CO 466/666: Continuous Optimization
Problem Set 2

H. Wolkowicz

Winter 2012.
Handed out: 2012-Feb-9.
Due: 2012-Feb-17, at the start of class.

Contents

1 Hyperplane Separation Theorems

1. Suppose that $\bar{x} \notin K$, a closed convex subset of $\mathbb{R}^n$. Show that there exists a hyperplane that strictly separates $\bar{x}$ from K.

2. Let K, L be two nonempty convex subsets of $\mathbb{R}^n$. Suppose that K, L are disjoint, $K \cap L = \emptyset$. Show that there exists a hyperplane that separates the two sets, i.e. there exists a $u \in \mathbb{R}^n$ such that

$$\langle u, k \rangle \leq \langle u, l \rangle, \forall k \in K, \forall l \in L.$$

2 Finitely Generated Cone and Theorems of the Alternative

1. Let $S \subset \mathbb{R}^n$ and define the <u>*convex cone generated by S*</u>

$$\operatorname{cone} S := \left\{ x = \sum_{i=1}^{k} \alpha_i s_i : \alpha_i \geq 0, s_i \in S, \forall i = 1, \ldots, k, \text{ and } k = 1, 2, \ldots \right\}$$

Let $V := \{v_1, \ldots, v_t\} \subset \mathbb{R}^n$. Show that the *finitely generated cone*, $K = \operatorname{cone} V$, is closed.

2. Prove the following Theorems of the Alternative:

   (a) Let A be a given $p \times n$ matrix, and $b \in \mathbb{R}^n$. Then: either

$$I: \qquad Ax \leq 0, b^T x > 0 \qquad \text{has a solution } x \in \mathbb{R}^n$$

   or

$$II: \qquad A^T y = b, y \geq 0, \text{ has a solution } y \in \mathbb{R}^p$$

   but never both.

   (b) Let A, C, D be given matrices with A nonvacuous (contains at least one element). Then: either

$$I: \qquad Ax > 0, Cx \geq 0, Dx = 0 \qquad \text{has a solution } x$$

   or

$$II: \qquad A^T y_1 + C^T y_2 + D^T y_3 = 0, 0 \neq y_1 \geq 0, y_2 \geq 0, y_3 \text{ has a solution}$$

   but never both.

   (c) Can you Use Item 1 and the Lemma in class on closed convex cones to prove both theorems of the alternative above?

3 Problems in the Text

12.5, 12.10, 12.19.