C&O367: Nonlinear Optimization
(Winter 2013)
Assignment 5
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Posted Monday, Mar. 4

Due: Thursday, Mar. 19 10:00AM (before class)

1 Hyperplanes

1.1 Separating Hyperplanes

First we recall some definitions:

(a) K is a <u>cone</u> if $\alpha K \subseteq K, \forall \alpha \geq 0$;

(b) K is a <u>convex cone</u> if it is a cone and $K + K \subseteq K$.

(c) For a set $S \subseteq \mathbb{R}^n$, the <u>*polar (or dual) cone of* S</u> is $S^+ := \{\phi : \langle \phi, s\rangle \geq 0, \forall s \in S\}$. (Note that we denote $(S^+)^+ = S^{++}$. Also, the text uses $S^* = S^+$.)

1. Let A, B, C be closed convex sets in $\mathbb{R}^n$ such that

$$A + C = B + C.$$

Prove that $A = B$.

Solution: Suppose that the conclusion fails, i.e., without loss of generality, we assume that there exists $\hat{a} \in A \backslash B$. Then by the separation theorem there exists ϕ such that $\langle \phi, \hat{a}\rangle < \alpha \leq \langle \phi, b\rangle, \forall b \in B$. But then $\langle \phi, \hat{a} + c\rangle < \langle \phi, b + c\rangle, \forall b \in B$. This contradicts the fact that $\hat{a} + c = b + c$, for some $b \in B, c \in C$.

2. Let $K \subseteq \mathbb{R}^n$. Show that K is a ccc (closed, convex, cone) if, and only if, $K = K^{++}$. (Hint: Try a *Hail Mary*.)

Solution: By the above definition, for any K, K^+ is the intersection of halfspaces passing through the origin, so it is always a closed and convex cone. Hence if $K = K^{++}$, K is ccc.

For the other side, assume that K is closed. We have $K^{++} := \{\phi : \langle \phi, s\rangle \geq 0, \forall s \in K^+\}$. For any $x \in K$, by definition, we have $\langle \phi, x\rangle \geq 0$ for all $\phi \in K^+$, so by definition $x \in K^{++}$. Hence $K \subseteq K^{++}$. To prove that $K^{++} \subseteq K$, we prove contrapositive; we prove that if $x \notin K$, then $x \notin K^{++}$. K is closed and $x \notin K$, so by separation theorem, there exists $0 \neq a \in \mathbb{R}^n$ and an $\alpha \in \mathbb{R}$ such that

$y \cdot a \geq \alpha$, for all $y \in K$, and $x \cdot a < \alpha$. Because K is a cone, I claim that we can choose $\alpha = 0$. Tho show that, for any $\delta > 0$ and $y \in K$, we have $\delta y \in K$, so we must have $\delta y \cdot a \geq \alpha$, or $y \cdot a \geq \frac{\alpha}{\delta}$. By sending δ to ∞, we have $y \cdot a \geq 0$, for all $y \in K$. By definition, $a \in K^+$. But we have $x \cdot a < 0$, so again by definition $x \notin K^{++}$, as we wanted to prove.

1.2 Supporting Hyperplanes

1. Suppose that C_1, C_2 are convex sets in $\mathbb{R}^n$ such that C_1 has interior points and C_2 does not contain any interior points of C_1. Prove that there is a hyperplane H in $\mathbb{R}^n$ such that C_1 and C_2 lie in the opposite closed half-spaces determined by H, i.e., there exist an $0 \neq a \in \mathbb{R}^n$ and an $\alpha \in \mathbb{R}$ such that

$$x \cdot a \leq \alpha \leq y \cdot a, \quad \forall x \in C_1, \forall y \in C_2.$$

(Hint: Note that $C_2 \cap \operatorname{int} C_1 = \emptyset$ if, and only if, $0 \notin C_2 - \operatorname{int} C_1$.)

