CO 466/666: Continuous Optimization
Problem Set 5

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Winter 2012.
Handed out: 2012-Feb-27.
Due: 2012-Mar-9, at the start of class.

Contents

1 Lagrange Multiplier Theorem

Consider the abstract convex program discussed in class, i.e.

$$\begin{array}{lll} p^* := & \inf & f(x) \\ & \text{s.t.} & g(x) \preceq_K 0 \\ & & x \in \Omega. \end{array} \tag{CP}$$

Here:

- X, Y are Hilbert spaces (wlog e.g., $\mathbb{R}^n, \mathbb{R}^m$, respectively) with inner-products denoted by $\langle \cdot, \cdot \rangle$;
- $\Omega \subseteq X$ is convex;
  $K \subseteq Y$ is a closed convex cone with non-empty interior;
- $f : X \to \mathbb{R}$ is a convex real valued function;
- $g : X \to Y$ is a K-convex function, i.e.,
  $g(\lambda x + (1-\lambda)y) \preceq_K \lambda g(x) + (1-\lambda)g(y)$, $\forall x, y \in X$, $\forall\, 0 \le \lambda \le 1$;
- The optimal value p^* is finite valued.

Show the following:

1. $g(x)$ is a K-convex function if, and only if, $x \mapsto \langle \phi, g(x) \rangle$ is a convex function for all $\phi \in K^+$, where K^+ is the (nonnegative) polar cone of K.

2. Show that Strong duality holds for (CP) under the Slater constraint qualification: $\exists \hat{x} \in \Omega$ such that $g(\hat{x}) \prec_K 0$[1], i.e., show that there exists an optimal Lagrange multiplier $\lambda^* \in K^+$ such that

$$p^* = \inf_{x \in \Omega} \{f(x) + \langle \lambda^*, g(x) \rangle\}. \tag{1}$$

(Use the hyperplane separation argument outlined in class.)

3. Moreover, suppose that we have attainment in (CP), i.e., there exists $x^* \in \Omega$ such that $g(x^*) \preceq_K 0$ and $f(x^*) = p^*$. Show that complementary slackness holds, i.e., that $\langle \lambda^*, g(x^*) \rangle = 0$.

4. Use the above Lagrange multiplier theorem to prove strong duality for (CP), i.e. show that

$$p^* = \min_{x \in \Omega} \max_{\lambda \in K^+} \{f(x) + \langle \lambda, g(x) \rangle\} = d^* := \max_{\lambda \in K^+} \min_{x \in \Omega} \{f(x) + \langle \lambda, g(x) \rangle\}.$$

[1] equivalently $g(\hat{x}) \in \operatorname{int}(-K)$