Lecture 1

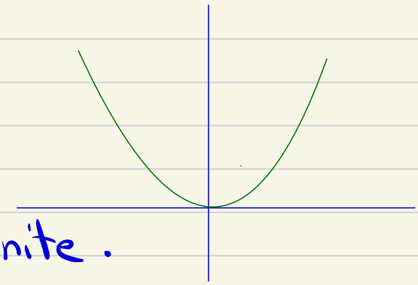
January 5

Co 769

Monotone operators, fixed point theory and splitting methods.

1 Monotone opertors generalize

* derivatives of convex function "concave up" functions e.g.,
  $f : \mathbb{R} \to \mathbb{R} : x \to x^2$



* Matrices whose symmetric part is positive semidefinite.
  "Let A be an $n \times n$ matrix. The symmetric part of A is
  $A_+ = \frac{1}{2}(A + A^T)$

* It has beautiful applications, e.g., Chebyshev sets & Klee sets

2 They are closely connected to (firmly) nonexpansive mappings. whose convergence properties we will study.

Example:

On the real line

$T: \mathbb{R} \to \mathbb{R}$ is firmly nonexpansive if T is non decreasing AND

$(\forall x \in \mathbb{R})\ (\forall y \in \mathbb{R})$

$$|Tx - Ty| \leq |x - y|,$$

e.g. $Tx = x \quad \forall x \in \mathbb{R}$.

Note that

$Tx = -x$ is Not firmly nonexpansive.

3 Putting pieces from 1 and 2 together we will consider algorithms to solve feasibility and optimization problems e.g.,

POCS : Projections onto convex sets.

DR : Douglas - Rachford

Dykstra.

In this Course :

X is a real $\boxed{\text{Hilbert space}}$

with inner product $\langle \cdot , \cdot \rangle$

and induced norm $\| \cdot \|$

$$\boxed{\begin{array}{l}\text{Recall that } (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy} \\ \text{sequence if } (\forall \varepsilon > 0)(\exists \; N \in \mathbb{N}) \\ \text{such that } (\forall n \in \mathbb{N}) \; (\forall m \in \mathbb{N}) \\ \qquad \| x_n - x_m \| < \varepsilon .\end{array}}$$

A Hilbert space is a complete inner product space.

* Complete : Every Cauchy sequence converges to a limit in X.
* inner product space : X is a vector space , $\langle \cdot , \cdot \rangle$ satisfies :

1. $(\forall x \in X)$ $\langle x, \cdot \rangle$ is linear

$$\langle x, \alpha y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$$

2. $(\forall x \in X)$ $(\forall y \in X)$

$$\langle x, y \rangle = \langle y, x \rangle \quad \text{Symmetry}$$

3. $\langle x, x \rangle \geqslant 0$,

$$\langle x, x \rangle = 0 \iff x = 0$$

Define the norm

$$\| x \| := \sqrt{\langle x, x \rangle}$$

satisfies

$$\| x \| \geqslant 0$$

$$\| \alpha x \| = |\alpha| \, \| x \|$$

$$\| x + y \| \leqslant \| x \| + \| y \|$$

* $X = \mathbb{R}^n$:

$x, y \in \mathbb{R}^n$ are column vectors.

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

$$\| x \| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

* The rational numbers :

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \,,\; b \neq 0 \right\}.$$

$\mathbb{Q}$ is an inner product space

However, $\mathbb{Q}$ is NOT complete.