Solution: If C_1 is convex, $\operatorname{int} C_1$ is also convex, then $C_2 - \operatorname{int} C_1$ is also convex. We have $0 \notin C_2 - \operatorname{int} C_1$, so by separation there exists an $0 \neq a \in \mathbb{R}^n$ and an $\beta \in \mathbb{R}$ such that $0 \leq \beta \leq (c_2 - c_1) \cdot a$, for all $c_2 \in C_2$ and $c_1 \in \operatorname{int} C_1$. So we have $c_1 \cdot a \leq c_2 \cdot a$. I claim that this is also true for any point in $c_1 \in C_1$. This is because if c_1 is on the boundry of C_1, then there is a sequence $c_1^k \in \operatorname{int} C_1$ such that $c_1^k \to c_1$. We have $c_1^k \cdot a \leq c_2 \cdot a$ for all k, so this is also true for the limit point c_1. Hence $c_1 \cdot a \leq c_2 \cdot a$, for all $c_1 \in C_1$ and $c_2 \in C_2$. Now α can be picked any point between $\sup\{c_1 \cdot a : c_1 \in C_1\}$ and $\inf\{c_2 \cdot a : c_1 \in C_2\}$.

2. Let $S = \{x \in \mathbb{R}^2_+ : x_1 x_2 \geq 1\}$. Show that the closed convex set S is an intersection of halfspaces.

Solution: The boundry of S is $\{x \in \mathbb{R}^2 : x_1 x_2 = 1\}$. I claim that S is equal to the intersection of all supporting halfspaces at points in its boundry. By symple calculus results, the supporting hyperplane at $x = (t, 1/t)$ is $\frac{x_1}{t^2} + x_2 = \frac{2}{t}$. So we can express S as

$$S = \bigcap_{t>0} \{x \in \mathbb{R}^2_+ : \frac{x_1}{t^2} + x_2 \geq \frac{2}{t}\}$$

For the answer to be complete, we have to check this equality. For one side, assume that $(x_1, x_2) \in \text{RHS}$. So we have $\frac{x_1}{t^2} + x_2 \geq \frac{2}{t}$ for all $t > 0$. If we hit both sides of the inequality with x_1 we have

$$\frac{x_1^2}{t^2} + x_2 x_1 \frac{2x_1}{t} \geq 0 \Rightarrow (\frac{x_1}{t} - 1)^2 - 1 + x_1 x_2 \geq 0$$

For this to be true for all $t > 0$, we must have $x_1 x_2 \geq 1$. The other side is also easy to prove.

3. Suppose that $S \subseteq \mathbb{R}^n$ is a closed set, has nonempty interior, and has a supporting hyperplane at every point in its boundary. Show that S is a convex set.

   Solution: For any point z on the boundry of S, define the supporting half-space as $H_z = \{x : \alpha_z \cdot x \leq b_z\}$, such that $S \subset H_z$, and $\alpha_z \cdot z = b_z$. I claim that $S = \cap_{z\in\partial(S)} H_z$, which is clearly convex. It is clear that $S \subseteq \cap_{z\in\partial(S)} H_z$. For the other side, assume that $\bar{y} \in \cap_{z\in\partial(S)} H_z$. If $\bar{y} \notin S$, pick a point $\bar{x} \in \text{int}(S)$. There is point $\bar{z}$ on the linesegment $[\bar{x}, \bar{y}]$ such that $\bar{z} \in \partial(S)$. By the hypothesis, $S \subset H_{\bar{z}}$. However $\bar{y} \in \cap_{z\in\partial(S)} H_z$, so $\bar{y} \in H_{\bar{z}}$. This means that the linesegment $[\bar{x}, \bar{y}]$ is on the boundry of $H_{\bar{z}}$; $[\bar{x}, \bar{y}] \subset \{x : \alpha_{\bar{z}} \cdot x = b_{\bar{z}}\}$, so we have $\alpha_{\bar{z}} \cdot \bar{x} = b_{\bar{z}}$. We picked $\bar{x}$ such that $\bar{x} \in \text{int}(S)$, so there is $r > 0$ such that $B(\bar{x}, r) \subset S$, which means $B(\bar{x}, r) \subset H_{\bar{z}}$. This is a contradiction to $\alpha_{\bar{z}} \cdot \bar{x} = b_{\bar{z}}$. Hence $S = \cap_{z\in\partial(S)} H_z$ is a convex set.

