Co 769

Lecture 5

January 19

Recall :

- X is a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and induced norm.
- $A : X \rightrightarrows X$ is mono. if
$(\forall (x, x^*) \in \operatorname{gr} A) \ (\forall (y, y^*) \in \operatorname{gr} A)$
$$\langle x - y , x^* - y^* \rangle \geq 0 .$$
- Recall

Characterization of Projection

Theorem L4 – 12

$\emptyset \neq C \subseteq X$ is convex closed. Let $x \in X$, let $p \in X$. Then $p = P_C x$ if and only if

i $\quad p \in C$

ii $\quad (\forall c \in C) \quad \langle c - p, x - p \rangle \leq 0 .$

Example LS_1

C = unit ball

$= \{ x \in X \mid \|x\| \leq 1 \}$.

$$P_C x = \begin{cases} x, & \text{if } x \in C; \\ \frac{x}{\|x\|}, & \text{otherwise} \end{cases}$$

$$= \frac{x}{\max\{1, \|x\|\}}.$$

Proof:

$p = P_C x \Leftrightarrow$

$\boxed{i}$ $p \in C$

$\boxed{ii}$ $(\forall c \in C)$ $\langle c-p, x-p \rangle \leq 0$.

Let $x \in X$, set

$$p = \frac{x}{\max\{1, \|x\|\}}.$$

i $p \in C$??

Observe that $p \in C \Leftrightarrow \|p\| \leq 1$.

Indeed, $\|p\| = \left\| \frac{x}{\max\{1, \|x\|\}} \right\| = \frac{\|x\|}{\max\{1, \|x\|\}} \leq 1$

$\Rightarrow p \in C$.

ii Let $c \in C$. Goal:

$\langle c-p, x-p \rangle \leq 0$.

Case 1: $x \in C$

$$\langle c-p, x-p\rangle = \langle c-x, x-x\rangle = 0 \quad \checkmark$$

Case 2: $x \notin C$.

$\Rightarrow p = \frac{x}{\|x\|}$ and.

$$\langle c-p, x-p\rangle \leq 0$$

$$\Leftrightarrow \langle c-\frac{x}{\|x\|}, \underbrace{x-\frac{x}{\|x\|}}_{=(1-\frac{1}{\|x\|})x}\rangle \leq 0$$

$$\Leftrightarrow \underbrace{\left(\boxed{1-\frac{1}{\|x\|}}\right)}_{\|x\|>1 \Rightarrow >0} \langle c-\frac{x}{\|x\|}, x\rangle \leq 0$$

$$\Leftrightarrow \langle c-\frac{x}{\|x\|}, x\rangle \leq 0.$$

?

$$\Leftrightarrow \boxed{\langle c, x\rangle} \leq \langle \frac{x}{\|x\|}, x\rangle = \frac{1}{\|x\|}\langle x, x\rangle \boxed{\leq \|x\|}.$$

Indeed, $c \in C \Rightarrow \boxed{\|c\| \leq 1}$

By Cauchy-Schwarz

$$\langle c, x\rangle \leq \|c\|\,\|x\| \boxed{\leq} 1 \cdot \|x\| = x \quad \boxed{\checkmark}$$

$\blacksquare$

Definition L5_2:

A subset $\varnothing \neq K \subseteq X$ is a cone if for $(\forall \lambda \geq 0)\ (\forall k \in K)\quad \lambda k \in K$.

Definition L5_3:

Let $K \subseteq X$ be a cone. The polar cone of K, denoted by $K^{\ominus}$ is

$$K^{\ominus} = \{ u \in X \mid \sup \langle K, u \rangle \leq 0 \}.$$

Proposition L5_4:

Let K be a nonempty closed convex cone of X, let $x \in X$, let $p \in X$. Then:

$$\boxed{p = P_K x} \iff \boxed{[\, p \in K,\ x - p \perp p,\ x - p \in K^{\ominus} \,]}.$$

Proof:

$p = P_K x \iff$ $\boxed{i}$ $p \in K$

$\boxed{ii}$ $(\forall k \in K)\quad \langle c - p, x - p \rangle \leq 0$.

