MATH 235, Winter 2016

Sample Midterm

1. Consider the linear operator $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$$T(ax^2 + bx + c) = 4cx^2 + 2bx + a,$$

   (a) For $B = \{1, x, x^2\}$, the standard basis, find the B-matrix $[T]_B = A$ that represents T, i.e., represents the linear operator T relative to basis B.

   (b) Find a basis $B^{'}$ of P_2 such that the B'-matrix representation of T satisfies $[T]_B = 3[T]_{B'}$.

2. In each part below a vector space $\mathbb{V}$ and a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ are given. Decide whether the function is an inner product on $\mathbb{V}$.

   (a) $\mathbb{V} = \mathbb{R}^2$, with the function $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1y_1 + 51x_1y_2 + 51x_2y_1 + y_1y_2.$

   (b) $\mathbb{V} = P_2(\mathbb{R})$, with the function $\langle p(x), q(x) \rangle = (p(0) + p(1) + p(2))\,(q(0) + q(1) + q(2))\,.$

3. Let L be the linear operator on $\mathbb{V} = P_2(\mathbb{R})$ defined by

$$L(a + bx + cx^2) = (3a + 2b + 2c) - 2a + b + 2c)x - (a + b)x^2.$$

   Consider the basis $\mathbb{B} = \{x - x^2, 1 - x, -1 + x + x^2\}$ for $P_2(\mathbb{R})$.

   (a) Compute the matrix $[L]_B$.

   (b) Find a basis for $\mathrm{Ker}(L)$.

4. Find an orthonormal basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 3 \\ 0 & 3 & 1 & -2 & 1 \end{bmatrix}$$

   with the property that one of the vectors in your basis is a scalar multiple of the first column of A^T.

5. Let Q be an $m \times n$ matrix with orthonormal columns.

   (a) What is $\mathrm{rank}(Q)$?

   (b) Prove that $Q^TQ = I_n$.

   (c) Prove that $\|Qx\| = \|x\|$ for all $x \in \mathbb{R}^n$.

6. Let $\mathbb{V}, \mathbb{W}$ be vector spaces and $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Prove or disprove the following statements.

   (a) If $\mathrm{rank}(L) = \dim \mathbb{W}$ then L is an isomorphism.

   (b) If $\mathrm{Span}\{v_1, ..., v_n\} = \mathbb{V}$, then $\mathrm{Span}\{L(v_1), ..., L(v_n)\} = \mathrm{Range}(L)$.

   (c) If $Ker(L) = \{0\}$ and $\{v_1, ..., v_n\}$ is a linearly independent subset of $\mathbb{V}$, then $\{L(v_1), ..., L(v_n)\}$ is a linearly independent subset of $\mathbb{W}$.