CO 769

Lecture 3

January 12

Recall that:

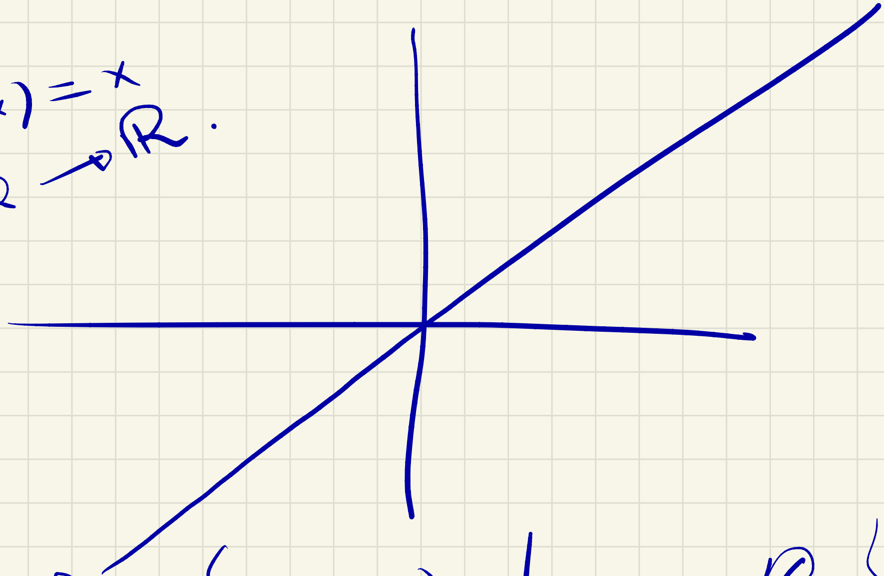
- X is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| = \sqrt{\langle x, x \rangle}$.
- A is monotone if
$$(\forall (x, x^*) \in \operatorname{gr} A) \quad (\forall (y, y^*) \in \operatorname{gr} A)$$
$$\langle x - y, x^* - y^* \rangle \geq 0.$$
- A is maximally monotone if $\operatorname{gr} A$ has no proper extension (in terms of set inclusion).

Theorem

Let $A : X \rightrightarrows X$ be monotone such that $\operatorname{gr} A \neq \emptyset$. Then there exists $\tilde{A} : X \rightrightarrows X$ such that $\tilde{A}$ is a maximal monotone extension of A, i.e., $\operatorname{gr} A \subseteq \operatorname{gr} \tilde{A}$.

$f(x) = x$

$f : \mathbb{R} \to \mathbb{R}.$



$\operatorname{gr} f = \{(x, x) \mid x \in \mathbb{R}\}.$

Minty thm.

A monotone.

A is max mono $\iff$ $(\mathrm{Id} + A)$ is surjective.

$(\mathrm{Id} + A)^{-1}$

∂F

Prox_F

We will use Zorn's lemma ☺.

Zorn's Lemma (Fact L3_1)

Let A be ① a partially ordered set (P.O.set) such that each ② chain of A has ③ an upper bound. Then A has ④ a maximal element.

Poset:

Let A be a set, and let "$\preccurlyeq$" be a binary relation on $A \times A$. Then A is partially ordered if $\forall\, a, b, c \in A$ we have:

1. $a \preccurlyeq a$
2. $a \preccurlyeq b,\ b \preccurlyeq c \Rightarrow a \preccurlyeq c$
3. $a \preccurlyeq b,\ b \preccurlyeq a \Rightarrow a = b$

Examples: $(\mathbb{R}, \leq)$, $(\mathbb{R}; \geq)$, $(\mathbb{R}; =)$

NOT posets: $(\mathbb{R}, <)$

If, in addition, we have

4 $(\forall a \in A)\ (\forall b \in A)$ either $a \preceq b$ or $b \preceq a$

Then A is **totally ordered set**.

