Theorem 3.1 ([10])**.** *Suppose that the optimal value* v_P^{conic} *in* (1.4) *is finite. Then strong duality holds for the pair* (3.1) *and* (3.2)*, or equivalently, for the pair* (1.4) *and* (3.2)*; i.e.,* $v_P^{\text{conic}} = v_{SP}^{\text{conic}} = v_{DSP}^{\text{conic}}$ *and the dual optimal value* v_{DSP}^{conic} *is attained.* ■

3.2 Theorems of the alternative

In this section, we state some theorems of the alternative for the Slater CQ of the conic convex program (1.4), which are essential to our reduction process. We first recall the notion of recession direction (for the dual (1.5)) and its relationship with the minimal face of the primal feasible region.

Definition 3.2. *The convex cone of* recession directions *for* (1.5) *is*

$$\mathcal{R}_{\mathrm{D}} := \{D \in \mathcal{V} : \mathcal{A}(D) = 0,\ \langle C, D\rangle = 0,\ D \succeq_{K^*} 0\}. \tag{3.3}$$

The cone $\mathcal{R}_{\mathrm{D}}$ consists of feasible directions for the homogeneous problem along which the dual objective function is constant.

Lemma 3.3. *Suppose that the feasible set* $\mathcal{F}_P \neq \emptyset$ *for* (1.4)*, and let* $0 \neq D \in \mathcal{R}_{\mathrm{D}}$*. Then the minimal face of* (1.4) *satisfies*

$$f_P \trianglelefteq K \cap \{D\}^\perp \lhd K.$$

Proof. We have

$$0 = \langle C, D\rangle - \langle \mathcal{F}_P, \mathcal{A}(D)\rangle = \langle C - \mathcal{A}^*(\mathcal{F}_P), D\rangle.$$

Hence $C - \mathcal{A}^*(\mathcal{F}_P) \subseteq \{D\}^\perp \cap K$, which is a face of K. It follows that $f_P \subseteq \{D\}^\perp \cap K$. The required result now follows from the fact that f_P is (by definition) a face of K, and D is nonzero. □

Lemma 3.3 indicates that if we are able to find an element $D \in \mathcal{R}_{\mathrm{D}}\backslash\{0\}$, then D gives us a smaller face of K that contains $\mathcal{F}_P^Z$. The following lemma shows that the existence of such a direction D is *equivalent* to the failure of the Slater CQ for a feasible program (1.4). The lemma specializes [12, Theorem 7.1] and forms the basis of our reduction process.

Lemma 3.4 ([12])**.** *Suppose that* $\operatorname{int} K \neq \emptyset$ *and* $\mathcal{F}_P \neq \emptyset$*. Then exactly one of the following two systems is consistent:*

1. $\mathcal{A}(D) = 0$*,* $\langle C, D\rangle = 0$*, and* $0 \neq D \succeq_{K^*} 0$ *(*$\mathcal{R}_{\mathrm{D}}\backslash\{0\}$*)*
2. $\mathcal{A}^* y \prec_K C$ *(Slater CQ)*

Proof. Suppose that D satisfies the system in Item 1. Then for all $y \in \mathcal{F}_P$, we have $\langle C - \mathcal{A}^* y, D\rangle = \langle C, D\rangle - \langle y, (\mathcal{A}(D))\rangle = 0$. Hence $\mathcal{F}_P^Z \subseteq K \cap \{D\}^\perp$. But $\{D\}^\perp \cap \operatorname{int} K = \emptyset$ as $0 \neq D \succeq_{K^*} 0$. This implies that the Slater CQ (as in Item 2) fails.

Conversely, suppose that the Slater CQ in Item 2 fails. We have $\operatorname{int} K \neq \emptyset$ and

$$0 \notin (\mathcal{A}^*(\mathbb{R}^m) - C) + \operatorname{int} K.$$

Therefore, we can find $D \neq 0$ to separate the open set $(\mathcal{A}^*(\mathbb{R}^m) - C) + \operatorname{int} K$ from 0. Hence we have

$$\langle D, Z\rangle \geq \langle D, C - \mathcal{A}^* y\rangle\,,$$

for all $Z \in K$ and $y \in \mathcal{W}$. This implies that $D \in K^*$ and $\langle D, C\rangle \leq \langle D, \mathcal{A}^* y\rangle$, for all $y \in \mathcal{W}$. This implies that $\langle \mathcal{A}(D), y\rangle = 0$ for all $y \in \mathcal{W}$; hence $\mathcal{A}(D) = 0$. To see that $\langle C, D\rangle = 0$, fix any $\hat{y} \in \mathcal{F}_P$. Then $0 \geq \langle D, C\rangle = \langle D, C - \mathcal{A}^* \hat{y}\rangle \geq 0$, so $\langle D, C\rangle = 0$. □