Success of SDP Relaxation of MC

Goemans-Williamson .878 approx. algor. for MC

MC is one of Karp's NP-complete problems (APX-hard);
G-W '94 showed (with nonnegative weights on edges):

$$.87856(\mathrm{bnd}_{SDP}) \leq \mathrm{optvalue}_{MC} \leq \mathrm{bnd}_{SDP}$$

Extensions/Numerics

This result has been extended (e.g. Nesterov/97) to more general quadratic functions to obtain a $\frac{\pi}{2}$ guarantee
In practice, the strength of the bound is much tighter; large problems can be solved (many authors).

SDP arise from general quadratic approximations?

General Quadratic Approximations

Approximations from quadratic functions are stronger than from linear functions. E.g.

$x \in \{\pm 1\}$ iff $x^2 = 1$	$x \in \{0, 1\}$ iff $x^2 - x = 0$

QQPs

Let

$$q_i(y) = \frac{1}{2} y^T Q_i y + y^T b_i + c_i, \; y \in \mathbb{R}^n$$

$$(QQP) \quad \begin{array}{rll} q^* = & \min & q_0(y) \\ & \text{s.t.} & q_i(y) \leq 0 \\ & & i = 1, \ldots m \end{array}$$

Lagrangian Relaxation

Lagrangian; x Lagrange multiplier vector

$$L(y,x) = q_0(y) + \sum_{i=1}^m x_i q_i(y)$$

or equivalently (combine quad./lin. terms)

$$\begin{aligned} L(y,x) = & \tfrac{1}{2} y^T (Q_0 + \sum_{i=1}^m x_i Q_i) y \\ & + y^T (b_0 + \sum_{i=1}^m x_i b_i) \\ & + (c_0 + \sum_{i=1}^m x_i c_i) \end{aligned}$$

Weak Duality

Use hidden constraints

$$d^* = \max_{x \geq 0} \min_y L(y,x) \leq q^* = \min_y \max_{x \geq 0} L(y,x)$$

Homogenization

Homogenize the Lagrangian

multiply linear term by new variable y_0:

$y_0 y^T(b_0 + \sum_{i=1}^m x_i b_i), \;\; y_0^2 = 1$

use: strong duality for TRS; hidden SDP constraints

$$
\begin{array}{rcll}
d^* & = & \max\limits_{x\geq 0} \min_y & L(y,x) \\
 & = & \max\limits_{x\geq 0} \min\limits_{y_0^2=1} & \frac{1}{2} y^T(Q_0 + \sum_{i=1}^m x_i Q_i)y + t y_0^2 \\
 & & & \quad + y_0 y^T(b_0 + \sum_{i=1}^m x_i b_i) \\
 & & & \quad + (c_0 + \sum_{i=1}^m x_i c_i) - t \\
 & = & \max\limits_{x\geq 0, t} \min\limits_{y} & \frac{1}{2} y^T(Q_0 + \sum_{i=1}^m x_i Q_i)y + t y_0^2 \\
 & & & \quad + y_0 y^T(b_0 + \sum_{i=1}^m x_i b_i) \\
 & & & \quad + (c_0 + \sum_{i=1}^m x_i c_i) - t
\end{array}
$$

Hidden SDP Constraint in Lagrangian Dual

Hessian is $\succeq 0$

$$B := \begin{pmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{pmatrix},$$

$$\mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} := -\begin{bmatrix} t & \sum_{i=1}^m x_i b_i^T \\ \sum_{i=1}^m x_i b_i & \sum_{i=1}^m x_i Q_i \end{bmatrix}, \quad : \mathbb{R}^{m+1} \to \mathcal{S}^{n+1}$$

and the SDP constraint

$$B - \mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} \succeq 0.$$

NOTE: There is NO hidden constraint needed in convex case; e.g. if all q_i are convex.

