C&O 769, Topics in Semidefinite Programming and Applications (Winter 2013)
Assignment 3
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Due: Tuesday Apr. 16, 2013

Directions/Instructions: The exam is *long*. Complete as much as you can.

Contents

1 Preliminaries/Definitions

$\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices equipped with the trace inner product $\langle A, B\rangle = \operatorname{trace} AB$. $\mathbb{S}^n_+$ denotes the cone of positive semidefinite matrices and we denote $A \succeq 0$ for $A \in \mathbb{S}^n_+$.

Definition 1.1 *The primal SDP is*

$$(\mathrm{P}) \qquad \begin{array}{rl} \min & \langle C, X\rangle \\ s.t. & \mathcal{A}X = b \\ & X \succeq 0 \end{array}$$

where $C \in \mathcal{S}^n$*, the space of real* $n \times n$ *symmetric matrices, and* $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ *is a linear transformation.*

Definition 1.2 *A set* $\mathcal{F}$ *is a* <u>*spectrahedron*</u> *if it has the form*

$$\mathcal{F} := \left\{ x \in \mathbb{R}^m : \mathcal{L}(x) := A_0 + \sum_{i=1}^m x_i A_i \succeq 0 \right\},$$

for some symmetric matrices $A_i, i = 0, 1, \ldots, m$*.*

Definition 1.3 *A set $\mathcal{F}$ is a* <u>*projected spectrahedron*</u> *if it has the form*

$$\mathcal{F} := \left\{ x \in \mathbb{R}^m : \exists y \in \mathbb{R}^p, \mathcal{L}(x) := A_0 + \sum_{i=1}^{m} x_i A_i + \sum_{j=1}^{p} y_j A_j \succeq 0 \right\},$$

for some symmetric matrices $A_i, i = 0, 1, \ldots, m, B_j, j = 1, \ldots, p$.

Similar definitions hold for spectrahedra in symmetric matrix space, i.e., symmetric matrices that satisfy $\mathcal{A}X = b, X \succeq 0$ (the intersection of an affine set and the cone of positive semidefinite matrices).

2 Spectrahedra

1. Consider the feasible set (spectrahedron) defined by

$$\mathcal{F} := \left\{ (x, y) \in \mathbb{R}^2 : \mathcal{L}(x, y) := \begin{bmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{bmatrix} \succeq 0 \right\}.$$

   (a) Draw the spectrahedron in $\mathbb{R}^2$.

   (b) Let $p(t) = \det(\lambda I - \mathcal{L}(x, y)) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$ be the characteristic polynomial of $\mathcal{L}(x, y)$. Find the expressions for $c_i, i = 0, 1, 2$ as functions of x, y and show that positive semidefiniteness is equivalent to

$$c_2 \leq 0, c_1 \geq 0, c_0 \leq 0.$$

   (c) Show that the boundary of the spectrahedron is determined by the determinant of the matrix inequality (a polynomial inequality). What is the role of the other two (polynomial) inequalities in Item 1b?

2. Show that spectrahedra are always closed sets. Are projected spectrahedra always closed sets?

3 Instances of SDP

1. For (P) the primal SDP, show that the linear transformation $\mathcal{A}X$ can always be expressed as $(\langle A_i, X \rangle) \in \mathbb{R}^m$, for some $A_i \in \mathcal{S}^n, i = 1, \ldots, m$.

2. Consider the primal SDP

$$\begin{array}{ll} \min & \langle C, X \rangle \\ \text{s.t.} & \mathcal{A}X = b \\ & X \succeq 0 \end{array}$$

   where $C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $b = 1 \in \mathbb{R}$, and $\mathcal{A}X = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X \right\rangle$.

(a) Draw the feasible set (spectrahedron) in $\mathbb{R}^2$, i.e., as a function in the two matrix elements x_{11}, x_{12}.

(b) Is there a rational optimal solution?

