Algebraic Combinatorics – algebra or combinatorics?

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Combinatorics

▶ enumerative combinatorics

> how many \{0, 1\}–strings of length \(n\)? \(2^n\), for each \(n = 0, 1, 2, \ldots\)

▶ how many \{N, S, E, W\}–strings of length \(n\), with no substring with equal number of N’s and S’s, and equal number of E’s and W’s?

▶ who knows? self-avoiding walks in the plane
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Generating series

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Algebra – formal power series

Consider two formal power series

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Definition: \[ A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n, \]

1 = 1 + 0 x + 0 x^2 + \cdots is a multiplicative identity, and \((1 - 2x)(1 + 2x + 4x^2 + 8x^3 + \ldots) = 1\), (since coefficient of \(x^n\) in this product is \(2^n - 2 \cdot 2^{n-1} = 0\) for each positive integer \(n\))
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Combinatorics of strings and matrix algebra

Simon Newcomb Problem: Consider the generating series

\[ R(x_1, \ldots, x_n, u) = \sum_{\sigma \in \{1, \ldots, n\}^*} x_1^{\text{num}(1's)} \cdots x_n^{\text{num}(n's)} u^{\text{num}(\text{rises})}, \]
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where a rise is a substring \( ij \) with \( i < j \)
Matrix encoding: Let

\[
A = \begin{pmatrix}
1 & u & \cdots & u & u \\
1 & 1 & \cdots & u & u \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & u \\
1 & 1 & \cdots & 1 & 1
\end{pmatrix},
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an \( n \) by \( n \) matrix, and \( X \) be a diagonal \( n \) by \( n \) matrix with entries \( x_1, \ldots, x_n \). Note that the monomial \( x_ia_{ij}x_ja_{jk}x_ka_{kl}x_la_{lm}x_ma_{mn}x_n \)
gives precisely the correct contribution to \( R \) for the string \( ijklmn \),
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gives precisely the correct contribution to \( R \) for the string \( ijklmn \), and that this monomial arises in the \( ij \)–entry of the matrix

\[ XAXAXAXAXAX. \]
We conclude that $R - 1$ is the sum of all the entries in the matrix

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**Sherman-Morrison formula:** If $P, Q$ are square matrices of the same size, with $P$ invertible and $Q$ of rank 1, then

$$(P + Q)^{-1} = P^{-1} - \frac{1}{1 + \text{trace}P^{-1}Q}P^{-1}Q,$$

if $1 + \text{trace}P^{-1}Q \neq 0.$
Circular sequences

Here, the generating series is

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the last equality from Jacobi’s identity adapted to formal power series
Symmetric functions and the symmetric group

A tableau of shape (5, 3, 2) is given below. Positive integers are placed in each cell so that they are weakly increasing in each row (left to right), and strictly increasing down each column (top to bottom).

\[
\begin{array}{cccc}
1 & 1 & 3 & 3 & 5 \\
2 & 3 & 5 \\
4 & 4 \\
\end{array}
\]

We call the weakly decreasing list (5, 3, 2) a partition of 10, with parts 5, 3, 2 (e.g., the partitions of 4 are (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)).

The Schur function indexed by a partition \( \lambda \) is the generating series \( s_\lambda(x_1, x_2, \ldots) = \sum T x_{\text{num}(1's)} x_{\text{num}(2's)} \cdots \), summed over all tableaux \( T \) of shape \( \lambda \).

Schur functions are symmetric in \( x_1, x_2, \ldots \).
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summed over all tableaux $T$ of shape $\lambda$. Schur functions are symmetric in $x_1, x_2, \ldots$. 
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Another symmetric function is the power sum $p_i = x_1^i + x_2^i + \ldots$, and we define the power sum indexed by a partition to be the product of the power sums indexed by the parts, so, for example, $p_{(5,3,2)} = p_5p_3p_2$. 
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\begin{pmatrix}
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5 & 6 & 10 & 1 & 12 & 2 & 7 & 11 & 8 & 3 & 9 & 4
\end{pmatrix},
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describes a permutation \( \sigma \), where \( \sigma(1) = 5, \sigma(2) = 6, \ldots \).
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C_\mu = |C_\mu| \sum_{\theta \vdash n} \frac{\chi^\theta(\mu)}{\chi^\theta(1^n)} F_\theta,
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where \( 1^n \) is the partition with \( n \) parts, each equal to 1, and \( \chi^\lambda(\mu) \) is an irreducible character of the symmetric group.
The combinatorial calculations for multiplying conjugacy classes can be translated into the language of symmetric functions, since

