Lecture 2

January 10

Co 769

Recall our setting:

X is a real Hilbert space with inner product and induced norm.

Notation::

$A : X \rightrightarrows X$ is a possibly set valued operator, i.e, Ax is a subset of X.

That is:

$Ax \in 2^X = \{ S \mid S \text{ is a subset of } X \}$

power set of X (the collection of subsets of X).

$A : X \longrightarrow X$

A is a mapping from X to X single-valued.

Let $A : X \rightrightarrows X$. Then

$$\operatorname{gr} A = \{ (x, x^*) \mid x^* \in Ax \} .$$
$$\subseteq X \times X$$

$$\operatorname{dom} A = \{ x \in X \mid Ax \neq \varphi \} .$$

$$\operatorname{ran} A = \bigcup_{x \in X} Ax$$

$$A^{-1} : X \rightrightarrows X$$

$$\operatorname{gr} A^{-1} = \{ (x^*, x) \mid (x, x^*) \in \operatorname{gr} A \}.$$

Definition L2-1:

Let $A : X \rightrightarrows X$. Then A is monotone if $(\forall (x, x^*) \in \operatorname{gr} A)$ $(\forall (y, y^*) \in \operatorname{gr} A)$

$$\langle x - y , x^* - y^* \rangle \geq 0 .$$

Example:

$$A : \mathbb{R} \to \mathbb{R} , \quad Ax = \begin{cases} [0, 1] , & x = 0 \\ \varphi , & x \neq 0 \end{cases}$$

$\operatorname{dom} A = \{0\}$, $\operatorname{ran} A = [0, 1]$

$\operatorname{gra} A = \{0\} \times [0, 1]$.

Lemma L2.2 :

1 A is monotone $\iff$ A^{-1} is mono.

2 $A, B : X \rightrightarrows X$ mono.

Then $A+B$ is mono.

3 $A : X \to X$ is mono, $\alpha \geq 0$.

Then αA is monotone.

Proof :

See HW 1.

Optimization break

We like to work with functions

$$f : X \longrightarrow]-\infty, +\infty]$$

one useful example is the indicator function of a set S; i_S.

$$i_S(x) = \begin{cases} 0, & x \in S; \\ +\infty, & x \notin S. \end{cases}$$

The essential domain of f is

$$\operatorname{dom} f := \{ x \in X \mid f(x) < +\infty \}.$$

If we care about minimizing f over a set S, then it is useful to observe the correspondance:

$$\begin{array}{l} \text{minimize } f(x) \\ \text{subject to } x \in S \end{array} \longleftrightarrow \underset{x \in X}{\text{minimize}}\ f(x) + i_S(x)$$

Let $f : X \longrightarrow]-\infty, +\infty]$ be proper, i.e., $\operatorname{dom} f \neq \emptyset$.

Definition L2-3:

The subdifferential of f at $x \in X$ is defined by:

$$\partial f(x) = \{ x^* \in X \mid (\forall y \in X)\ f(y) \geq f(x) + \langle x^*, y - x \rangle \}$$

$$\partial f : X \rightrightarrows X.$$

Lemma L2-4 ::

Suppose $f : X \to]-\infty, +\infty]$ is proper.
Then $\operatorname{dom} \partial f \subseteq \operatorname{dom} f$.

Proof ::

Suppose that $x \in \operatorname{dom} \partial f$, $x^* \in \partial f(x)$.
let $z \in \operatorname{dom} f \neq \varphi$ (f is proper).
Then

$$f(z) \geq f(x) + \langle x^*, z - x \rangle$$

Suppose for eventual contradiction that
$x \notin \operatorname{dom} f \Rightarrow f(x) = +\infty$.

$$+\infty = f(x) \leq f(z) - \langle x^*, z - x \rangle < +\infty,$$

which is absurd.

Hence $x \in \operatorname{dom} f$.

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Proposition L2-5 :

Suppose $f : X \to]-\infty, +\infty]$ is proper.
Then ∂f is monotone.

Proof :

Suppose that $\{(x, x^*), (y, y^*)\} \subseteq \operatorname{gr} \partial f$.

Goal: $\langle x - y, x^* - y^* \rangle \geq 0$??

Indeed, observe that $(\forall z \in X)$

$$f(z) \geq f(x) + \langle x^*, z - x \rangle \quad (1)$$

$$f(z) \geq f(y) + \langle y^*, z - y \rangle \quad 2$$

Applying (1) (with z replaced by x)
(2) (" " " by y)

$$f(y) \geq f(x) + \langle x^*, y - x \rangle \quad (3)$$

$$f(x) \geq f(y) + \langle y^*, x - y \rangle \quad (4)$$

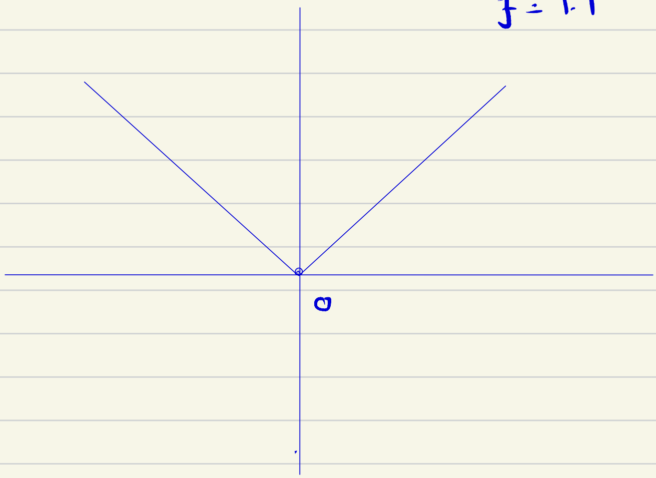
Adding (3), (4)

$$f(y) + f(x) \geq f(x) + f(y) + \langle x^* - y^*, y - x \rangle$$

$$\Rightarrow \langle x - y, x^* - y^* \rangle \geq 0. \quad \blacksquare$$

The subdifferential: A geometric intuition.