4 Duality

1. Consider the primal cone program with additional constraints

$$
(\mathrm{CP}) \qquad \begin{array}{ll} \min & \langle C, X\rangle \\ \text{s.t.} & \mathcal{A}X = b \\ & \mathcal{B}X \le c \\ & X \succeq 0, X \in P, \end{array}
$$

where $\mathcal{B} : \mathcal{S}^n \to \mathbb{R}^p$ is another linear transformation and P is a closed convex cone.

(a) Using the game theory approach, derive the dual program for (CP).

(b) State an appropriate constraint qualification for (CP).

(c) Using the notions of minimal face, provide a dual program for (CP) for which strong duality holds without any constraint qualification.

(d) For the following SDP, with $\alpha > 0$, first show that a duality gap exists and then find a dual program for which strong duality holds.

$$
\begin{array}{ll} \min & \alpha X_{11} \\ \text{s.t.} & X_{22} = 0 \\ & X_{11} + 2X_{23} = 1 \\ & X \in \mathbb{S}^3_+ \end{array}
$$

2. Consider the two matrix norms on the space of real $m \times n$ matrices: the operator norm given by the largest singular value $\|A\| = \sigma_{\max}(A)$; and the nuclear (trace class) norm given by the sum of the singular values $\|A\|_* = \sum_{i=1}^r \sigma_i(A)$, where r denotes the rank.

(a) Show that the operator norm is found by solving the SDP

$$
(\mathrm{SDPoper}) \qquad \begin{array}{ll} \max & \operatorname{trace} 2A^T X_{12} \\ \text{s.t.} & \operatorname{trace} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = 1 \\ & X \succeq 0 \end{array}
$$

(b) Find the dual of (SDPoper) and then find an SDP primal-dual pair for calculating the nuclear norm.

5 Euclidean Distance Matrices, EDMs

1. Let the rows of $P \in \mathbb{M}^{nr}$ contain the position of n vectors in $\mathbb{R}^r$.

   (a) Derive the linear operator $\mathcal{K}$ on $\mathcal{S}^n$ that sends the Gram matrix $B = PP^T$ to the Euclidean distance matrix D. Derive the Moore-Penrose generalized inverse of $\mathcal{K}$.

   (b) Given $B \succeq 0$ and $D = \mathcal{K}(B)$, prove that the elements $d_{ij} = \sqrt{D_{ij}}$ satisfy the triangle inequalities $d_{ik} \leq d_{ij} + d_{jk}$. Is the converse true?

2. Suppose that a partial EDM $D \in \mathcal{S}^n$ is given with exact partial distance information; $G = (V, E)$ is the corresponding weighted graph where an edge $ij \in E$ if, and only if, the corresponding weight D_{ij} is known. The embedding dimension of the problem is r.

   (a) Phrase the problem of finding the nearest EDM problem using the linear operator $\mathcal{K}$ and the unknown Gram matrix B.

   (b) Suppose that the graph G has a clique $\bar{G} = (\bar{V}, \bar{E})$, where the cardinality $|\bar{V}| = k$. Using facial reduction, find an equivalent problem with a smaller Gram matrix. Carefully show why the two problems are equivalent, i.e. show how to obtain the optimal solution of the original problem given the optimal solution of the facially reduced problem.

6 Algorithms

1. Let $\alpha(G)$ denote the stability number of an undirected graph $G = (V, E)$. The Lovasz theta function provides an upper bound on the stability number and can be found by solving the following SDP.

$$
(\mathrm{TP}) \qquad \begin{array}{cc} \max & \langle J, X \rangle \\ \text{s.t.} & \operatorname{trace} X = 1 \\ & X_{ij} = 0, \forall ij \in E \\ & X \succeq 0 \end{array}
$$

   (a) Find the dual program of (TP).

   (b) Derive the equations/procedure for the HKM direction.

   (c) Derive the equations/procedure for the GN direction.