Example :

Let S be a set of two or more elements. Set $A = 2^S = \{ U \mid U \subseteq S \}$.

Define a binary relation A by :

$$U \preceq V \ :\Longleftrightarrow\ U \subseteq V.$$

Then
1. $U \subseteq U \quad \forall U \in A$
2. $U \subseteq V,\ V \subseteq W \Rightarrow U \subseteq W$
3. $U \subseteq V,\ V \subseteq U \Rightarrow U = V$

Hence, $\preceq$ defines a partial order on A.

Is it a total order? NO!

$S = \{0, 1\}$, $A = \{ \phi, \{0\}, \{1\}, S \} = 2^S$

$\{1\} \not\subseteq \{0\} \not\subseteq \{1\}$.

Chain :

Let A be a P.O. set. Then $C \subseteq A$ is a chain if C is totally ordered.

Revisiting the previous example:

$S = \{0, 1\}$, $A = \{\phi, \{0\}, \{1\}, S\} = 2^S$

$\{1\} \nsubseteq \{0\} \nsubseteq \{1\}$.

$C_1 = \{\phi, \{0\}\}$.

C_1 is a chain since $\phi \subseteq \{0\}$.

$C_2 = \{\phi, \{0\}, S\}$ is a chain.

since $\phi \subseteq \{0\}$, $\phi \subseteq S$,

$\{0\} \subseteq S$.

Upper bound:

Let $B \subseteq A$. Then $a \in A$ is an upper bound of B if $(\forall b \in B)\ b \preccurlyeq a$.

$a \in A$ is maximal if

$$\left.\begin{array}{l} c \in A \\ a \preccurlyeq c \end{array}\right\} \Longrightarrow a = c.$$

$S = \{0, 1\}$, $A = \{\varphi, \{0\}, \{1\}, S\} = 2^S$

$\{1\} \npreccurlyeq \{0\} \npreccurlyeq \{1\}$.

$B_1 = \{\varphi, \{1\}\}$

$\{1\}$, S are upper bounds of B_1

$B_2 = \{\{0\}, \{1\}\}$

S is the only upper bound of B_2

S is maximal element of A.

Zorn's Lemma (Fact L3-1)

Let A be a partially ordered set (POSET) such that each chain of A has an upper bound. Then A has a maximal element.

Applications:

Theorem L3-2

V is a vector space. Then V has a basis, i.e., a linearly independant spanning set.

Proof:

Define

$$\mathcal{M} := \{ B \subseteq V \mid B \text{ is linearly independant} \}.$$

Then $\mathcal{M}$ is a p.o. set with partial order $\subseteq$

Let $\mathcal{C} = \{ B_i \in \mathcal{M} \mid i \in I \}$ be a chain.

$\mathcal{M} := \{ B \subseteq V \mid B \text{ is linearly independant} \}$

Then $\mathcal{M}$ is a po.set with partial order $\subseteq$

Let $\mathcal{C} = \{ B_i \in \mathcal{M} \mid i \in I \}$ be a chain.

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Claim 1: $\mathcal{C}$ has an upper bound.

Set $\mathcal{U} := \bigcup_{i \in I} B_i$.

Clearly, $(\forall i \in I) \quad B_i \overset{\leq}{\subseteq} \mathcal{U}$ 1

It remains to verify that $\mathcal{U} \in \mathcal{M}$.

Take $\{u_1, \ldots, u_n\} \subseteq \mathcal{U}$,
and consider

$$\lambda_1 u_1 + \lambda_2 u_2 \cdots + \lambda_n u_n = 0$$

Goal:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \; ?$$

Indeed, $\forall k \in \{1, \ldots, n\}$ there exists i_k such that $u_k \in B_{i_k} \in \mathcal{C}$.

All the sets B_{i_k}, $k \in \{1, \ldots n\}$ are comparable (because it is a chain).