Suppose $\square \Rightarrow \boxed{p \in K}$ ① ✓ and

$$\boxed{(\forall y \in K)\quad \langle y - p, x - p \rangle \leq 0}$$

$(\forall y \in K) \quad \langle y-p, x-p \rangle \leq 0$

set $y = 0 \quad \langle -p, x-p \rangle \leq 0$

$\quad\; y = 2p \quad \langle p, x-p \rangle \leq 0$

$\Rightarrow \quad \langle p, x-p \rangle = 0$

$\Rightarrow \quad \boxed{p \perp x-p}$ 2

Let $y \in K$

$\boxed{\langle y, x-p \rangle} = \langle y, x-p \rangle - \langle p, x-p \rangle$

$= \langle y-p, x-p \rangle \boxed{\leq 0}$

$\Rightarrow \quad \boxed{x-p \in K^{\ominus}}$ 3

Conversely, suppose $[p \in K, \; x-p \perp p, \; x-p \in K^{\ominus}]$.

$p \in K$

Now, let $y \in K$

$\langle y-p, x-p \rangle = \langle \underbrace{y}_{\in K}, \underbrace{x-p}_{\in K^{\ominus}} \rangle - \langle p, x-p \rangle$

$\langle y, x-p \rangle \leq 0, \quad p \perp x-p, \; \langle p, x-p \rangle = 0$

$\leq \quad 0 \quad + 0 \quad = 0$

Hence, $\quad P_K x = p$ by

Characterization of Projection

Theorem L4-12

$\emptyset \neq C \subseteq X$ is convex closed. Let $x \in X$, let $p \in X$. Then $p = P_C x$ if and only if

i $\quad p \in C$

ii $\quad (\forall c \in C) \quad \langle c-p, x-p \rangle \leq 0$.

Theorem L 5_5 Moreau

Let K be nonempty closed convex cone in X and let $x \in X$. Then the following hold:

i $\quad x = P_K x + P_{K^\ominus} x$

ii $\quad P_K x \perp P_{K^\ominus} x$

iii $\quad \|x\|^2 = d_K^2(x) + d_{K^\ominus}^2(x)$

Proof

i

Set $p = P_K x$, set $q = x - p$.

Recalling $\boxed{p = P_K x} \Leftrightarrow [p \in K,\ x-p \perp p,\ x-p \in K^\ominus]$.

1 $\boxed{q \in K^\ominus}$, $\boxed{x - q = P_K x \perp x - P_K x = q}$ 2

$\boxed{x-q} = P_K x \in K \ \boxed{\subseteq K^{\ominus}}{}^{\ominus}$?? 3 (See HW2).

Using with K replaced by $K^\ominus$, p replaced by q yields

$$q = P_{K^\ominus} x$$

ii $\quad$ By i and $P_K x \perp \underbrace{x - P_K x}_{\overset{i}{=} P_{K^\ominus} x}$

$\boxed{i}$ $x = P_K x + P_{K^\ominus} x$

$\boxed{ii}$ $P_K x \perp P_{K^\ominus} x$

$\boxed{iii}$

$$\|x\|^2 \overset{(i)}{=} \|P_K x + P_{K^\ominus} x\|^2$$

$$= \|P_K x\|^2 + \|P_{K^\ominus} x\|^2 + 2 \underbrace{\langle P_K x , P_{K^\ominus} x \rangle}_{=0}$$

$$\overset{(ii)}{=} \|P_K x\|^2 + \|P_{K^\ominus} x\|^2$$

$$\overset{(i)}{=} \|x - P_{K^\ominus} x\|^2 + \|x - P_K x\|^2$$

$$= d^2_{K^\ominus} x + d^2_K x$$

■

Example L5_6

Suppose $X = S^n$, let $K = S^n_+$ be the closed convex cone of symmetric PSD matrices, let $A \in S^n$. Then there exists a diagonal matrix D and a unitary matrix U such that $A = UDU^T$. Moreover:

$$P_K A = U D_+ U^T, \quad (D_+)_{ii} = \max\{0, d_{ii}\}.$$
$$(D_+)_{ij} = 0, \quad i \neq j$$

