± 1 QUADRATIC PROGRAMMING

Formulations

$$(P) \quad \mu^* := \max\ q(x) := x^t Q x + c^t x, \\ x \in F := \{-1, 1\}^n,$$

Q $n \times n$ symmetric matrix, $c \in \mathbb{R}^n$.

Relaxations, $K \supset F$

$$(RP) \quad f(u) = \max_{x \in K} q_u(x) := x^t (Q - \operatorname{diag}(u)) x + u^t e + c^t x,$$

Solve Tractable Problem $B := \min_{u \in L} f(u)$
Upper Bound $\mu^* \leq B$

Three Relaxations

relaxed problem RP1

$$(RP_v^1) \quad f_1(v) := \max_{-1 \leq x \leq 1} q_v(x).$$

bound $B_1 := \min_{Q - \mathrm{diag}(v) \preceq 0} f_1(v)$.

relaxed problem RP2

$$(RP_u^2) \quad f_2(u) := \max_{||y||^2 = n} q_u(y).$$

bound $B_2 := \min_{u^t e = 0} f_2(u) = \min_u f_2(u)$.

Bound 3

$$Q^c := \begin{bmatrix} 0 & \frac{1}{2}c^t \\ \frac{1}{2}c & Q \end{bmatrix}.$$

$$q_u^c(y) := y^t(Q^c - \operatorname{diag}(u))y + u^t e$$

relaxed problem RP3

$$(RP_u^3) \qquad \begin{array}{rcl} f_3(u) & := & \max_{||y||^2=n+1} q_u^c(y) \\ & = & (n+1)\lambda_{\max}(Q^c - \operatorname{diag}(u)) + u^t e. \end{array}$$

bound $B_3 := \min_{u^t e=0} f_3(u) = \min_u f_3(u)$.

BOUND 1 - Convex Quadratic Programming

Consider the shifted function

$$q_v(x) := x^t(Q - \mathrm{diag}(v))x + v^t e + c^t x,$$

and the relaxed problem

$$(RP_v^1) \quad f_1(v) := \max_{-1 \le x \le 1} q_v(x).$$

Then a bound for (P) is

$$B_1 := \min_{Q - \mathrm{diag}(v) \preceq 0} f_1(v).$$

Properties for Bound B_1.

1. $\mu^* \leq B_1$.
2. f_1 is convex, finite valued, with subdifferential

$$\partial f_1(v) = \operatorname{conv}\{z = (1 - x_i^2) \in \mathbb{R}^n : \\ x = (x_i) \in \mathbb{R}^n - 1 \leq x \leq 1,\ f_1(v) = q_v(x)\}.$$

3. B_1 is attained for some $v \in \mathbb{R}^n$ such that $\lambda_{max}(Q - \operatorname{diag}(v)) = 0$.
4. $$B_1 = \inf_{Q - \operatorname{diag}(v) \prec 0} f_1(v).$$

Optimality conditions

Lagrangian

$$L_1(v, \Lambda) := f_1(v) + \operatorname{trace} \Lambda(Q - \operatorname{diag}(v)), \qquad \Lambda \succeq 0$$

Optimality conditions:

$$\begin{array}{cr} 0 \in \partial f_1(v) - \operatorname{diag}(\Lambda) & \text{(stationarity)} \\ \operatorname{trace} \Lambda(Q - \operatorname{diag}(v)) = 0 & \text{(complementary slack)} \\ \Lambda \succeq 0 & \text{(multiplier sign).} \end{array}$$

BOUND 2 - Optimization Over Sphere

$$q_u(y) := y^t(Q - \text{diag}(u))y + u^t e + c^t y,$$

the second relaxed problem

$$(RP_u^2) \quad f_2(u) := \max_{||y||^2 = n} q_u(y).$$

Now a bound for (P) is

$$B_2 := \min_{u^t e = 0} f_2(u) = \min_u f_2(u).$$

Properties for bound B_2

1. $\mu^* \leq B_2$.
2. f_2 is convex, finite valued, with subdifferential

$$\partial f_2(u) = \text{conv}\{z = (1 - y_i^2) \in \mathbb{R}^n : \\ y = (y_i) \in \mathbb{R}^n,\ ||y||^2 = n,\ f_2(u) = q_v(y)\}.$$

3. The bound B_2 is attained for some $u \in \mathbb{R}^n$. Moreover, if $B_2 > \mu^*$, then the hard case holds for (RP_u^2).

