Matrix Completions

Henry Wolkowicz

Dept. of Combinatorics and Optimization
University of Waterloo

CO769/2013

Simple Completion Problem

Hermitian positive semidefinite completion

Given: $A = A^* \succeq 0,$ with $\mathrm{diag}(A) > 0.$
Consider the *completion problem*

$$\max \{\det(B) : B \succeq 0,\ \mathrm{diag}(B) = \mathrm{diag}(A)\}.$$

(Hadamard's Inequality: $\det(B) \leq \prod_{i=1}^{n} B_{ii}$) implies maximum occurs when B^{-1}, and so B, is diagonal.

Proof? Optimum is interior! Differentiate log-det !!

$$0 = \nabla \left(\log\det(B) + \sum_i \lambda_i \,\mathrm{trace}\, E_{ii}(B - A) \right) = B^{-1} + \sum_i \lambda_i E_{ii}$$

I.e., B^{-1} must be diagonal.

Banded Completions

$2k+1$ bands are specified in A

$a_{ij} \forall |i-j| \leq k$ are fixed; all principal submatrices within $2k+1$ bands are positive definite.

THEOREM (Dym and Gohberg, 1981)

1. If
$$\mathcal{D}_k = \left\{ B : B \succeq 0,\ b_{ij} = a_{ij}, \forall |i-j| \leq k \right\},$$
then $\mathcal{D}_k \neq \emptyset$.
2. $\max \left\{ \det(B) : B \in \mathcal{D}_k \right\}$ occurs at the unique matrix $B \in \mathcal{D}_k$ having the property that for all $|i-j| \geq k+1$, the (i,j)-entry of B^{-1} equals 0.

Partial Hermitian Matrix, $A = A^*$, Completion

components $A_{ij}, ij \in \mathcal{E}$ are specified (fixed) (others free)

$$A = A(\mathcal{G}), \qquad \text{undirected graph } \mathcal{G} = (N, \mathcal{E})$$

positive semidefinite completion problem

find components $A_{ij}, ij \notin \mathcal{E}$, such that the resulting matrix $A \succeq 0$, is positive semidefinite.

Chordal Graphs

Definitions:

1. *chordal graph*: no minimal cycles of length ≥ 4.
2. A graph $\mathcal{G}$ is *completable* if and only if any $\mathcal{G}$-partial positive definite matrix has a positive definite completion.

THEOREMS (GJSW, 1984)

1. $\mathcal{G}$ is completable if and only if $\mathcal{G}$ is chordal.
2. Suppose that $\mathrm{diag}(A)$ is fixed and $A(\mathcal{G})$ has a positive definite completion. Then there is a unique such completion, B, with maximum determinant; and it is characterized by B^{-1} having 0 components in positions corresponding to all $kl \notin \mathcal{E}$.

Nearest Matrix Problems

Best Approximate Completion Problem

Let $H = H^T \geq 0$ be matrix of weights with $diag(H) > 0$; let $A = A^*$ be given given Hermitian matrix; let $\mathcal{L}$ be a linear transformation;

$$f(P) := \|H \circ (A - P)\|_F^2,$$

(Hadamard product, Frobenius norm)

$$\text{(AC)} \qquad \begin{array}{rcc} \mu^* := & \min & f(P) \\ & \text{subject to} & \mathcal{L}P = b \\ & & P \succeq 0, \end{array}$$

Optimal Solution $\bar{P}$ of AC in Unconstrained Case

$$\nabla f(\bar{P}) \in (\mathcal{S}^n_+ - \bar{P})^+ \qquad \text{(polar cone)}$$

THEOREM

The matrix $\bar{P} \succeq 0$ solves AC if and only if

$$\text{trace}(H^{(2)} \circ (\bar{P} - A))(P - \bar{P}) \geq 0, \ \forall P \succeq 0,$$

where $H^{(2)} = H \circ H$ is the Hadamard squared term.

