Co 769

Lecture $\boxed{4}$

January 17

Recall :

- X is a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and induced norm.
- $A : X \rightrightarrows X$ is mono. if
  $(\forall (x, x^*) \in \operatorname{gr} A)$ $(\forall (y, y^*) \in \operatorname{gr} A)$
  $$\langle x - y , x^* - y^* \rangle \geq 0 .$$
- Let $A : X \rightrightarrows X$ is mono. Then A is MAX mono if $B : X \rightrightarrows X$ is mono,
  $$\operatorname{gr} A \subseteq \operatorname{gr} B \Rightarrow A = B .$$

Fact L4-1 : Minty

$A: X \rightrightarrows X$ is mono. Then

A is max. mono $\Leftrightarrow$ $\operatorname{ran}(\mathrm{Id}+A) = X$.

Proof:

"$\Rightarrow$" is hard. Doable when A is $n\times n$ matrix. See HW2.

"$\Leftarrow$" Suppose $\operatorname{ran}(\mathrm{Id}+A) = X$.

Goal: A is max mono.

Let $(x, x^*) \in X \times X$ be monotonically related to $\operatorname{gr} A$, i.e., $\forall (a, a^*) \in \operatorname{gr} A$

$$\langle x - a, x^* - a^* \rangle \geq 0$$

Done if: $(x, x^*) \in \operatorname{gr} A$.

Observe that $x + x^* \in X = \operatorname{ran}(\mathrm{Id}+A)$
i.e., $(\exists\, a \in \operatorname{dom} A = \operatorname{dom}(\mathrm{Id}+A))$
such that

$$x + x^* \in (\mathrm{Id}+A)a$$

Let $a^* \in Aa$ such that

$$x + x^* = a + a^*$$

Then
$$\boxed{\begin{aligned} x^* - a^* &= a - x \\ &= -(x-a) \end{aligned}}$$

By and

$$\begin{aligned}
0 &\leq \langle x-a, x^* - a^* \rangle \\
&= \langle x-a, -(x-a) \rangle \\
&= -\|x-a\|^2
\end{aligned}$$

$$\iff \|x-a\|^2 \leq 0$$

$$\iff \|x-a\|^2 = 0$$

$$\left.\begin{aligned} &\iff x = a \\ &\iff x^* = a^* \end{aligned}\right\}$$

$$\iff \boxed{(x, x^*) = (a, a^*) \in \operatorname{gr} A}$$

$\blacksquare$

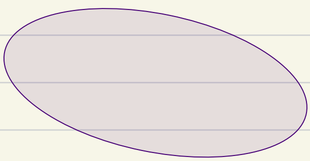
Convex Sets.

Definition L4-2

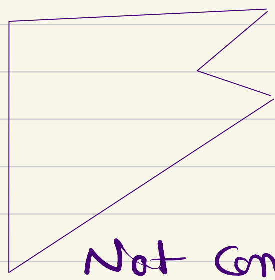
V is a vector space, $S \subseteq V$. Then

S is convex if $(\forall s_1 \in S)(\forall s_2 \in S)(\forall \lambda \in]0,1[)$

$$\lambda s_1 + (1-\lambda) s_2 \in S.$$



Convex



Not convex

Let $f : X \longrightarrow]-\infty, +\infty]$

The epigraph of f is

$$\operatorname{epi} f = \underbrace{\{ (x,\alpha) \mid f(x) \leq \alpha \}}_{\subseteq X \times \mathbb{R}}.$$

f is convex $\Leftrightarrow$ $\operatorname{epi} f$ is convex.

Equivalently, $(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f)(\forall \lambda \in]0,1[)$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

$f : X \longrightarrow]-\infty, +\infty]$ is lower semicontinuous if $(x_n)_{n\in\mathbb{N}}$ in X, $x_n \longrightarrow x$

$$\Rightarrow \liminf_{n\to\infty} f(x_n) \geq f(x).$$

Fact L4_3

f is l.s.c. $\Leftrightarrow$ epi f is closed.