Proof:

Set $A_+ = U D_+ U^T$, $D_- = D - D_+$

$$\boxed{p = P_K x} \Leftrightarrow \boxed{[\overset{1}{p \in K},\ \overset{2}{x-p \perp p},\ \overset{3}{x-p \in K^\ominus}]}$$

Clearly, $A_+ \in S^n_+$. 1

$A - A_+ = UDU^T - UD_+U^T = UD_-U^T \in -K \overset{??}{=} K^\ominus$ 3

HW 2

Finally:

$$\langle A_+, A - A_+\rangle = tr(A_+(A - A_+))$$
$$= tr(UD_+U^T(UD_-U^T))$$
$$= tr(UD_+\boxed{U^TU}D_-U^T)$$
$$= tr(UD_+D_-U^T) = tr(D_+D_-)$$
$$= 0 \quad 2$$

$\boxed{\text{Example L5-7}}$

Let $a \in X \setminus \{0\}$, let $\beta \in \mathbb{R}$. The hyperplane

$H = \{ x \in X \mid \langle a, x\rangle = \beta \}$ has

$$P_H x = x - \frac{\langle a, x\rangle - \beta}{\|a\|^2} a$$

Proof:

Let $x \in X$, set $p = x - \dfrac{(\langle a, x\rangle - \beta)}{\|a\|^2} a$

$p \in H$??

$$\boxed{\langle a, p\rangle =} \langle a, x\rangle - \frac{(\langle a, x\rangle - \beta)}{\|a\|^2} \underbrace{\langle a, a\rangle}^{= \|a\|^2}$$

$$= \langle a, x\rangle - (\langle a, x\rangle - \beta) \boxed{= \beta} \quad ✓$$

Let $\boxed{y \in H}$. Examine

$$\langle y - p, x - p\rangle = \left\langle y - x + \frac{\langle a, x\rangle - \beta}{\|a\|^2} a, \underbrace{\frac{\langle a, x\rangle - \beta}{\|a\|^2}}\, a \right\rangle$$

$$= \frac{\langle a, x\rangle - \beta}{\|a\|^2} \left[\underbrace{\langle y, a\rangle}^{= \beta} - \cancel{\langle x, a\rangle} + \frac{\cancel{\langle a, x\rangle} - \beta}{\cancel{\|a\|^2}} \cancel{\|a\|^2} \right]$$

$$= \frac{\langle a, x\rangle - \beta}{\|a\|^2} \underbrace{[\beta - \beta]}_{= 0} = 0$$

$$\leq 0 \quad ✓$$

∎

Remember A is max mono

$\overline{\text{dom}\, A}$, $\overline{\text{ran}\, A}$ are convex,

of course $\neq \phi$, closed.

Let $\phi \neq C \subseteq X$ be closed.

$$\Pi_C(x) = \{ p \in C \mid d_C(x) = \|x - p\| \}.$$

$\Pi_C(x) \neq \phi$ because C is closed.

$\Pi_C(x)$ is possibly set valued.

Proposition L5-8:

$C \neq \phi$ and closed $\Rightarrow \Pi_C$ is monotone.

Proof:

Let $\boxed{\{(x, x^*), (y, y^*)\} \subseteq \text{gr}\, \Pi_C}$

$$\Leftrightarrow \begin{array}{l} x^* \in \Pi_C(x) \\ y^* \in \Pi_C(y) \end{array} \begin{array}{c} \overset{\text{def}}{\Longleftrightarrow} \\ \Longleftrightarrow \end{array} \begin{array}{l} \|x - x^*\| = d_C(x) \leq \|x - y^*\| \\ \|y - y^*\| = d_C(y) \leq \|y - x^*\| \end{array}$$

That is
$$\boxed{\begin{array}{l} \|x - x^*\|^2 \leq \|x - y^*\|^2 \\ \|y - y^*\|^2 \leq \|y - x^*\|^2 \end{array}}$$

$$\boxed{\begin{aligned} \|x - x^*\|^2 &\leq \|x - y^*\|^2 \\ \|y - y^*\|^2 &\leq \|y - x^*\|^2 \end{aligned}}$$