$f = |\cdot|$



$$\partial f(x) = \begin{cases} 1, & x > 0; \\ -1, & x < 0; \\ [-1, 1], & x = 0. \end{cases}$$

$u \in \partial f(x) \iff f(y) \geq f(x) + \langle u, y-x \rangle$

Fix $x \in X$

$g(y) = f(x) + \langle u, y-x \rangle$

$\hookrightarrow$ linear function, linearization of f at x, $g(x) = f(x)$, $g(y) \leq f(y) \quad \forall y$.

Zeros of monotone operators :-

Consider $A = \partial f : X \rightrightarrows X$.

$0 \in \partial f(x)$

$\Leftrightarrow (\forall y \in X) \quad f(y) \geq f(x) + \langle 0, y-x \rangle$

$\Leftrightarrow (\forall y \in X) \quad f(y) \geq f(x)$

$\Leftrightarrow$ x is a global minimizer of f.

We now know:

f proper $\Rightarrow$ ∂f is mono.

(not very useful).

f proper not convex yields a very tiny subdifferential. .

Back to monotone operators!

Proposition L2-6:

If f is differentiable at $x \in X$, and $x \in \operatorname{dom} \partial f$.

Then $\partial f(x) = \{\nabla f(x)\}$.

Proof:

Take $\boxed{x^* \in \partial f(x)}$. Then $(\forall z \in X)$
$$f(z) \geq f(x) + \langle x^*, z - x \rangle.$$
Fix $h \in X$, consider $z = x + th$, $t > 0$.
$$f(x+th) \geq f(x) + \langle x^*, x + th - x \rangle$$
$$= f(x) + t\langle x^*, h \rangle.$$
Rearranging
$$\langle x^*, h \rangle \leq \frac{f(x+th) - f(x)}{t}$$
$$\Rightarrow \langle x^*, h \rangle \leq \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}$$
$$= \langle \nabla f(x), h \rangle$$
$$\Rightarrow \langle x^* - \nabla f(x), h \rangle \leq 0.$$
Setting $h = x^* - \nabla f(x) \Rightarrow \|x^* - \nabla f(x)\|^2 \leq 0$
$$\Rightarrow \boxed{\nabla f(x) = x^*}$$
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Definition L2-7:

Let $A: X \rightrightarrows X$ be monotone. Then A is maximally monotone if:

$$\left.\begin{array}{l} B: X \rightrightarrows X \text{ is monotone} \\ \operatorname{gr} A \subseteq \operatorname{gr} B \end{array}\right\} \Rightarrow \quad A = B.$$

We say that $(x, x^*) \in X \times X$ is monotonically related to $\operatorname{gr} A$ if

$(\forall (a, a^*) \in \operatorname{gr} A)$

$$\langle x - a, x^* - a^* \rangle \geq 0.$$

Lemma L2-8:

Let $A: X \rightrightarrows X$ be monotone. Then A is maximally monotone iff

$[(x, x^*)$ is monotonically related to $\operatorname{gr} A \Rightarrow (x, x^*) \in \operatorname{gr} A]$.

<u>Proof</u>: See HW 1.

Theorem L2-9:

Let $A : X \longrightarrow X$ be:
1. monotone
2. $\operatorname{dom} A = X$
3. single-valued
4. A continuous.

Then A is maximally monotone.

Proof:

Let (x, x^*) be mono. related to $\operatorname{gr} A$.
Goal:

$$(x, x^*) \in \operatorname{gr} A \;??$$

By 3 2, 1

$$(\forall y \in X) \quad \langle x - y, x^* - Ay \rangle \geq 0 \quad *$$

Set $(\forall \alpha > 0)$ $\quad y_\alpha = x + \alpha(x^* - Ax)$

$$* \Rightarrow (\forall \alpha > 0) \; \langle x - y_\alpha, x^* - Ay_\alpha \rangle \geq 0$$

$$\Rightarrow (\forall \alpha > 0) \; -\alpha \langle x^* - Ax, x^* - Ay_\alpha \rangle \geq 0$$

$$\xRightarrow{\alpha > 0} (\forall \alpha > 0) \; -\langle x^* - Ax, x^* - Ay_\alpha \rangle \geq 0$$

$$\xRightarrow{4} -\langle x^* - Ax, x^* - \lim_{\alpha \to 0} Ay_\alpha \rangle \geq 0$$

$$\Rightarrow -\langle x^* - Ax, x^* - Ax \rangle \geq 0$$

$$\Leftrightarrow -\|x^* - Ax\|^2 \geq 0 \Leftrightarrow \|x^* - Ax\|^2 \leq 0$$

$$\Leftrightarrow x^* = Ax$$

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useful to check maximality!

Let $f : \mathbb{R} \to \mathbb{R} : x \to x^3$.

Then f is max. mono.

Example :

Let A be an $n \times n$ matrix.

Suppose that A is monotone.

Show that A is max. mono.

Proof :

See HW 1.