Consider $x_n = \left(1 + \frac{1}{n}\right)^n$

$(x_n)_{n \in \mathbb{N}}$ is a sequence of rational numbers,

(x_n) is a Cauchy sequence.

$$x_n \longrightarrow e \in \mathbb{R} \setminus \mathbb{Q}$$

$X = \mathbb{R}^{n \times n}$

$A, B \in \mathbb{R}^{n \times n}$

$\langle A, B \rangle = tr(A^T B)$

$X = L^2[0,1]$

space of measurable functions

$x : [0,1] \to \mathbb{R}$ such

that $|x|^2$ is Lebsegue integrable

$\langle x, y \rangle = \int_0^1 x(t) y(t) \, dt$

$X = \ell^2$

$= \{ x = (x_i)_{i \in \mathbb{N}} \mid \sum_{i=1}^{\infty} x_i^2 < +\infty \}.$

$(1, 1, 0, 0, 0, \cdots) \in \ell^2$

<u>But</u>

$(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots) \notin \ell^2$

as $\sum \frac{1}{n} = +\infty$

Monotone map:

$A : X \longrightarrow X$ is monotone if

$(\forall x \in X)(\forall y \in X)$

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

Example

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone if

$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})$

$$(f(x) - f(y))(x - y) \geq 0.$$

This means f is nondecreasing!

Indeed, let $x, y \in \mathbb{R}$, $\boxed{x \leq y}$.

$\Rightarrow x - y \leq 0 \Rightarrow f(x) - f(y) \leq 0$

$\Rightarrow \boxed{f(x) \leq f(y)}$

- We can easily then see that $-Id$ is NOT monotone.

Suppose now that $X = \mathbb{R}^n$,
A is linear, i.e., $A \in \mathbb{R}^{n \times n}$.
Recall that for $x, y \in \mathbb{R}^n$

$$\langle x, y \rangle = x^T y .$$

Therefore

$$\begin{aligned} \langle x, Ay \rangle &= x^T Ay \\ &= (x^T Ay)^T \\ &= y^T A^T (x^T)^T \\ &= y^T A^T x \\ &= \langle y, A^T x \rangle . \end{aligned}$$

Proposition L1

A is mono $\iff$ $A_+ := \frac{1}{2}(A + A^T)$ is positive semidefinite.

Proof :

A is monotone $\iff (\forall x)(\forall y)$

$$\langle Ax - Ay, x-y \rangle \geq 0$$

$$\|$$

$$\langle A(x-y), x-y \rangle \geq 0$$

$$\iff (\forall z) \ \langle Az, z \rangle \geq 0$$

$$\iff (\forall z) \ \langle A^T z, z \rangle \geq 0$$

$$\iff (\forall z) \left\langle \frac{Az + A^T z}{2}, z \right\rangle \geq 0$$

$$\iff (\forall z \in \mathbb{R}^n) \ \langle A_+ z, z \rangle \geq 0$$

We proved

A is mono $\iff A_+ = \dfrac{A + A^T}{2}$ is positive semidefinite.

$\square$

Examples :

1 $A = Id$ (In $\mathbb{R}^n$ $A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$)

i.e., ($\forall x \in X$)

$$Ax = x.$$

Then A is monotone.

Indeed: $\forall z \in X$

$$\langle Az, z \rangle = \langle z, z \rangle = \|z\|^2 \geq 0$$

Observe that

$$A = \nabla \left(\tfrac{1}{2} \|\cdot\|^2 \right).$$

2 $A \equiv 0$ is monotone.

Indeed: $\forall z \in X$

$$\langle Az, z \rangle = \langle 0, z \rangle = 0 \geq 0$$

Observe that

$$A = \nabla(0)$$

3 $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates by $\pi/2$ in the plane.

Then A is monotone

Indeed: $(\forall (z_1, z_2) \in \mathbb{R}^2)$

$$\langle Az, z \rangle = (Az)^T z$$

$$= (-z_2, z_1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= -z_2 z_1 + z_1 z_2$$

$$= 0$$

Observe that

$$A_+ = A + A^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad ✓$$

Nonetheless :

$\nexists f$ such that

$$A = \nabla f .$$

Indeed, suppose that $\exists f : \mathbb{R}^2 \to \mathbb{R}$ such that $\forall z \in \mathbb{R}^2$

$$\nabla f(z) = A z$$

Then

$$\nabla^2 f(z) = A$$

not symmetric

which is absurd.

" Recall the symmetry of the Hessian matrix $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ >

$z = (x, y)$ " .