Adding yields

$$\|x - x^*\|^2 + \|y - y^*\|^2 \leq \|x - y^*\|^2 + \|y - x^*\|^2$$

Expanding yields

$$\cancel{\|x\|^2} + \cancel{\|x^*\|^2} - 2\langle x, x^*\rangle + \|y\|^2 + \|y^*\|^2 - 2\langle y, y^*\rangle$$

$$\leq \cancel{\|x\|^2} + \|y^*\|^2 - 2\langle x, y^*\rangle + \|y\|^2 + \cancel{\|x^*\|^2} - 2\langle x^*, y\rangle$$

$$\Leftrightarrow \langle x, x^*\rangle + \langle y, y^*\rangle - \langle x, y^*\rangle - \langle x^*, y\rangle \geq 0$$

$$\Leftrightarrow \boxed{\langle x - y, x^* - y^*\rangle \geq 0.}$$

Remember

Let $\boxed{\{(x, x^*), (y, y^*)\} \subset \operatorname{gr} \Pi_C}$

The Chebyshev problem:

Suppose $C \neq \varnothing$, closed such that $(\forall x \in X)$ $\Pi_C(x)$ is single-valued.

Must C be convex?

Bunt–Motzkin (1930's) L5_10

Yes! when X is finite-dimensional.

Proof: Recall: **Theorem L2_9:**

Let $A: X \to X$ be:
1. monotone
2. $\operatorname{dom} A = X$
3. single-valued
4. A continuous

Then A is maximally monotone.

We proved Π_C is mono,
By assumption $\operatorname{dom} \Pi_C = X$
Π_C is single-valued.
If we can show that Π_C is continuous then Π_C is max. mono.
Done ☺ because: $\overline{\operatorname{ran} \Pi_C}$ is convex. But: $\overline{\operatorname{ran} \Pi_C} = \overline{C} = C$ ✓

Suppose $C \neq \phi$, closed such that $(\forall x \in X)$ $\Pi_C(x)$ is single-valued.

We now show that $\Pi_C(x)$ is continuous.
Write $\Pi_C = P$.
P is mono, single-valued, $\operatorname{dom} P = X$

$d_C : X \longrightarrow \mathbb{R} : x \longrightarrow \inf_{c \in C} \|x - c\|$

is continuous (See HW 2).

Let $x \in X$, $x_n \longrightarrow x$.

$$\boxed{\|x_n - P_C x_n\| = d_C(x_n) \longrightarrow d_C(x)}$$

$(x_n)_{n\in\mathbb{N}}$ converges $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is bounded.
$(d_C x_n)_{n\in\mathbb{N}}$ " $\Rightarrow$ $(d_C x_n)_{n\in\mathbb{N}}$ " "

$\Rightarrow$ $\boxed{(P_C x_n)_{n\in\mathbb{N}} \text{ must be bounded}}$.
Done if: we can show we have a unique cluster point.

Let $\bar{c}$ be a cluster point of $P_C x_n$
say $\boxed{P_C x_{k_n} \longrightarrow \bar{c} \in C}$

C closed, $(P_C x_n)_{n\in\mathbb{N}}$ lies in C!

$$\boxed{\| x_n - P_C x_n \| = d_C(x_n) \longrightarrow d_C(x)}$$

$$\boxed{P_C x_{k_n} \longrightarrow \bar{c} \in C}$$

$$\Rightarrow \| x_{k_n} - P_C x_{k_n} \| = d_C(x_{k_n}) \longrightarrow d_C(x)$$

$$\xRightarrow[\text{limit } n \to \infty]{\text{taking the}} \| x - \bar{c} \| = d_C(x)$$

By assumption, C is Chebyshev set, i.e., the projection is unique $\Rightarrow \bar{c} = P_C x$

That is all the cluster points of $P_C x_n$ are equal to $P_C x$.

Therefore $P_C x_n \longrightarrow P_C x$. Hence P_C is continuous, and mono by single valued, full domain by assumption $\Rightarrow P_C$ is max mono

$$C = \overline{C} = \overline{\operatorname{ran}} \, P_C \text{ is convex!}$$

$\square$