THEOREM: The bound $B_1 = B_2$

In fact

$$\begin{aligned} f_2(u) &= q_u(y_u) \\ &\geq \max_{(-1 \leq y \leq 1)} q_{(u+\lambda e)}(y) \\ &= f_1(u + \lambda e) \end{aligned}$$

and
$B_2 = f_2(u)$, with Lagrange multiplier λ for (RP_u^2) if and only if $B_2 = f_2(u) = f_1(u + \lambda e) = B_1$.

THEOREM:

$$B_1 = B_2 = B_3 = B_4$$

Suppose that $f_4(v) := \max_{-1 \leq x \leq 1} q_v^c(x)$ and

$$B_4 := \min_{Q^c - \mathrm{diag}(v) \preceq 0} f_4(v).$$

Then

$$B_1 = B_2 = B_3 = B_4.$$

Corollary

Let B be an $(n-1) \times (n-1)$ real symmetric matrix, and consider the perturbation of B given by

$$A = \begin{pmatrix} B & -\alpha \end{pmatrix}$$

± 1 Quadratic Programming Problem

(P) $\mu^* := \max q(x) \quad x \in F \cap S$

where: $F = \{\pm 1\}^n$, $S \subset \mathbb{R}^n$, $F \cap S \neq \emptyset$
quadr. obj.: $q(x) := x^t Q x - 2c^t x$, $\quad Q \in \mathcal{S}^n, c \in \mathbb{R}^n$

EXAMPLES

Quadratic Assignment Problem, QAP, (in trace formulation)

$$(QAP) \quad \max_{X \in \Pi} q(X) := \operatorname{trace}(AXB - 2C)X^t$$

Max-Cut Problem, MC

$$(MC) \quad \max \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j), \quad x \in F$$

HIDDEN SEMIDEFINITE CONSTRAINTS

Trust Region Subproblem (TRS)

$$\begin{aligned}
\mu^* &= \min_x q(x) \text{ s.t. } x^t x = s^2 \ (\le s^2) \\
&= \min_x \max_\lambda L(x,\lambda) \\
&\ge \max_\lambda \min_x L(x,\lambda) \\
&= \max_{Q-\lambda \succeq 0} \min_x L(x,\lambda) \\
&= \max_{Q-\lambda \succeq 0} h(\lambda) = \mu^*
\end{aligned}$$

where

$x_\lambda = (Q - \lambda I)^\dagger c; \quad h(\lambda) = L(x,\lambda) = -c^t (Q - \lambda I)^\dagger c + \lambda s^2$

nonconvex objective but min-max = max-min

hidden convexity provides the hidden convex dual program

Relaxations of ± 1-Quadratic Progr.

use perturbations

$$q_u(x) := q(x) + x^t \operatorname{Diag}(u)x - u^t e$$

Relaxation 0:

$$f_0(u) := \max_x q_u(x)$$

$$\begin{array}{rcl} \mu^* \leq B_0 & := & \min_{u^t e=0} f_0(u) = \min_u f_0(u) \\ & = & \min_{Q+\operatorname{Diag}(u) \preceq 0} f_0(u) \end{array}$$

B_0 is Lagrangian dual of original quadratic program in the following form with u_i as Lagrange multipliers

$$\min q(x) \text{ s.t. } x_i^2 = 1, \ \forall i$$

Relaxation 1

sphere of radius $\sqrt{n}$

$$f_1(u) := \max_{||x||^2=n} q_u(x)$$

$$\mu^* \leq B_1 := \min_{u^t e=0} f_1(u) = \min_u f_1(u)$$

$$\begin{aligned} B_1 &= \min_u \max_{x^t x=n} q_u(x) \\ &= \min_{u,\lambda} \max_x q_u(x) + \lambda(x^t x - n) \\ &= \min_{v^t e=0} \max_x q_v(x), \text{ with } v = u + \lambda e \\ &= B_0 \end{aligned}$$