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and

\[ p_\mu = \sum_{\theta \vdash n} \chi^\theta(\mu) s_\theta. \]
Permutations and rooted hypermaps in orientable surfaces
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The green faces are hyperedges, the white faces are hyperfaces.
\( V = (1\ 2\ 7)(3\ 4)(5\ 6)(8)(9), \)
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\( W = (1\ 5\ 8)(2\ 6\ 3\ 9)(4\ 7), \)
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acts transitively on \( \{1, \ldots, 9\} \) (the hypermap is connected).
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\[ \langle V, G, W \rangle \text{ acts transitively on } \{1, \ldots, 9\} \text{ (the hypermap is connected).} \]
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Hurwitz numbers and the KP hierarchy

- For a partition $\alpha$ of $n$ and a nonnegative integer $r$, let $H^r_\alpha$ be the number of tuples $(\sigma, \pi_1, \ldots, \pi_r)$ of permutations on $\{1, \ldots, n\}$ such that

- Branched covers of the sphere with branch points $\infty, X_1, \ldots, X_r$, at which we have branching $\sigma, \pi_1, \ldots, \pi_r$, respectively. (The branching at $\pi_1, \ldots, \pi_r$ is simple.)

- (The product equal to the identity permutation is a monodromy condition, and the transitivity condition means that the covers are connected.)

- The genus $g$ of the cover is given by $r = l(\alpha) + n + 2g - 2$, from the Riemann-Hurwitz formula.
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For a partition $\alpha$ of $n$ and a nonnegative integer $r$, let $H_{\alpha}^r$ be the number of tuples $(\sigma, \pi_1, \ldots, \pi_r)$ of permutations on \{1, \ldots, n\} such that $\sigma \in C_\alpha$, $\pi_1, \ldots, \pi_r$ are transpositions,
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Applying the relationship between Schur functions and conjugacy classes, we can evaluate the generating series for Hurwitz numbers, and prove that it is of the form

$$\log \left( \sum_{\lambda} a_{\lambda} s_{\lambda} \right),$$

where \(\{a_{\lambda}\}\) satisfies the Plücker relations from algebraic geometry.
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This implies that the Hurwitz generating series is a solution to the KP hierarchy.
Consider two independent sets of indeterminates \( p = (p_1, p_2, \ldots) \) and \( \hat{p} = (\hat{p}_1, \hat{p}_2, \ldots) \). Then \( \log \tau \) satisfies the KP hierarchy if and only if

\[
[t^{-1}] \exp \left( \sum_{k \geq 1} \frac{t^k}{k} (p_k - \hat{p}_k) \right) \exp \left( - \sum_{i \geq 1} t^{-i} \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial \hat{p}_i} \right) \right) \tau(p) \tau(\hat{p}) = 0.
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The KP hierarchy is a simultaneous system of quadratic pde's:

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F_{2,2} - F_{3,1} + \frac{1}{12} F_{1,1,1,1} + \frac{1}{2} F_{1,1}^2 = 0,
\]
\[
F_{3,2} - F_{4,1} + \frac{1}{6} F_{2,1,1,1} + F_{1,1} F_{2,1} = 0,
\]
\[
F_{4,2} - F_{5,1} + \frac{1}{4} F_{3,1,1,1} - \frac{1}{120} F_{1,1,1,1,1,1} + F_{1,1} F_{3,1} + \frac{1}{2} F_{2,1}^2,
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where \( F_{2,1} \) denotes \( \frac{\partial^2}{\partial p_1 \partial p_2} F \).