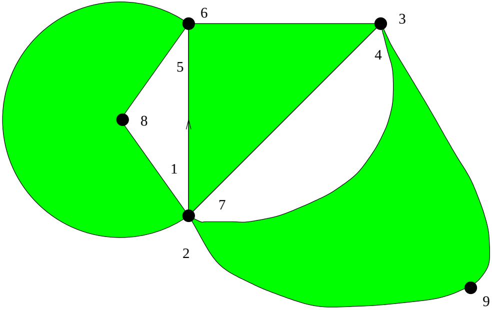


$V = (1\,2\,7)(3\,4)(5\,6)(8)(9),$



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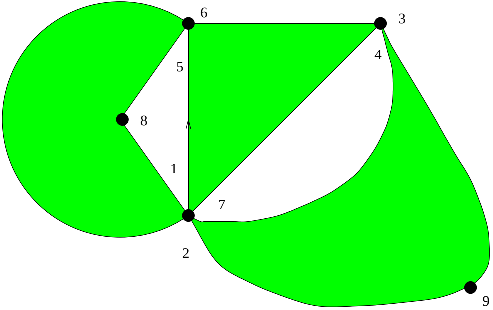

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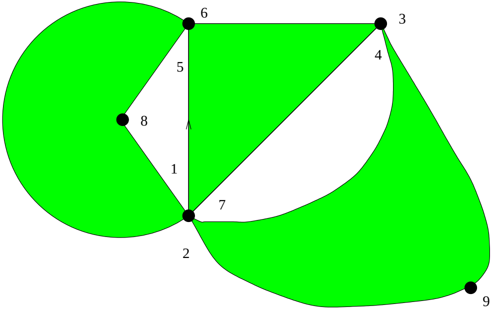

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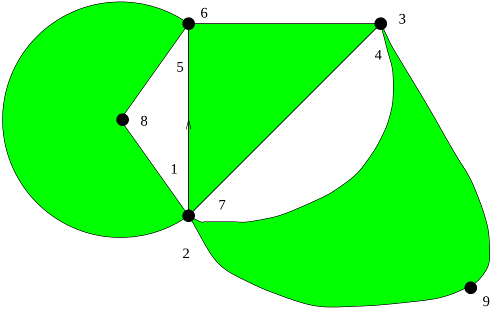

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$\langle V, G, W \rangle$ acts transitively on $\{1, \ldots, 9\}$ (the hypermap is connected).

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Hurwitz numbers and the KP hierarchy

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- Branched covers of the sphere with branch points $\infty, X_1, \ldots, X_r$, at which we have branching $\sigma, \pi_1, \ldots, \pi_r$, respectively. (The branching at $\pi_1, \ldots, \pi_r$ is simple.) (The product equal to the identity permutation is a monodromy condition, and the transitivity condition means that the covers are connected.) The genus g of the cover is given by $r = l(\alpha) + n + 2g - 2$, from the Riemann-Hurwitz formula.

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- This implies that the Hurwitz generating series is a solution to the KP hierarchy.

Consider two independent sets of indeterminates $p = (p_1, p_2, \ldots)$ and $\widehat{p} = (\widehat{p}_1, \widehat{p}_2, \ldots)$. Then $\log \tau$ satisfies the KP hierarchy if and only if

$$[t^{-1}] \exp\left(\sum_{k\geq 1} \frac{t^k}{k}(p_k - \widehat{p}_k)\right) \exp\left(-\sum_{i\geq 1} t^{-i}\left(\frac{\partial}{\partial p_i} - \frac{\partial}{\partial \widehat{p}_i}\right)\right) \tau(p)\tau(\widehat{p}) = 0.$$

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The KP hierarchy is a simultaneous system of quadratic pde's:

$$F_{2,2} - F_{3,1} + \tfrac{1}{12} F_{1,1,1,1} + \tfrac{1}{2} F_{1,1}^2 = 0,$$

$$F_{3,2} - F_{4,1} + \tfrac{1}{6} F_{2,1,1,1} + F_{1,1} F_{2,1} = 0,$$

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where $F_{2,1}$ denotes $\frac{\partial^2}{\partial p_1 \partial p_2} F$.