Relaxation 2

Unit Box

$$f_1(u) := \max_{|x_i|\leq 1} q_u(x)$$

$$\mu^* \leq B_2 := \min_{Q+\mathrm{Diag}(u)\preceq 0} f_2(u) = \min_u f_2(u)$$

$$\begin{aligned} B_2 &= \min_u \max_{x_i^2\leq 1} q_u(x) \\ &= \min_u \min_{\lambda\geq 0} \max_x q_u(x) + \sum_i \lambda_i(1-x_i^2) \\ &= B_0 \text{ after } v = u - \lambda \end{aligned}$$

i.e. same bound again

Relaxation 1^c

Homogenization; sphere radius $= \sqrt{n+1}$

$$Q^c := \begin{bmatrix} 0 & -c^t \\ -c & Q \end{bmatrix}; \quad q_u^c(y) := y^t(Q^c + \text{diag}(u))y - u^t e$$

$$\begin{aligned} f_1^c(u) &:= \max_{\|y\|^2=n+1} q_u^c(y) \\ &= (n+1)\lambda_{\max}(Q^c + \text{diag}(u)) - u^t e \end{aligned}$$

$\mu^* \le B_1^c := \min_{u^t e=0} f_1^c(u) = \min_u f_1^c(u)$

$$\begin{aligned} B_1^c &= \min_v \max_{y^t y=n+1} q_v^c(y) = \min_v \max_y q_v^c(y) \\ &= \min_{u,u_0} \max_{x,x_0} u_0(x_0^2 - 1) + x^t(Q + \text{Diag}(u))x \\ &\qquad -2x_0 c^t x - u^t e \\ &= \min_u \max_{x,x_0^2=1} x^t(Q + \text{Diag}(u))x - 2x_0 c^t x - u^t e \\ &= B_0 \end{aligned}$$

i.e. same bound again; similarly for B_2^c

Relaxation 3

Semidefinite SDP ($c = 0$)

$q(x) = x^t Q x = \operatorname{trace} Q x x^t, \qquad Y = x x^T$

$B_3 := \max \operatorname{trace} QY \text{ s.t } \operatorname{diag}(Y) = e, Y \succeq 0$

The dual is

$B_3 = \min y^t e \text{ s.t. } Q - \operatorname{Diag}(y) \preceq 0$

$Q - \operatorname{Diag}(y - \frac{e^t y}{n} e) \preceq \frac{e^t y}{n} I$, with $w = y - \frac{e^t y}{n} e$ and $z = \frac{e^t y}{n}$

$$\begin{array}{lll} B_3 = & \text{minimize} & nz \\ & \text{subject to} & Q - \operatorname{Diag}(w) \preceq zI \\ & & w^t e = 0 \end{array}$$

So: $B_3 = B_1^c$, i.e. all bounds are equal to Lagrangian relaxation of the equivalent quadratically constrained program. The SDP relaxation is the dual of the Lagrangian dual.

Allow more General Perturbations?

$q_{V,d}(x) := x^t(Q+V)x + (c+d)^t x$

THEOREM: Suppose that $q_{V,d}(x) \geq q(x), \quad \forall x \in F$.
Then

$$V = P + U, \text{ with } P \succeq 0, \; U \text{ is diagonal,}$$
$$\text{and trace}\, U = 0.$$

Moreover, there exists a diagonal matrix W, with trace $W = 0$, such that

$$\max_x q_{V,d}(x) \geq \max_x q_{W,0}(x).$$

□

Therefore, we need only consider diagonal perturbations, i.e. we have the best quadratic approximation - by duality we have the best SDP relaxation.