COROLLARY

COROLLARIES

1. If $H = E$, the matrix of ones, then the (unique) optimal solution of AC is $\bar{P} = A_{\succ}$, where $A_{\succ}$ is the positive part of A in the Loewner partial order.
2. If $H \succ 0$, then the (unique) optimal solution $\bar{P}$ of AC is the solution of

$$H \circ \bar{P} = (H \circ A)_{\succ}.$$

General Constrained Case

Lagrangian

For $\Lambda \in \mathcal{H}^n$ (Hermitian) and $y \in \mathcal{C}^m$, let

$$L(P, y, \Lambda) = f(P) + \langle y, b - \mathcal{L}P \rangle - \text{trace}\, \Lambda P$$

denote the *Lagrangian* of AC.
primal program AC is equivalent to

$$\mu^* = \min_P \max_{\substack{y \\ \Lambda \succeq 0}} L(P, y, \Lambda).$$

Assume generalized Slater's constraint qualification

$$\exists\, P \succ 0 \text{ with } \mathcal{L}P = b\ ,$$

holds for AC.

Duals

Lagrangian Dual

Then

$$\mu^* = \max_{\substack{y \\ \Lambda \succeq 0}} \min_P L(P, y, \Lambda)$$

differentiate and get the Wolfe dual

(DAC)

$$\begin{array}{rcc} \mu^* := & \max & f(P) + \langle y, b - \mathcal{L}P \rangle - \operatorname{trace} \Lambda P \\ & \text{subject to} & \nabla f(P) - \mathcal{L}^* y - \Lambda = 0 \\ & & \Lambda \succeq 0. \end{array}$$

THEOREM

Primal-Dual Framework!!

The matrix $\bar{P} \succeq 0$ and vector-matrix $\bar{y}, \bar{\Lambda} \succeq 0$ solve AC and DAC if and only if

$$\begin{array}{cc} \mathcal{L}\bar{P} = b & \text{primal feas} \\ 2H^{(2)} \circ (\bar{P} - A) - \mathcal{L}^* \bar{y} - \bar{\Lambda} = 0 & \text{dual feas} \\ \text{trace}\, \bar{\Lambda}\bar{P} = 0 & \text{compl slack} \end{array}$$

compl slack value= duality gap value

Optimal solution of AC

Solve the two equations

$$\bar{P} - (\bar{P} - 2H^{(2)} \circ (\bar{P} - A) + \mathcal{L}^* y)_{\succ} = 0$$

and

$$\mathcal{L}\left(\bar{P} - 2H^{(2)} \circ (\bar{P} - A) + \mathcal{L}^* y\right)_{\succ} = b.$$

Suppose $A = 0$ and $H = \frac{1}{\sqrt{2}}E$.

the optimal solution of AC is:

$$\bar{P} = (\mathcal{L}^* y)_{\succ},$$

where

$$\mathcal{L}(\mathcal{L}^* y)_{\succ} = b.$$

$\mathcal{L} = 0$ case

Unconstrained - Using only Weights

$$2H^{(2)} \circ (P - A) - \Lambda = 0 \qquad \text{dual feas}$$
$$-P + \mu\Lambda^{-1} = 0 \qquad \text{perturbed compl slack}$$

Note

$$H_{ii} = 0 \ (P_{ii} \text{ free }) \ \Rightarrow \Lambda_{ii} = 0$$

(dual variable Λ takes role of P^{-1})

Primal-Dual Interior-Point Algorithms

Linearization of the second equation

$$-(P+h)+\mu\Lambda^{-1}-\mu\Lambda^{-1}l\Lambda^{-1}=0$$

We get

$$h=\mu\Lambda^{-1}-\mu\Lambda^{-1}l\Lambda^{-1}-P.$$

and

$$l=\frac{1}{\mu}\{-\Lambda(P+h)\Lambda\}+\Lambda$$

Linearization of the dual feasibility equation

$$2H^{(2)}\circ h-l=-(2H^{(2)}\circ(P-A)-\Lambda).$$

Dual-Step-First

Eliminate the primal step h and solve for the dual step l

$$\begin{aligned} l &= 2H^{(2)} \circ h + (2H^{(2)} \circ (P - A) - \Lambda) \\ &= 2H^{(2)} \circ (\mu\Lambda^{-1} - \mu\Lambda^{-1} l \Lambda^{-1} - P) \\ &\quad + (2H^{(2)} \circ (P - A) - \Lambda). \end{aligned}$$