Example L4_4

$f : \mathbb{R} \longrightarrow]-\infty, +\infty]$

$$f = i_{[-1,1]} = \begin{cases} 0, & x \in [-1,1]; \\ +\infty & \text{otherwise.} \end{cases}$$

$f(1) = 0$

$$\lim_{h\to 0^+} (h+1) = +\infty$$

Similarly at $x = -1$

$\forall$ sequences $x_n \longrightarrow x$ we have

$$\liminf_{n\to\infty} f(x_n) \geq x.$$

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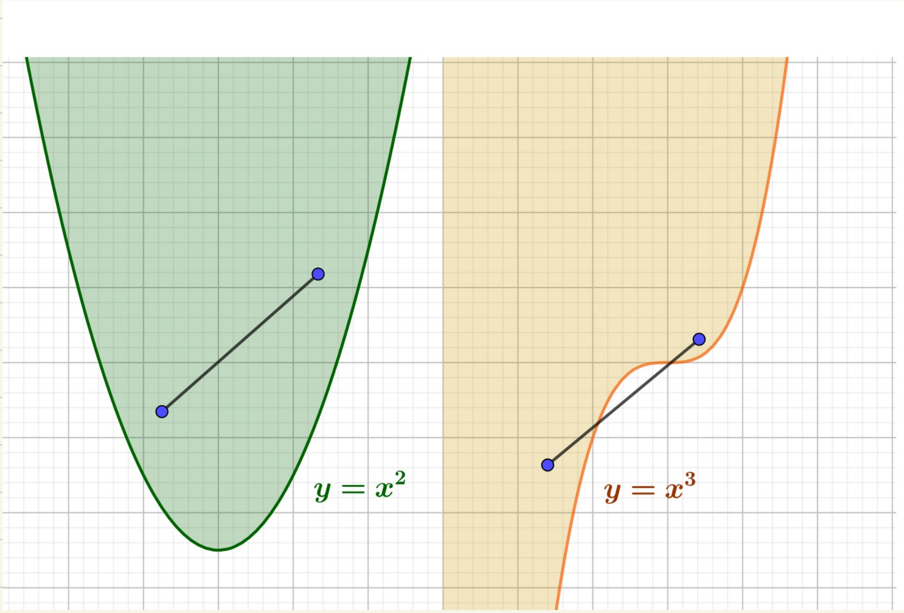
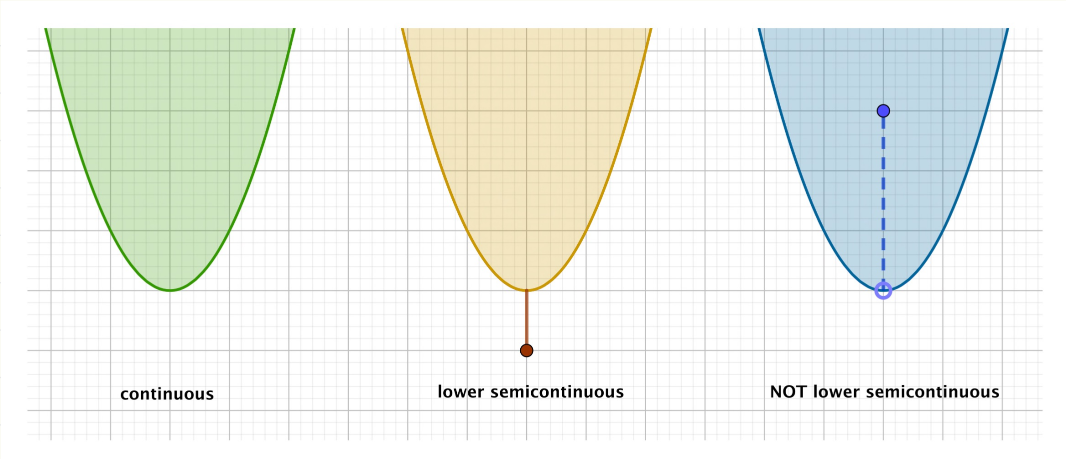


$y = x^2$
$y = x^3$



continuous
lower semicontinuous
NOT lower semicontinuous

Fact L4 - 5

$f : X \longrightarrow]-\infty, +\infty]$ convex (1), lsc (2), proper (3).