2 Optimality Conditions and Duality

Consider the primal optimization problem

$$\begin{array}{ll} \min & x^2 + 1 \\ \text{s.t.} & (x-2)(x-4) \leq 0 \\ & x \in \mathbb{R} \end{array}$$

1. What is the feasible set, the optimal value, and the optimal solution?

   Solution: The feasible set is the interval $[2, 4]$. $x^2 + 1$ is increasing on the feasible region, so the optimal solution is $x^* = 2$ and the optimal value is $p^* = 5$.

2. Plots and Values:

   (a) Plot the objective value $x^2 + 1$ versus x.

   Solution: See next question.

   (b) On the same plot show the feasible set, optimal point and value p^*, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ.

   Solution: The Lagrangian is $L(x, \lambda) = (1 + \lambda)x^2 - 6\lambda x + (1 + 8\lambda)$. Let's define $f_0 := x^2 + 1$ and $f_1 := x^2 - 6x + 8$, then we have $L(x, \lambda) = f_0 + \lambda f_1$. The objective value versus x is shown as f_0 in Figure 1. The Lagrangian is also plotted for some other values of λ.

   (c) Verify the lower bound property $p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$.

   Solution: We have $f_1(2) = 0$, so all the lagrangians for $\lambda \geq 0$ pass through the point $(2, 5)$. Hence $5 = p^* \geq \inf_x L(x, \lambda)$.

   (d) Derive and sketch the Lagrange dual function as a function of λ.

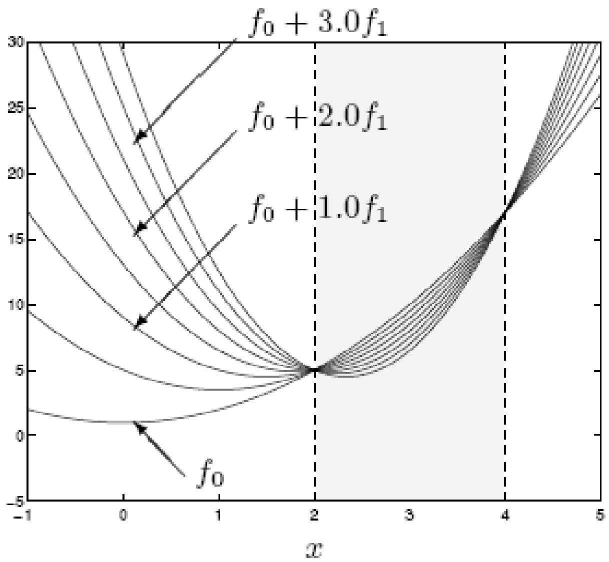

Figure 1: Objective function and Lagrangian versus x.

Solution: In the lagrangian function, the coefficient of x^2 is $(1+\lambda)$, so the lagrangian is unbounded for $\lambda \leq -1$. For $\lambda > -1$, by a simple derivation, the minimum of the lagrangian is get at $x = \frac{3\lambda}{1+\lambda}$. By substitution, we get:

$$g(\lambda) = \begin{cases} \frac{-9\lambda^2}{1+\lambda} + 1 + 8\lambda & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

The plot is in Figure 2.

3. State the dual problem and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?

Solution: The Lagrangian dual problem is:

$$\begin{array}{ll} \max & \dfrac{-9\lambda^2}{1+\lambda} + 1 + 8\lambda \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

The second derivative of the objective function in $\frac{-18}{(1+\lambda)^3}$ which is nagative for $\lambda \geq 0$, so the objective function is concave. By elementary calculus, we can see that the optimal solution is $\lambda^* = 2$ with the dual optimal solution equal to $d^* = 5$, so here strong duality holds.