Newton equation

$$2H^{(2)} \circ (\mu\Lambda^{-1} l \Lambda^{-1}) + l = 2H^{(2)} \circ (\mu\Lambda^{-1} - A) - \Lambda.$$

or equivalently

$$\begin{aligned} &\left[2\,\mathrm{Diag}(\mathrm{vec}(H^{(2)})\mu(\Lambda^{-1} \otimes \Lambda^{-1}) + I\right] \mathrm{vec}(l) \\ &= \mathrm{vec}\left(2H^{(2)} \circ (\mu\Lambda^{-1} - A) - \Lambda\right). \end{aligned}$$

Matrix Representations

F denotes $k \times 2$ matrix with row s denoting the s-th nonzero element of H

for $s = 1, \dots, k$: $(F_{s1}, F_{s2})_{s=1,\dots k} = \{ij : H_{ij} \neq 0\}$.

(kl, ij) component of the Hadamard-Kronecker product of $2H^{(2)} \circ (\mu\Lambda^{-1} I \Lambda^{-1})$

$$
\begin{aligned}
&2\mu \operatorname{trace} E_{kl} \left(H^{(2)} \circ \Lambda^{-1} E_{ij} \Lambda^{-1}\right) \\
&\quad = 2\mu \operatorname{trace} e_k e_l^t \left(H^{(2)} \circ \Lambda^{-1} e_i e_j^t \Lambda^{-1}\right) \\
&\quad = 2\mu \operatorname{trace} e_l^t \left(H^{(2)} \circ \Lambda_{:,i}^{-1} \Lambda_{j:}^{-1}\right) e_k \\
&\quad = 2\mu H_{lk}^{(2)} \Lambda_{li}^{-1} \Lambda_{jk}^{-1}.
\end{aligned}
$$

This can be “vectorized” in MATLAB. The “sparsity” (free variables of A) of H can be exploited.

Primal-Step-First

Many elements of H are sufficiently large

eliminate l and solve for h first.

$$2H^{(2)} \circ h + \frac{1}{\mu}\Lambda h \Lambda = \Lambda - \frac{1}{\mu}\Lambda P \Lambda - \left(2H^{(2)} \circ (P - A) - \Lambda\right)$$

or equivalently

$$\left[2\,\mathrm{Diag}(\mathrm{vec}(H^{(2)})) + \tfrac{1}{\mu}(\Lambda \otimes \Lambda)\right] \mathrm{vec}(h) \\ \mathrm{vec}\left(\Lambda - \tfrac{1}{\mu}\Lambda P \Lambda - (2H^{(2)} \circ (P - A) - \Lambda)\right).$$

The (kl, ij) component of the Kronecker product $\Lambda \otimes \Lambda$ is found from

$$\mathrm{trace}\, E_{kl}\Lambda E_{ij}\Lambda = \mathrm{trace}\, e_k e_l^t \Lambda e_i e_j^t \Lambda = \Lambda_{li}\Lambda_{jk}.$$

This can be "vectorized" in MATLAB. The "infinities" (fixed variables of A) of H can be exploited.

The Diagonal of P and H

dual feasibility

$$2H^{(2)} \circ (P - A) - \Lambda = 0$$

For $\Lambda \succ 0$, we cannot fix ($P_{ii} = A_{ii}$) or free ($H_{ii} = 0$) the diagonal of A

Thanks for your attention!

Matrix Completions

Henry Wolkowicz

Dept. of Combinatorics and Optimization
University of Waterloo

CO769/2013