(1) epi f closed (2) epi f convex

(3) $\operatorname{dom} f = \{ x \mid f(x) < +\infty \} \neq \phi$.

Then ∂f is maximally monotone.

Fact L4 - 6

$f : X \longrightarrow]-\infty, +\infty]$ is convex

i. $\operatorname{int} \operatorname{dom} f \subseteq \operatorname{dom} \partial f \subseteq \operatorname{dom} f \subseteq \overline{\operatorname{dom}} f$

ii. $\overline{\operatorname{dom}} \, \partial f = \overline{\operatorname{dom}} \, f$

iii. If f is differentiable at x then
$$\partial f(x) = \{ \nabla f(x) \}.$$

Example L4 - 7

$C \subseteq X$. Recall $i_C(x) = \begin{cases} 0, & x \in C; \\ +\infty, & x \notin C. \end{cases}$

i_C is proper $\iff C \neq \phi$.

i_C is l.s.c $\iff C$ is closed

i_C is convex $\iff C$ is convex.

Definition L4 - 8

$C \neq \phi$. The normal cone operator of C at x is

$$N_C(x) = \begin{cases} \{ u \mid \sup_{c \in C} \langle u, c - x \rangle \leq 0 \}, & x \in C; \\ \phi, & x \notin C. \end{cases}$$

Theorem L4-9

$\phi \neq C \subseteq X$ convex and closed. So i_C is convex lsc and proper. Moreover

$$N_C(x) = \partial i_C(x)$$

Proof:

Indeed, if $x \notin C \Rightarrow N_C(x) = \phi$.

Now let $x \in C$, and let $u \in X$. Then

$$\boxed{u \in \partial i_C(x)} \iff (\forall y \in X)\ i_C(y) \geq i_C(x) + \langle u, y-x \rangle$$

$$\iff (\forall y \in C)\ i_C(y) \geq i_C(x) + \langle u, y-x \rangle$$

$$\iff (\forall y \in C)\ 0 \geq \langle u, y-x \rangle$$

$$\iff \boxed{u \in N_C(x)}.$$

■

No kinks on the boundry of C

$\Rightarrow N_C(x)$ is a ray.

Counterexample $N_{\mathbb{R}^2_+}(0,0)$

Example L4 – 10

Let U be a linear subspace of X. Then

$$N_U(x) = \begin{cases} U^{\perp}, & x \in U; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The Projection Theorem

Theorem L4 — 11

$C \subseteq X$, $C \neq \emptyset$, convex, closed.

Then for every $x \in X$, there exists a unique point $p \in C$ such that

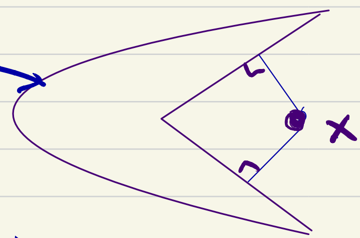
$$\|x-p\| = \inf_{c \in C} \|x-c\| =: d_C(x)$$

distance from x to C

p is called the projection of x onto C, also written as $P_C(x) = P_C x$.

Convex for uniqueness:

Not convex



two projections.

Closed for existence:

$$]0,1[\subseteq \mathbb{R}$$

Note that $0 \in [0,1] = \overline{]0,1[}$

$$d_{]0,1[}(0) = 0 = \inf_{c \in]0,1[} \|0-c\|$$

Characterization of Projection

Theorem L4 – 12

$\emptyset \neq C \subseteq X$ is convex closed. Let $x \in X$, let $p \in X$. Then $p = P_C x$ if and only if

i $\quad p \in C$

ii $(\forall c \in C) \quad \langle c-p, x-p \rangle \leq 0$.