Mutually comparing by the chain property

(4) $(\forall a \in A) \; (\forall b \in A)$ either $a \leq b$ or $b \leq a$

$\exists \; k^* \in \{1, \ldots, n\}$ such that $\forall k \in \{1, \ldots, n\}$

$B_{i_k} \subseteq B_{i_{k^*}}$.

Hence,

$$\{u_1, u_2, \dots, u_n\} = \bigcup_{i=1}^{n} \{u_i\}$$

$$\subseteq \bigcup_{k=1}^{n} B_{i_k} \subseteq B_{i_{k^*}} \in \mathcal{M}$$

Because $B_{i_{k^*}} \in \mathcal{M}$

$\xrightarrow[\text{lin indep vectors}]{B_{i_{k^*}}}$ $\boxed{\lambda_1 = \lambda_2 = \dots = \lambda_n = 0}$

That is $\mathcal{U}$ is a set of linearly indep. vectors

$\Rightarrow$ $\boxed{\mathcal{U} \in \mathcal{M}}$ $\boxed{2}$

$\boxed{1}$, $\boxed{2}$ $\Rightarrow$ C has an upper bound

That is: $\mathcal{M}$ is a po. set

Every chain has an upper bound

$\xrightarrow[\text{Lemma}]{\text{Zorn's}}$ $\mathcal{M}$ has a maximal element

say $B \in \mathcal{M}$. Hence, B is linearly indep.

$\left.\begin{array}{l} \tilde{B} \in \mathcal{M} \\ B \subseteq \tilde{B} \end{array}\right\}$ maximal element $\Rightarrow B = \tilde{B}$

Claim 2: $\text{Span}(B) = V$

Suppose for eventual contradiction that $\text{span } B \neq V$. Get $v \in V \setminus \text{span } B$ and observe that $\overline{B} := \underbrace{B \cup \{v\} \in \mathcal{N}}_{\text{lin. indep.}}$

Indeed, suppose $\lambda_i \in \mathbb{R}$, $b_i \in B$, $i \in \{1, \dots, n\}$ such that

$$\lambda_1 b_1 + \dots + \lambda_n b_n + \mu v = 0.$$

If $\mu = 0 \Rightarrow \lambda_1 b_1 + \dots + \lambda_n b_n = 0$ done because B is lin. indep.

Otherwise, if $\mu \neq 0$ then

$$v = \sum_{i=1}^{n} -\frac{\lambda_i}{\mu} b_i \in \text{span } B \ (\text{absurd})$$

Hence $B \cup \{v\}$ is lin. indep. set but $\overline{B} = B \cup \{v\} \supsetneqq B$ which contradicts the maximality of B. Hence, $\text{span}(B) = V$ and we have a basis! $\square$

Theorem L3.3:

Let $A : X \rightrightarrows X$ be monotone such that $\mathrm{gr}\, A \neq \emptyset$. Then there exists $\tilde{A} : X \rightrightarrows X$ such that $\tilde{A}$ is a maximal monotone extension of A, i.e., $\mathrm{gr}\, A \subseteq \mathrm{gr}\, \tilde{A}$.

Zorn's Lemma (Fact L3.1)

Let A be a partially ordered set (POSET) such that each chain of A has an upper bound. Then A has a maximal element.

Proof:

Set $\mathcal{M} := \{ B : X \rightrightarrows X \mid B \text{ is mono, } \mathrm{gr}\, A \subseteq \mathrm{gr}\, B, \text{ i.e. } B \text{ extends } A \}$.

Then $\mathcal{M} \neq \emptyset$ $(A \in \mathcal{M})$.

Define a partial order on $\mathcal{M}$ via:

$$B_1 \preceq B_2 \quad :\Leftrightarrow \quad \mathrm{gr}\, B_1 \subseteq \mathrm{gr}\, B_2.$$

(verify it is a partial order).

