C&O 769, Topics in Semidefinite Programming and Applications (Winter 2013)
Assignment 1
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Due: Thursday Jan. 31, 1:00PM (before class),

1 Slater Condition and Minimal Face

Consider the primal SDP

$$\text{(PSDP)} \qquad \max\{\operatorname{trace} CX : \mathcal{A}X = b, X \succeq 0\}, \tag{1}$$

where $\mathcal{A} : \mathbb{S}^n_+ \to \mathbb{R}^m$ is the appropriate linear transformation and $X \succeq 0$ denotes positive semidefinite. The Slater constraint qualification, SCQ, refers to the existence of a positive definite $\hat{X} \succ 0$ such that $\mathcal{A}\hat{X} = b$.

1. Show that the SCQ holds for PSDP if and only if the minimal face for PSDP is all of $\mathbb{S}^n_+$.

   Solution: First we note that $F \trianglelefteq \mathbb{S}^n_+$ if, and only if, $F = U\mathbb{S}^r_+U^T$, for some U with $U^TU = I_r$. We let $Q = [U\ V]$ be an orthogonal matrix. Then $X \in F$ if, and only if, $\text{Range}(X) \subseteq \text{Range}(U)$ if, and only if, $\text{Null}(X) \supseteq \text{Range}(V)$.

   Now, for necessity, suppose that SCQ holds, i.e., suppose that there exists $\hat{X} \succ 0$ with $\mathcal{A}\hat{X} = b$. Let the minimal face satisfy $f_p \trianglelefteq F \trianglelefteq \mathbb{S}^n_+$, for some face F. Then $\hat{X} \in f_p$ and $f_p = U\mathbb{S}^n_+U^T = \mathbb{S}^n_+$, with $U = X$.

   For sufficiency, suppose that $f_p = \mathbb{S}^n_+$ but SCQ fails, i.e., there exists $\hat{X}$ with rank $r < n$ and it is the maximum rank over all X feasible. Let U be full column rank r with $\text{Range}(U) = \text{Range}(\hat{X})$. Then X feasible means that $X = \hat{X} + \mathcal{N}(y) \succeq 0$, for some y, where the $\text{Range}(\mathcal{N}) = \text{Null}(\mathcal{A})$. Then necessarily the $\text{Range}(\mathcal{N}(y)) \subseteq \text{Range}(U)$, or else the rank of X is greater than r. Therefore the feasible set $\mathcal{F} \subseteq U\mathbb{S}^r_+U^T \subsetneq \mathbb{S}^n_+$, a contradiction to $f_p = \mathbb{S}^n_+$.

2. Find the dual, DSDP. State the appropriate SCQ for DSDP, i.e. strict feasibility. Then show that the same result holds for DSDP, i.e. SCQ holds if, and only if, the minimal face (of slacks) is all of $\mathbb{S}^n_+$.

   Solution: The dual can be found using the min-max approach and the hidden constraint. The dual is

$$\text{(DSDP)} \qquad \min\{\operatorname{trace} b^Ty : \mathcal{A}^*y \succeq C\}. \tag{2}$$

2 Perturbed Optimality Conditions

Consider the primal dual pair from Problem 1 above. Suppose that the Slater CQ (strict feasibility) holds for both. Also assume that the linear transformation $\mathcal{A}$ is onto. Show that for each fixed barrier parameter $\mu > 0$, the perturbed optimality conditions (dual feasibility, primal feasibility, perturbed complementary slackness) has a unique primal-dual solution (X_μ, y_μ, Z_μ) with $X_\mu, Z_\mu \succ 0$.

> **Solution:** A proof can be found in the Handbook on SDP in the Chapter by Monteiro and Todd.

3 Matrix Inequalities

Suppose that $X \succ 0$. Show that $X - vv^T \succ 0$ if, and only if, $v^T X^{-1} v < 1$.

> **Solution:** A proof follows from an application of the Schur complement.

4 Bellman and Fan, 1963

Suppose that each individual nonnegative variable in a standard-form LP is replaced by a positive semidefinte matrix to get

$$\begin{array}{lll} \min & \sum_{j=1}^n \operatorname{trace} C_j X_j & \\ \text{s.t.} & \sum_{j=1}^n \operatorname{trace} A_{ij} X_j = b_i, & \forall i = 1, \ldots, m \\ & X_j \succeq 0, & \forall j = 1, \ldots, n. \end{array}$$

Show how this can be written as an SDP.

5 Numerical Solutions

Consider the SDP:

$$\begin{array}{ll} \min & \operatorname{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X \\ \text{s.t.} & \operatorname{trace} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X = 4 \\ & \operatorname{trace} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X = 0 \\ & X \succeq 0 \end{array}$$

1. State the dual problem.
2. State the perturbed optimality conditions that define the central path for a given $\mu > 0$.

3. Start from the strictly feasible points

$$X = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

and perform two iterations of the primal-dual interior-point method using the HKM search direction, i.e., find the search direction and the *exact* step to the boundary and then backtrack (using $.9$) to stay strictly feasible. (You can use MATLAB to help with the arithmetic.) State the value of the primal and dual objectives at each iteration.

4. Solve the SDP using CVX.