Proof $\quad \boxed{\|x-p\| = \inf_{c \in C} \|x-c\| =: d_C(x)}$

$(\Rightarrow)$ By defn. of inf, get a sequence $(c_n)_{n \in \mathbb{N}}$ such that $\boxed{\|x - c_n\| \to d_C(x) = \inf_{c \in C} \|x-c\|}$ **1**

Note that:
$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})$

$$\boxed{\left\| x - \frac{c_n + c_m}{2} \right\| \geq d_C(x)} \quad \mathbf{2}$$

$\frac{c_n + c_m}{2} \in C$ because C is convex

Apollonius: For any vectors a, b, c

$$\|a-b\|^2 = 2\|a-c\|^2 + 2\|c-b\|^2 - 4\left\|c - \frac{a+b}{2}\right\|^2$$

[Proof: See HW2]

$$\boxed{\|a-b\|^2 = 2\|a-c\|^2 + 2\|c-b\|^2 - 4\left\|c - \frac{a+b}{2}\right\|^2}$$

Hence,

$$0 \leq \|c_n - c_m\|^2 \overset{\text{Apol.}}{=} 2\|c_n - x\|^2 + 2\|c_m - x\|^2 - 4\left\|x - \frac{c_n + c_m}{2}\right\|^2$$

$$\overset{\boxed{2}}{\leq} 2\|c_n - x\|^2 + 2\|c_m - x\|^2 - 4 d_c^2(x)$$

$$\xrightarrow{\boxed{1}} 2 d_c^2(x) + 2 d_c^2(x) - 4 d_c^2(x) = 0$$

as $(n,m) \to (\infty, \infty)$

By the squeeze theorem

$\|c_n - c_m\| \to 0$ as $(n,m) \to (\infty,\infty)$

That is $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since X is Hilbert, and hence complete, $(c_n)_{n\in\mathbb{N}}$ converges say $\underbrace{c_n}_{\text{seq. in } C} \to \boxed{p \in \underbrace{C}_{\text{Closed.}}}$

Moreover

$0 \leq \big|\, \|p - x\| - \|c_n - x\| \,\big| \leq \|p - c_n\|$ "by Δ ineq."

$\to 0$

Again by the squeeze thm.

$d_c(x) \leftarrow \|x - c_n\| \longrightarrow \|x - p\|$ "by the uniqueness of the limit".

$\Rightarrow \|x - p\| = d_c(x)$

$\overset{\boxed{}}{\underset{p\in C}{\Longrightarrow}}$ p is "a" projection (nearest point) to x in C.

Existence is proved ✓

We now turn to uniqueness:

Assume that $q \in C$ also satisfies that
$$\boxed{\|x - q\| = d_C(x).}$$
To show uniqueness it suffices to show that

Define $(\tilde{c}_n)_{n \in \mathbb{N}} = (p, q, p, q, \dots\dots)$.
Then $(\forall n \in \mathbb{N})$
$$\|x - \tilde{c}_n\| = d_C(x)$$
and $(\tilde{c}_n)_{n \in \mathbb{N}}$ lies in C.

Proceeding as before, $(\tilde{c}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence (by what we just proved). Hence, $(\tilde{c}_n)_{n \in \mathbb{N}}$ converges. In particular,
$$0 \longleftarrow \tilde{c}_n - \tilde{c}_{n+1} = p - q$$
$$\Rightarrow \quad p = q$$
and $P_C x$ is unique.

$(\Rightarrow)$ Done. ✓

We now prove $(\Leftarrow)$. First we recall the following lemma.