Let $\mathcal{C}$ be a chain of $\mathcal{M}$ (every two elements are comparable).

Define $B : X \rightrightarrows X$ via its graph

$$\mathrm{gr}\, B = \bigcup_{C \in \mathcal{C}} \mathrm{gr}\, C$$

Clearly, $(\forall\ C \in \mathcal{C})$

$$\boxed{\operatorname{gr} C \subseteq \bigcup_{C \in \mathcal{C}} \operatorname{gr} C = \operatorname{gr} B}$$

It remains to show that $B \in \mathcal{M}$.

a Indeed, let $\bar{C} \in \mathcal{C} \subseteq \mathcal{M}$

Then $\operatorname{gr} A \subseteq \operatorname{gr} \bar{C} \subseteq \bigcup_{C \in \mathcal{C}} \operatorname{gr} C = \operatorname{gr} B$

b - We show that B is monotone.

Indeed, take $(x, x^*), (y, y^*) \in \operatorname{gr} B = \bigcup_{C \in \mathcal{C}} \operatorname{gr} C$

Then there exist $C_1, C_2 \in \mathcal{C}$ such that

$(x, x^*) \in \operatorname{gr} C_1$, $(y, y^*) \in \operatorname{gr} C_2$

$\xRightarrow[\text{chain}]{\mathcal{C} \text{ is a}}$ Without loss of generality, we may and do assume

that: $\operatorname{gr} C_1 \subseteq \operatorname{gr} C_2$

$$\Rightarrow \{(x, x^*), (y, y^*)\} \subseteq \operatorname{gr} C_2$$

$$\xRightarrow[\text{mono.}]{C_2} \langle x - y, x^* - y^* \rangle \geq 0$$

a and b $\Rightarrow B \in \mathcal{M}$

Together with $\xRightarrow[\text{Lemma}]{\text{Zorns}}$ $\exists$ a maximal element $\tilde{A} \in \mathcal{M}$

claim :

$\tilde{A}$ is maximally monotone.

$\tilde{A} \in \mathcal{M} \implies \tilde{A}$ is mono, $\operatorname{gr} A \subseteq \operatorname{gr} \tilde{A}$

We only need to verify maximality.

Suppose for eventual contradiction that $\tilde{A}$ is not maximally monotone. Then there exists $(z, z^*) \in X \times X \setminus \operatorname{gr} \tilde{A}$ such that (z, z^*) is monotonically related to $\operatorname{gr} \tilde{A}$, i.e.,

$(\forall (a, a^*) \in \operatorname{gr} \tilde{A}) \ \langle z - a, z^* - a^* \rangle \geq 0$.

Define $\tilde{B}$ via its graph by

$$\boxed{\operatorname{gr} \tilde{B}} := \operatorname{gr} \tilde{A} \cup \{(z, z^*)\}$$

$$\boxed{\supsetneq \operatorname{gr} \tilde{A}} \supseteq \operatorname{gr} A$$

observe that $\tilde{B}$ is monotone (verify) and extends $A \Rightarrow \tilde{B} \in \mathcal{M}$, and

$$\tilde{A} \preccurlyeq \tilde{B} \xRightarrow[\text{maximal element}]{\tilde{A} \text{ is a}} \tilde{A} = \tilde{B}$$

$$\Leftrightarrow \operatorname{gr} \tilde{A} = \operatorname{gr} \tilde{B}$$

which is absurd since

$$(z, z^*) \in \operatorname{gr} \tilde{B} \setminus \operatorname{gr} \tilde{A}.$$

Therefore, $\tilde{A}$ is maximally mono.

$\square$

Connection to optimization:

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} +\infty, & x < 0; \\ 1, & x = 0; \\ 0, & x > 0. \end{cases}$$

Find ∂f.

Verify that ∂f is monotone but Not maximally mono.

Next lecture:

Minty's theorem

Maximality is crucial for algorithms!
maximality and convexity, l.s.c.