UNIVERSITY OF WATERLOO

MIDTERM EXAMINATION

WINTER 2013

Surname: ____________________

First Name: ____________________

Id.#: ____________________

Course	C&O 367
Course Title	Nonlinear Optimization
Instructor	Henry Wolkowicz Sect. 001
Date of Exam	Wed, Feb. 13, 2013
Time Period	6:00-8:30 P.M., MC 4064
Number of Exam Pages (including this cover sheet)	8
Exam Type	Closed Book/NO calculators
Additional Instructions	Write your name AND answers in the booklets provided. Provide careful justification for your answers. Show your work/reasoning/arithmetic.

Contents

1 Convex Functions: ==== 15 Marks ===

Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be a convex set and let $f : C \to \mathbb{R}$.

1. Define:

   (a) f is a convex function on C;

   Solution:

   $$\{x, y \in C, \lambda \in (0,1)\} \implies f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y). \tag{1}$$

   (b) epi(f), epigraph of f.

   Solution:

   $$\text{epi}(f) = \left\{(r, x) \in \mathbb{R}^{n+1} : r \geq f(x)\right\} \qquad \text{region above the graph}$$

2. Prove that f is a convex function if, and only if, epi(f) is a convex set.

   Solution: Suppose that (1) above holds and $(r, x), (s, y) \in$ epi(f), and $\lambda \in (0,1)$. Then by definition of epi(f) and λ, we have $\lambda r + (1-\lambda)s \geq \lambda f(x) + (1-\lambda)f(y)$. And by (1) we now conclude that $\lambda r + (1-\lambda)s \geq f(\lambda x + (1-\lambda)y)$. This proves that epi$(f)$ is a convex set.

   Now suppose that epi(f) is a convex set. Let $x, y \in C$ and $\lambda \in (0,1)$. Then the points $(f(x), x), (f(y), y)$ are both in epi(f). By convexity, we conclude that $\lambda(f(x), x) + (1 - \lambda)(f(y), y) \in$ epi(f), i.e., that (1) holds.

3. Let $\emptyset \neq S \subseteq \mathbb{R}^n$. Let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$. Prove that the distance to the farthest point of S

   $$f(x) := \sup_{y \in S} \|x - y\|$$

   is a convex function.

   Solution: We observe that the epigraph of the supremum of convex functions is the intersection of the epigraphs, i.e., if $g(x) := \sup_{y \in T} h(x, y)$, then

   $$r \geq g(x) \ \underline{\text{iff}} \ r \geq h(x, y), \forall y \in T.$$

And we use the fact that the intersection of convex sets is a convex set. We conclude using the previous result on the characterization of convex functions and epigraphs that $f(x)$ is a convex function.

2 Second Order Approximation: ==== 20 Marks ===

1. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = x_1x_2 + e^{(x_1-x_2)} + \log(x_1x_3)$$

   (a) What is the gradient and Hessian of f at $x = (1, 1, 1)^T$.

   (b) Where is f differentiable? Twice differentiable?

   (c) Give a second order approximation of f near $x = (1, 1, 1)^T$. (Use Taylor's Theorem.)

   (d) Is the direction $d = (1, 1, 1)^T$ a descent direction or ascent direction at the point $x = (1, 1, 1)^T$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ and x be defined by

$$f(x) = 8x_1^2 + 8x_2^2 - x_1^4 - x_2^4 - 1, \qquad x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

   (a) What is the gradient and Hessian of f at x?

   Solution: $\nabla f(x) = \begin{pmatrix} 16x_1 - 4x_1^3 \\ 16x_2 - 4x_2^3 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 7.5 \end{pmatrix}$, $\nabla^2 f(x) = \begin{bmatrix} 16 - 12x_1^2 & 0 \\ 0 & 16 - 12x_2^2 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$ We note that the Hessian is positive definite at x.

   (b) Provide a second order approximation for f near x using Taylor's Theorem. Then compute the minimum x^* of this quadratic approximation. Justify whether or not this is a local or global minimum.

   Solution: The quadratic approximation is

$$m_2(y) \;=\; f(x) + (y - x)^T \nabla f(x) + \tfrac{1}{2}(y - x)^T \nabla^2 f(x)(y - x) \;.$$

   The global minimum can be obtained by taking one step of Newton's method from x, i.e.,

$$x^* = x - (\nabla^2 f(x))^{-1} \nabla f(x) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}^{-1} \begin{pmatrix} 7.5 \\ 7.5 \end{pmatrix},$$

This follows from the fact that the function is a strictly convex quadratic and the theorem in the text and in class notes, that states that Newton's method finds the global minimum of a strictly convex quadratic function in one iteration.

3 Positive Semidefinite Matrices: ==== 5 Marks ===

For each of the following matrices A, find a positive number μ such that $A+\mu I$ is positive semidefinite but not positive definite. (Justify your solutions.)

1. $A = \begin{bmatrix} 3 & 7 & 0 \\ 7 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

4 Local Optimum: ==== 5 Marks ===

Show that the function $f(x) = (x_2 - x_1^2)^2 + x_1^5$ has only one stationary point. Show that it is neither a local maximum or a local minimum.

Solution: Set the gradient $\nabla f(x) = \begin{pmatrix} -4x_1(x_2 - x_1^2) + 4x_1^4 \\ 2(x_2 - x_1^2) \end{pmatrix} = 0$. The second equation implies that $x_2 = x_1^2$. Substituting into the first equation yields $x_1 = x_2 = 0$. The Hessian at $x = 0$ is $\nabla^2 f(x) = \begin{bmatrix} -4x_2 + 12x_1^2 + 16x_1^3 & -4x_1 \\ -4x_1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, i.e., it is positive semidefinite but $x = 0$ is a saddle point as a descent direction can be found.

5 AGM: ==== 10 Marks ===

Find the minimizer of

$$f(x_1, x_2) = 4x_1 + \frac{x_1}{x_2^2} + \frac{4x_2}{x_1}$$

Solution: See page 65 of the text.

6 Quadratic Functions: ==== 15 Marks ===

Suppose that $q(x)$ is a quadratic function of n variables. Find necessary and sufficient conditions for $q(x)$ to be bounded below. (Prove the results.)

Solution: See problem 6 on assignment 2.