Lemma L 4-13

Let $a, b \in X$. The following are equivalent:

$\boxed{i}$ $\quad \langle a, b \rangle \leq 0$

$\boxed{ii}$ $\quad (\forall \alpha > 0) \quad \|a\| \leq \|a - \alpha b\|$

$\boxed{iii}$ $\quad (\forall \alpha \in [0,1]) \quad \|a\| \leq \|a - \alpha b\|$.

Proof: The key identity is $(\forall \alpha \in \mathbb{R})$

$$\|a - \alpha b\|^2 - \|a\|^2 = \|a\|^2 - 2\alpha \langle a, b \rangle + \alpha^2 \|b\|^2 - \|a\|^2$$

$$= \alpha \left(\alpha \|b\|^2 - 2 \langle a, b \rangle \right). \qquad \boxed{*}$$

$\boxed{i} \Rightarrow \boxed{ii}$: $\quad \langle a, b \rangle \leq 0, \quad \alpha > 0$

$$\Rightarrow \underbrace{\alpha}_{>0} \left(\underbrace{\alpha \|b\|^2}_{\geq 0} \underbrace{- 2 \langle a, b \rangle}_{\geq 0} \right) \geq 0$$

$$\Rightarrow \|a - \alpha b\|^2 \geq \|a\|^2$$

$\boxed{ii} \Rightarrow \boxed{iii}$: Obvious.

$\boxed{iii} \Rightarrow \boxed{i}$: $\boxed{*}$ is true for all $\alpha \in \mathbb{R}$.

$$\Rightarrow \alpha \|b\|^2 - 2 \langle a, b \rangle \geq 0$$

In particular, $\alpha = 0$ gives

$$\langle a, b \rangle \leq 0.$$

✓

Back to the proof:

Let $p \in X$. Then

$$p = P_C x \iff \|x-p\| = \inf_{c \in C} \|x-c\|$$

$$\iff (\forall c \in C) \ \|x-p\| \leq \|x-c\| \text{ and } p \in C$$

$$\overset{C \text{ is}}{\underset{\text{convex}}{\iff}} \ p \in C \text{ and } (\forall c \in C)(\forall \alpha \in]0,1[)$$

$$\|\underbrace{x-p}_{=:a}\| \leq \|x-((1-\alpha)p+\alpha c)\|$$

$$= \|\underbrace{(x-p)}_{=:a} - \alpha \underbrace{(c-p)}_{=:b}\|$$

$\boxed{\text{Lemma} \quad L\,4-13}$

$$\iff p \in C \text{ and } (\forall c \in C) \ \langle \underbrace{x-p}_{a}, \underbrace{c-p}_{b} \rangle \leq 0.$$

Remark L4-14

We saw $\boxed{p = P_C x} \iff p \in C$ and $(\forall c \in C)$
$\langle c-p, x-p \rangle \leq 0$

$$\iff (\forall c \in C)\ \langle x-p, c-p\rangle + \iota_C(p) \leq \iota_C(c)$$

$$\iff (\forall c \in X)\ \langle x-p, c-p\rangle + \iota_C(p) \leq \iota_C(c)$$

$$\iff x - p \in \partial \iota_C(p) = N_C(p)$$

$$\iff x \in p + N_C(p)$$

$$= (Id + N_C)(p)$$

$$\overset{\text{def}}{\underset{\text{of } (.)^{-1}}{\iff}} \boxed{p \cancel{\in} = (Id + N_C)^{-1}(x)}$$

So $P_C = (Id + N_C)^{-1}$

$$X \underset{\substack{\text{by the} \\ \text{projection} \\ \text{theorem}}}{=} \operatorname{dom} P_C = \operatorname{dom}(Id + N_C)^{-1}$$

$$= \operatorname{ran}(Id + \underbrace{N_C}_{\text{mono}})$$

$N_C = \partial \iota_C$.

mono because it's a subdiff.
$C \neq \emptyset$.
ι_C proper.

Minty $\Rightarrow$ N_C is max mono.