4. Let $p^*(u)$ denote the optimal value of the perturbed problem

$$\begin{array}{ll} \min & x^2 + 1 \\ \text{s.t.} & (x-2)(x-4) \leq u \\ & x \in \mathbb{R} \end{array}$$

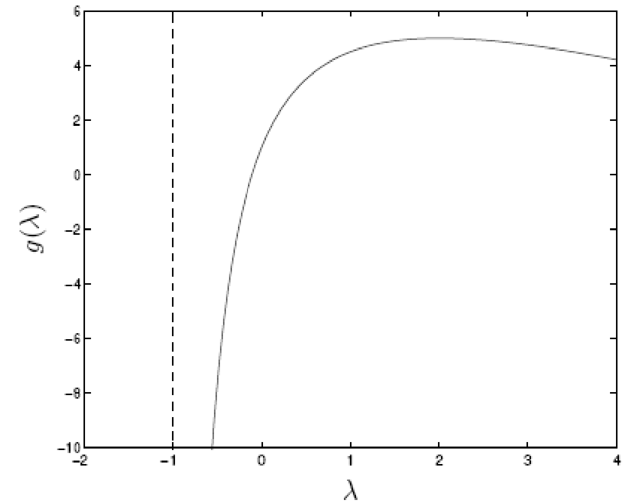

Figure 2: Lagrange dual function as a function of λ.

as a function of u. Plot $p^*(u)$. Verify that $\frac{\partial p^*(0)}{\partial u} = -\lambda^*$.

Solution: The minimum of $(x-2)(x-4)$ is -1, so the purtued problem is feasible for $u \geq -1$. For every $u \geq -1$, the feasible region is spcified by the roots of $x^2 - 6x - 6 = u$ and is $[3 - \sqrt{1+u}, 3 + \sqrt{1+u}]$. The function $x^2 + 1$ is increasing for positive values of x. For $-1 \leq u \leq 8$, the optimal solution is $x^*(u) = 3 - \sqrt{1+u}$. For $u \geq 8$, the lower bound $3 - \sqrt{1+u}$ goes below zero, so $x^*(u) = 0$. We can write:

$$p^*(u) = \begin{cases} \infty & u < -1 \\ 11 + u - 6\sqrt{1+u} & -1 \leq u \leq 8 \\ 1 & u \geq 8. \end{cases}$$

The plot is in Figure 3. It is easy to check that $\frac{\partial p^*(0)}{\partial u} = -2 = -\lambda^*$

3 CVX, Numerical Solutions

Consider the LP

$$\begin{array}{ll} \text{min} & 4x_1 + 15x_2 + 12x_3 + 2x_4 \\ \text{s.t.} & 2x_2 + 3x_3 + x_4 \geq 1 \\ & x_1 + 3x_2 + x_3 - x_4 \geq 1 \\ & x \geq 0 \end{array}$$

Solve this LP using CVX installed in MATLAB.
(See http://cvxr.com/cvx/ on how to install CVX inside of MATLAB. Hand in your input and output with the optimal value and solution.)

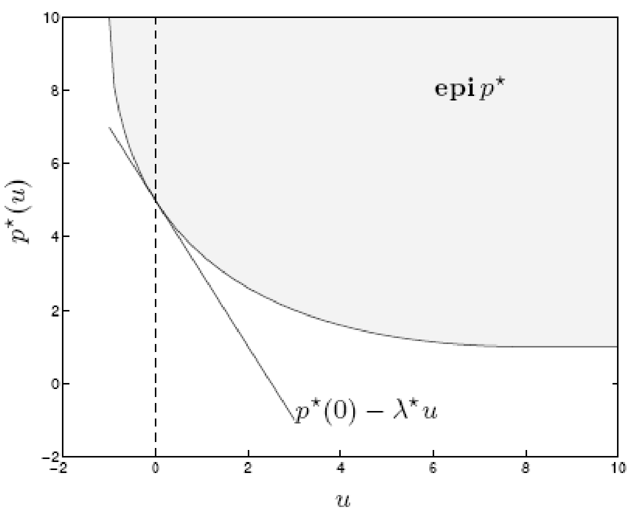

Figure 3: $\mathtt{p}^*(\mathtt{u})$.