Lagrangian Relaxation and Equivalent SDP

Dual-Primal Programs

Lagrangian Relaxation is equivalent to SDP (with $c_0 = 0$)

$$
(\textbf{DSDP}) \qquad \begin{array}{rll} d^* = & \sup & -t + \sum_{i=1}^m x_i c_i \\ & \text{s.t.} & \mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} \preceq B \\ & & x \in \mathbb{R}^m, t \in \mathbb{R} \end{array}
$$

As in LP, Dual of Dual; Use Opt. Strategy of Competing Player

$$
(\textbf{DD}) \qquad \begin{array}{rll} d^* \leq p^* := & \inf & \text{trace}\, BY \\ & \text{s.t.} & \mathcal{A}^* Y = \begin{pmatrix} -1 \\ c \end{pmatrix} \\ & & Y \succeq 0. \end{array}
$$

Quadratic Assignment Problem, (QAP)

QAP Problem

- n facilities i, l, A_{il} flow or weight;
  n locations j, k, B_{jk} distances;
  C_{ij} location costs
- $n \geq 16$ considered hard; SDP provides strong (though expensive) bounds/1998;
- Nugent $n = 30$ solved for first time using weakened SDP relaxation on computational grids (CONDOR)/2002;
- Exploit group symmetry in SDP relaxation of QAP; major advance in size and efficiency/2007

QAP Formulation

QAP Applications

designing of facility layouts; VLSI design (location of modules on chips); campus planning; scheduling; process communication; turbine balancing; typewriter keyboard design; many more ...

QAP Trace Formulation/Model

$$\text{(QAP)} \quad \mu^* := \min_{X \in \Pi} \ \text{trace}\, AXBX^T - 2CX^T$$

$A, B, C \in \mathcal{M}^n$; Π set of permutaion matrices.

Quadratic Assignment Problem, (QAP)

Permutation Matrices

$$
\begin{aligned}
\Pi &= \{n \times n : (0,1), \text{row/col sums } 1\} \\
&= \{X \in \mathcal{M}^n : X \circ X = X, Xe = X^T e = e\} \\
&= \{X \in \mathcal{M}^n : X^T X = I, Xe = X^T e = e, X \geq 0\} \\
&= \{X \in \mathcal{M}^n : X^T X = XX^T = I, Xe = X^T e = e, \\
&\qquad X \circ X = X, X \geq 0\}
\end{aligned}
$$

QQP Model of QAP/Add Redundant Constraints

$$
(\text{QAP}_E) \qquad
\begin{array}{rll}
\mu^* := & \min & \operatorname{trace} AXBX^T - 2CX^T \\
& \text{s.t.} & XX^T = I, X^T X = I \\
& & Xe = X^T e = e \\
& & X_{ij}^2 - X_{ij} = 0, \quad \forall i, j.
\end{array}
$$

Lagrangian Relaxation of QAP

Find SDP relaxation of QAP by taking dual of dual

(ignore $Xe = X^T e = e$ for now)

- Add $(0,1)$-constraints to objective function; use Lagrange multipliers W_{ij}

$$\mu_{\mathcal{O}} = \min_{\substack{XX^T=I \\ X^TX=I}} \max_{W} \operatorname{trace} AXBX^T - 2CX^T + \sum_{ij} W_{ij}(X_{ij}^2 - X_{ij})$$

homogenize obj. fn; multiply by a constrained scalar x_0

$$\mu_{\mathcal{O}} \geq \mu_R = \max_{W} \min_{\substack{XX^T=X^TX=I \\ x_0^2=1}} \operatorname{trace}\left[AXBX^T + W(X \circ X)^T - x_0(2C + W)X^T\right].$$

Lagrangian Relaxation/Dual

Grouping: quadratic, linear, constant terms

Lagrange multiplier w_0 for constraint on x_0; Lagrange multipliers S_b for $XX^T = I$, S_o for $X^TX = I$

$$\begin{aligned} \mu_{\mathcal{O}} \geq \mu_R \quad := \quad \max_{W} \min_{X,\, x_0} \quad & \operatorname{trace}\left[AXBX^T + W(X \circ X)^T + w_0 x_0^2 \right. \\ & \left. + S_b XX^T + S_o X^T X\right] \\ & - \operatorname{trace} x_0 (2C + W) X^T \\ & - w_0 - \operatorname{trace} S_b - \operatorname{trace} S_o . \end{aligned}$$

Lagrangian Relaxation/Dual

Vectorize X

define $x := \operatorname{vec} X$, $y^T := (x_0, x^T)$ and $w^T := (w_0, \operatorname{vec} W^T)$

$$\begin{aligned}\mu_R \;=\; \max_{W}\min_{y}\;\; & y^T\left[L_Q + Arrow(w) + \mathrm{B}^0\mathrm{Diag}\,(S_b) + \right. \\ & \left. \mathrm{O}^0\mathrm{Diag}\,(S_o)\right] y \\ & -w_0 - \operatorname{trace} S_b - \operatorname{trace} S_o\end{aligned}$$

Linear Transformations

$$L_Q := \begin{bmatrix} 0 & -\operatorname{vec}(C)^T \\ -\operatorname{vec}(C) & B \otimes A \end{bmatrix}, \quad (n^2+1) \times (n^2+1)$$

$$\mathrm{Arrow}\,(w) := \begin{bmatrix} w_0 & -\frac{1}{2} w_{1:n^2}^T \\ -\frac{1}{2} w_{1:n^2} & \operatorname{Diag}\,(w_{1:n^2}) \end{bmatrix},$$

$$\mathrm{B}^0\mathrm{Diag}\,(S) := \begin{bmatrix} 0 & 0 \\ 0 & I \otimes S_b \end{bmatrix}$$

and

$$\mathrm{O}^0\mathrm{Diag}\,(S) := \begin{bmatrix} 0 & 0 \\ 0 & S_o \otimes I \end{bmatrix}.$$

Dual of Dual

Hidden Semidefinite Constraint Yields the Equivalend SDP

$$
(D_{\mathcal{O}}) \quad \begin{array}{ll} \max & -w_0 - \operatorname{trace} S_b - \operatorname{trace} S_o \\ \text{s.t.} & L_Q + \operatorname{Arrow}(w) + \\ & \operatorname{B}^0\operatorname{Diag}(S_b) + \operatorname{O}^0\operatorname{Diag}(S_o) \succeq 0, \end{array}
$$

dual of this dual yields the semidefinite relaxation; $Y \succeq 0$ is $(n^2+1) \times (n^2+1)$, the dual matrix variable

$$
(SDP_{\mathcal{O}}) \quad \begin{array}{lll} \min & \operatorname{trace} L_Q Y & \\ \text{s.t.} & \operatorname{b}^0\operatorname{diag}(Y) = I & \operatorname{o}^0\operatorname{diag}(Y) = I \\ & \operatorname{arrow}(Y) = e_0 & Y \succeq 0 \end{array}
$$

Lagrangian Relaxation/Dual

Adjoint Operators

$$\operatorname{arrow}(Y) := \operatorname{diag}(Y) - (0, (Y_{0,1:n^2})^T.$$

$$\mathrm{b}^0\mathrm{diag}(Y) := \sum_{k=1}^{n} Y_{(k-1)n+1:kn,(k-1)n+1:kn}$$

$$[\mathrm{o}^0\mathrm{diag}(Y)]_{ij} := \operatorname{trace} Y_{(i-1)n+1:in,(j-1)n+1:jn}$$

Vectorize Permutation Matrix

$X \in \Pi$; $x = \text{vec}(X)$, $c = \text{vec}(C)$.

$$
\begin{aligned}
q(X) &= \text{trace}\, AXBX^T - 2CX^T \\
&= x^T(B \otimes A)x - 2c^T x \\
&= \text{trace}\, xx^T(B \otimes A) - 2c^T x \\
&= \text{trace}\, L_Q Y_X,
\end{aligned}
$$

$$
Y_X := \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix}
$$

Loss of Slater CQ

Comments

After adding the row/column sum constraints $Xe = X^T e = e$, we get that Slater's CQ fails; but we can explicitly regularize, i.e. find the smallest face/minimal cone. the SDP relaxation provides strong bounds, but expensive. Exploit group symmetries/special structure.